ON THE CURVE STRAIGHTENING FLOW OF INEXTENSIBLE, OPEN, PLANAR CURVES

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Abstract. We consider the curve straightening flow of inextensible, open, planar curves generated by the Kirchhoff bending energy. It can be considered as a model for the motion of elastic, inextensible rods in a high friction regime. We derive governing equations, namely a semilinear fourth order parabolic equation for the indicatrix and a second order elliptic equation for the Lagrange multiplier. We prove existence and regularity of solutions, compute the energy dissipation, prove its coercivity and conclude convergence to equilibrium, namely to a straight curve, at an exponential rate.

1. Introduction

We consider the L^2-gradient flow generated by the Kirchhoff bending energy (cp. [Kir59, Dil92])

$$\mathcal{E}[z] := \frac{1}{2}\int_0^1 |z''|^2\, ds \tag{1.1}$$

on the manifold

$$\mathcal{A} := \{z \in H^2([0,1],\mathbb{R}^2) : |z'| \equiv 1\}$$

of free, open, planar curves of length 1 which are parametrised by their arclength. The L^2-gradient flow is described by the system (cp. Appendix A)

$$\begin{cases} \partial_t z + z'''' - (\lambda_1 z')' = 0\ , \\ z''|_{s=0,1} = 0\ , \\ z''' - \lambda_1 z'|_{s=0,1} = 0\ , \\ |z'| = 1\ , \\ z(t=0,.) = z_I(.)\ , \end{cases} \tag{1.2}$$

where $\lambda_1 = \lambda_1(t,s) \in \mathbb{R}$ is a Lagrange multiplier function determined by the constraint on the arclength

$$|z'| \equiv 1 \tag{1.3}$$

and $z_I \in \mathcal{A}$ represents the initial datum of the evolution. Here and throughout the paper $'$ denotes derivatives with respect to the arclength.

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The system (1.2) can be considered as a model for the motion of elastic, inextensible rods in a high friction regime and has appeared in the modelling of Actin-filaments in biological cells (cp. [OS09c], [OS09b]). Currently the mathematical modelling of biopolymers and biopolymer networks is a field of high scientific interest and elastic rod models have recently also been used for the modelling of the DNA (cp [BMS06, BRMRE06]). The primary motivation for this study is to obtain qualitative results on the behaviour of these models which have the potential to give insight into the behaviour of the respective biological systems.

Continuing the discussion of the system (1.2), observe that the centre of mass, $\int_0^1 z\,ds$, is a conserved quantity of solutions to the problem (1.2). Furthermore the energy of the initial datum is given by

$$\mathcal{E}_I := \frac{1}{2}\int_0^1 |z_I''|^2\,ds\,. \tag{1.4}$$

Specialising Theorem 8 and Lemma 4 in [OS09a] implies (see also Appendix A for more details):

Theorem 1. *Let $z_I \in \mathcal{A}$, then there exist $T > 0$ and $z \in L^\infty\left((0,T); H^2(0,1)\right) \cap C^{0,1/8}\left([0,T]; C^1([0,1])\right) \cap H^1\left((0,T); L^2(0,1)\right)$ and $\lambda_1 \in L^2\left((0,T); \mathcal{M}(0,1)\right)$, satisfying $z(t=0,.) = z_I$, $|z'| \equiv 1$ and*

$$\int_0^T \int_0^1 \left[z'' \cdot v'' + \partial_t z \cdot v + \lambda_1 z' \cdot v'\right] ds\,dt = 0\,,$$

for every smooth $v : [0,T] \times [0,1] \to \mathbb{R}^2$.

In the theorem above we use the notation $\mathcal{M}(0,1)$ for the space of Radon measures on the interval $(0,1)$. The fact that the existence Theorem 1 is only local in time is due to the fact that in the more complex setting of [OS09a] additional geometric assumptions on z are required which cannot be guaranteed for all times. In the present special case (1.2), however, these assumptions are redundant and the proof of local existence of the above theorem can be extended. Here and in the rest of the paper we abbreviate the notation of function spaces writing the subscripts t for function spaces on $\{t \in [0,\infty)\}$ and the subscript s for function spaces on $\{s \in [0,1]\}$.

Corollary 2. *Let $z_I \in \mathcal{A}$, then there are $z \in H^1_{t,\mathrm{loc}} L^2_s \cap \mathcal{C}_t^{0,\frac{1}{2}} L^2_s \cap \mathcal{C}_t^{0,\frac{1}{8}} \mathcal{C}^1_s \cap L^\infty_t H^2_s$ and $\lambda_1 \in L^2_{\mathrm{loc},t} \mathcal{M}_s$ such that the pair (z, λ_1) is a weak solution of (1.2) satisfying*

$$\int_0^\infty \int_0^1 \left[z'' \cdot v'' + \partial_t z \cdot v + \lambda_1 z' \cdot v'\right] ds\,dt \tag{1.5}$$

for all $v \in \mathcal{C}_c^\infty(\mathbb{R}_+, \mathcal{C}_s^\infty)$, $z(t=0,.) = z_I$ and the constraint (1.3) in a pointwise sense. It holds that $\partial_t z \in L^2_t L^2_s$ with $\|\partial_t z\|_{L^2_t L^2_s} \le \sqrt{2\,\mathcal{E}_I}$.

Proof. A scetch of the proof, which is presented in detail in [OS09a], can be found in Appendix A. The crucial difference between the present setting and the more complex problem in [OS09a] is that the inequality (A.2) holds without additional terms. Hence the infinite sum for $n = 1, 2, ...$ is bounded and yields a telescoping series which in the limit as the step size converges to zero becomes (A.3), the estimate on $\|\partial_t z\|_{L^2_t L^2_s}$. □

In the present study we show that the problem (1.2) is equivalent to the system

$$\begin{cases} \partial_t \omega + \omega'''' - \omega'^2 \omega'' - (\omega' \lambda)' - \omega' \lambda' = 0\,, \\ -\lambda'' + \omega'^2 \lambda = \omega''' \omega' + (\omega'' \omega')'\,, \\ \omega', \omega'', \lambda\big|_{s=0,1} = 0\,, \\ \omega(t=0,.) = \omega_I(.)\,. \end{cases} \tag{1.6}$$

where $\lambda = \lambda_1 + \omega'^2$ and $\omega = \omega(t,s) \in \mathbb{R}$ represents the "indicatrix" of the curve z (e.g. see [Lin89]) so that

$$z' = (\cos(\omega), \sin(\omega)) \ . \tag{1.7}$$

The reconstruction of the curve z from the indicatrix ω has to be done in such a way, that the centre of mass of the initial datum $\bar{z}_I := \int_0^1 z_I \, ds \in \mathbb{R}^2$ is conserved,

$$z(t,s) = \bar{z}_I - \int_0^1 \int_0^{\tilde{s}} \begin{pmatrix} \cos(\omega(t,\bar{s})) \\ \sin(\omega(t,\bar{s})) \end{pmatrix} d\bar{s} \, d\tilde{s} + \int_0^s \begin{pmatrix} \cos(\omega(t,\bar{s})) \\ \sin(\omega(t,\bar{s})) \end{pmatrix} d\bar{s} \ .$$

The curvature energy (1.1), in a abuse of notation, can be written as

$$\mathcal{E}[\omega] := \frac{1}{2} \int_0^1 (\omega')^2 \, ds \ , \tag{1.8}$$

such that $\mathcal{E}[z] = \mathcal{E}[\omega]$ if the constraint (1.3) is satisfied.

In this work we prove

Theorem 3. *The solution according to Corollary 2 gives a distributional sense to the system* (1.6)*, i.e. for ω being the indicatrix of z and $\lambda = \lambda_1 + \omega'^2$ it holds that*

$$\int_0^\infty \int_0^1 \left[-\omega \partial_t \psi - \omega''' \psi' - \omega'' (\omega')^2 \psi - \lambda' \omega' \psi + \lambda \, \omega' \psi' \right] ds \, dt = 0 \quad \textit{and} \tag{1.9}$$

$$\int_0^\infty \int_0^1 \left[-\omega''' \omega' \phi + \omega'' \omega' \phi' + \lambda' \phi' + \lambda \, (\omega')^2 \phi \right] ds \, dt = 0 \ . \tag{1.10}$$

for all $\psi \in H^1_{0,t} L^2_s \cap L^2_t H^1_{0,s}$ and $\phi \in L^2_t H^1_{0,s}$ and $\lambda, \omega', \omega''\big|_{0,1} = 0$ a.e. on $\mathbb{R}_+$. Furthermore It holds that $\omega \in \mathcal{C}^{0,1/8}_t \mathcal{C}^0_s$ with $\omega' \in L^\infty_t L^2_s \cap L^2_t H^2_s$ and $\lambda \in L^2_t H^1_s$.

The regularity results are better than in the preliminary result Corollary 2, most notably we get L^2-regularity up to the third spatial derivative of the indicatrix, which partly corresponds to the fourth spatial derivative of the curve z, and we are able to prove that the Lagrange multiplier is a function $\lambda \in L^2_t H^1_s$. Furthermore we prove:

Theorem 4. *Let $z_I \in \mathcal{A}$, let (z, λ_1) be a solution of problem* (1.2) *according to Corollary 2 and let (ω, λ) be the corresponding solution to* (1.6) *according to Theorem 3, then the energy dissipation is given by*

$$\frac{d}{dt} \mathcal{E} = -\mathcal{D} \ , \tag{1.11}$$

in the sense that (1.11) *holds weakly in time. The curvature energy is given by* (1.1) *and* (1.8) *respectively and the energy dissipation $\mathcal{D}$ too can, in an abuse of notation, be alternatively written in terms of z or ω,*

$$\mathcal{D} = \mathcal{D}[z] = \mathcal{D}[\omega] \quad \textit{with}$$

$$\mathcal{D}[z] := \int_0^1 \left| z'''' - (\lambda_1 z')' \right|^2 \, ds \quad \textit{and} \quad \mathcal{D}[\omega] := \int_0^1 \left[\left(\omega' \omega'' + \lambda' \right)^2 + \left(\omega''' - \omega' \lambda \right)^2 \right] ds \ . \tag{1.12}$$

Finally we prove coercivity of (1.12) with respect to the curvature energy (a Poincaré type inequality) obtaining the exponential decay of the energy.

Theorem 5. *(Poincaré-type) Under the assumptions of Theorem 4, let the energy of the initial datum be given by* (1.4)*, then it holds that*

$$\mathcal{E} \leq \mathcal{E}_I \exp(-2\pi^4 t) ,$$

where again the curvature energy is alternatively given by (1.1) *and* (1.8)*.*

This result and the regularity statements in Theorem 3 are finally used to prove the convergence of solutions towards straight lines.

Theorem 6. *Under the assumptions of Theorem 4, let the energy of the initial datum be given by* (1.4)*, then there is $\omega_\infty \in \mathbb{R}$ such that*

$$\|\omega(t,.) - \omega_\infty\|_{L^2_s} \leq C_1 \exp(-\pi^4 t)\, \mathcal{E}_I^{1/2} + C_2 \exp(-3\pi^4 t)\, \mathcal{E}_I^{3/2} ,$$

where $C_1 := \sqrt{2}\left(\frac{1}{\pi} + 1\right) + \frac{3}{2\pi^2} + \frac{3}{4}$ and $C_2 := \frac{9\sqrt{2}+3}{2\pi^2} + \frac{3}{4}$.

Curve straightening flows have been investigated intensively since the 1980s. The paper [LS85] deals with global in time existence of solutions of the curve straightening flow of closed curves and with the stability of stationary limit shapes (elasticae). In [LS87] this work is generalised to curves in arbitrary Riemannian manifolds and in [LS84] and [Lin98] the authors deal with categorising the elasticae. In [Lin89] the development of self-intersections was investigated and in [Lin03] the focus was on finding a Riemannian structure that ensures that the curve-straightening flow preserves certain symmetries.

In more recent works, [Wen93] and [Wen95], the viewpoint shifted more towards PDE-methods. In [Wen93] a gradient flow acting directly on the indicatrix was investigated and in [Wen95] the L^2-gradient flow of the square curvature functional was investigated in the space of smooth closed curves and a fourth order semilinear parabolic equation was derived as a consequence of the indicatrix representation. The author proves global in time existence and a convergence result that involves the asymptotic exponential convergence rate due to the spectrum of the linearised model (cp. [Hen81]). In [Koi96] the evolution of closed, inextensible curves in $\mathbb{R}^3$ under the curve straightening flow with the same curvature functional as in the present study was considered, but without making use of the indicatrix representation. In the same way in [Oka07] the motion of elastic planar closed curves under the additional constraint of area-preservation was considered. Finally in [LS05] and in [DKS02] efficient ways to compute the evolution towards the stationary points called elasticae were investigated.

In contrast to the above mentioned studies, the present one considers open curves as opposed to closed ones. Although the indicatrix representation was also used in earlier works ([LS85, Lin89, LS05, Wen93, Wen95]) we consider as new the approach pursued in this paper: to derive a system of scalar valued equations for the indicatrix and the Lagrange multiplier and to use it in order to compute, respectively represent the energy dissipation. Doing so we are able to prove coercivity and, finally, convergence to a straight line at an exponential rate. These results are derived from the non-linear dynamics and therefore reveal part of the nature of problem (1.2), (1.6) respectively. Most notably the coercivity of (1.12) with respect to the square curvature energy is global and leads beyond asymptotic results which one can get by linearisation at the steady state.

The paper is organised as follows:

In Section 2 we derive formally the system (1.6) and prove the existence theorem 3.

In Section 3 we show the formal derivation of the energy dissipation and prove the Theorems 4-6.

Finally, in Section 4, we demonstrate a sequence of numerically computed snapshots of the evolution under consideration and discuss the observed exponential convergence rate of the curvature energy.

The paper finishes with an appendix which includes technical computations.

2. Indicatrix formulation

As a consequence of the definition of the indicatrix ω such that (1.7) holds the curvature of a curve $z \in \mathcal{A}$ can be written as

$$\omega' = z'^{\perp} \cdot z'' \,. \tag{2.1}$$

Here and in the sequel the superscript $\perp$ denotes the rotation of a vector by 90° to the left, $(x, y)^{\perp} = (-y, x)$. Frequently we will make use of the relations

$$z'' = \omega' z'^{\perp} \quad \text{and} \quad z''^{\perp} = -\omega' z' \,, \tag{2.2}$$

which holds whenever $|z'| \equiv 1$ and which implies the equality of (1.1) and (1.8) in this case. Additionally the initial energy (1.4) can also be written in terms the indicatrix ω_I of z_I,

$$\mathcal{E}_I = \frac{1}{2} \int_0^1 (\omega_I')^2 \, ds \,.$$

We use various formulations of the bending energy of a curve $z \in H^2((0,1);\mathbb{R}^2)$. On the one hand we rely on the straightforward formulation of the Kirchhoff bending energy (1.1). On the other hand we define

$$\mathcal{E}_p[z] := \frac{1}{2} \int_0^1 \left(\frac{z'^{\perp} \cdot z''}{|z'|^p} \right)^2 ds = \frac{1}{2} \int_0^1 \left(\frac{\omega'}{|z'|^{p-2}} \right)^2 ds \,, \tag{2.3}$$

which is a generalisation of the square curvature functional $\mathcal{E}_{5/2}[z] = \int_0^1 (z'^{\perp} \cdot z''/|z'|^3)^2 dz$ that is invariant with respect to reparametrisations of z (see Appendix B). The functional $\mathcal{E}_p$ represents, for every $p \in \mathbb{R}$, a notion of bending energy, which coincides with the Kirchhoff energy (1.1) and (1.8) respectively whenever the constraint (1.3) is satisfied,

$$\mathcal{E}[z] = \mathcal{E}[\omega] = \mathcal{E}_p[z] \quad \text{if} \quad |z'| \equiv 1 \,. \tag{2.4}$$

The energy functional (2.3) defines a gradient flow, which, in its weak formulation, reads

$$\int_0^{\infty} \left(\int_0^1 \partial_t z \cdot v \, ds + \delta\mathcal{E}_p[z] v + \int_0^1 \lambda_p z' \cdot v' \, ds \right) dt = 0 \,, \tag{2.5}$$

for all $v \in \mathcal{C}_c^{\infty}(\mathbb{R}_+, \mathcal{C}_s^{\infty})$ with the Lagrange multiplier function $\lambda_p = \lambda_p(t, s) \in \mathbb{R}$ to enforce the constraint (1.3) (see Appendix A for the derivation of the variational equation and Appendix B for the computation of the variation $\delta\mathcal{E}_p[z]$). Its solution coincides with the solution of the system (1.2), whereas the strong formulation of (2.5) is given by

$$\begin{cases} \partial_t z + \left(\omega'' z'^{\perp}\right)' + \left(\left((p-2)\omega'^2 - \lambda_p\right) z'\right)' = 0 \,, \\ \omega', \omega'', \lambda_p \Big|_{s=0,1} = 0 \,, \\ |z'| \equiv 1 \,, \\ z(t=0, .) = z_I(.) \,. \end{cases} \tag{2.6}$$

Due to (2.4) both systems, (1.2) and (2.6), describe the same evolution of z and the existence result Corollary 2 applies to the system (2.6). However, since the variation of the functional

(2.3) in non-admissible directions depends on the exponent p, the Lagrange multipliers λ_p are different for different p and (2.6) implies that

$$\lambda_{p_1} - \lambda_{p_2} = (p_1 - p_2)\omega'^2 \tag{2.7}$$

for different values of p. Furthermore in equation (B.2) of the Appendix B we show that non-admissible variations of (1.1) and (2.3) coincide when $p = 1$ which justifies the notation λ_1 for the Lagrange-multiplier function in system (1.2). Summarizing we obtain

Lemma 7. *Let $z_I \in \mathcal{A}$, let (z, λ_1) be a weak solution to problem (1.2) according to Corollary 2 and let $\lambda_p = \lambda_1 + (p-1)\omega'^2$, then there is $\omega \in \mathcal{C}_t^{0,1/8}\mathcal{C}_s^0$ with $\omega' \in L_t^\infty L_s^2$ such that ω is an indicatrix of z and (z, ω, λ_p) constitute a weak solution of (2.6) satisfying (2.5) for all $v \in \mathcal{C}_c^\infty(\mathbb{R}_+, \mathcal{C}_s^\infty)$.*

Proof. Furthermore the regularity of z' according to Corollary 2 allows to identify $\omega \in \mathcal{C}_t^{0,1/8}\mathcal{C}_s^0$ up to an additive constant being a multiple of 2π. The rest of the statement is then an consequence of the discussion above, most notably (2.7). □

The case $p = 2$ has an especially simple structure, since the forces generated by the curvature and by the Lagrange multiplier are orthogonal in some sense. This is reflected by the identities

$$\frac{\delta \mathcal{E}_2}{\delta z} \cdot \delta z = -\int_0^1 \omega'' z'^\perp \cdot \delta z' \, ds \quad \text{and} \quad \frac{\delta \mathcal{E}^L}{\delta z} \cdot \delta z = \int_0^1 \lambda z' \cdot \delta z' \, ds \ , \tag{2.8}$$

where $\mathcal{E}^L[z] := \int_0^1 \lambda(s)(|z'|^2 - 1)/2 \, ds$ is a potential term such that the effect of the Lagrange multiplier can be considered as its variation. For this reason we choose the notation

$$\lambda := \lambda_2 \ . \tag{2.9}$$

It holds that

Lemma 8. *(Regularity, a-priori estimates) Let (z, ω, λ) bet a weak solution of (2.6) in the case $p = 2$ in the sense of Lemma 7, then it holds that $\omega'' \in L_t^2 L_s^\infty$, $\omega'\omega'' \in L_t^2 L_s^2$, $\lambda \in L_t^2 L_s^\infty$, $\omega'\lambda \in L_t^2 L_s^2$, $\omega''' \in L_t^2 L_s^2$ and $\lambda' \in L_t^2 L_s^2$. It also holds that*

$$\int_0^\infty \int_0^1 \left[\left(\omega''' - \lambda\omega'\right)^2 + \left(\lambda' + \omega'\omega''\right)^2 \right] ds \, dt = \|\partial_t z\|_{L_t^2 L_s^2}^2 \ .$$

Proof. First note that the primitive $\int_0^s \partial_t z \, d\tilde{s} \in L_t^2 L_s^\infty$ since

$$\int_0^\infty \left\| \int_0^s \partial_t z \, d\tilde{s} \right\|_\infty^2 dt \le \int_0^\infty \left(\int_0^1 |\partial_t z| \, ds \right)^2 dt \le \int_0^\infty \int_0^1 |\partial_t z|^2 \, ds \, dt < 2 \, \mathcal{E}_I$$

by (A.3). We go back to the integrated version of (2.6) for $p = 2$

$$\begin{cases} \int_0^s \partial_t z \, ds + \left(\omega'' z'^\perp\right) - (\lambda z') = 0 \ , \\ \omega'\big|_{s=0,1} = 0 \ , \\ z(t=0,.) = z_I(.) \ . \end{cases}$$

and obtain

$$\omega'' = -z'^\perp \cdot \int_0^s \partial_t z \, ds \quad \text{and} \quad \lambda = z' \cdot \int_0^s \partial_t z \, ds \ , \tag{2.10}$$

which implies $\omega'' \in L_t^2 L_s^\infty$ and $\lambda \in L_t^2 L_s^\infty$. Moreover we find that

$$\int_0^\infty \int_0^1 \left(\lambda\omega'\right)^2 \, ds \, dt \le \int_0^\infty \|\lambda\|_{L_s^\infty}^2 \int_0^1 \left(\omega'\right)^2 \, ds \, dt \le 2 \, \mathcal{E}_I \int_0^\infty \|\lambda\|_{L_s^\infty}^2 \, dt < (2 \, \mathcal{E}_I)^2 \ . \tag{2.11}$$

This computation and an analogous one for $\omega'\omega''$ imply

$$\omega'\lambda \in L^2_t L^2_s \quad \text{and} \quad \omega'\omega'' \in L^2_t L^2_s \,. \tag{2.12}$$

We write the weak formulation (2.5) of problem (2.6) for $p = 2$ after two integrations by parts,

$$\int_0^\infty \int_0^1 \left(\partial_t z + \left(\omega'' z'^\perp\right)' - (\lambda z')'\right) \cdot v \, ds\, dt + \int_0^\infty \left[-\omega'' z'^\perp \cdot v + \lambda z' \cdot v + \omega' z'^\perp \cdot v'\right]_0^1 dt = 0 \,, \tag{2.13}$$

for all $v \in \mathcal{C}^\infty_c(\mathbb{R}_+, \mathcal{C}^\infty_s)$. The uniform estimates we obtained allows to set $v = z'^\perp\phi$ for a testfunction $\phi \in \mathcal{D}(([0,\infty) \times [0,1])$ and to obtain $\int_0^\infty \int_0^1 \left[(\partial_t z) z'^\perp \phi + \omega'''\phi - \lambda\omega'\phi\right] ds\ dt = 0$. Specialising (2.13) for $v = z'\psi$ with $\psi \in \mathcal{D}([0,\infty) \times [0,1])$ we conclude by an analogous computation $\int_0^\infty \int_0^1 [(\partial_t z) z'\psi - \omega'\omega''\psi + \lambda\psi]\, ds\, dt = 0$. Due to (A.3) and (2.12) this implies

$$z'^\perp \cdot \partial_t z + \omega''' - \lambda\omega' = 0 \quad \text{a.e.} \quad \text{and} \quad z' \cdot \partial_t z - \omega'\omega'' - \lambda' = 0 \quad \text{a.e.,} \tag{2.14}$$

hence $\omega''' \in L^2_t L^2_s$ and $\lambda' \in L^2_t L^2_s$.

Finally (2.14) implies

$$\int_0^\infty \int_0^1 \left[\left(\omega''' - \lambda\omega'\right)^2 + \left(\lambda' + \omega'\omega''\right)^2\right] ds\, dt = \tag{2.15}$$

$$= \int_0^\infty \int_0^1 \left[\left(z'^\perp \cdot \partial_t z\right)^2 + \left(z' \cdot \partial_t z\right)^2\right] ds\, dt = \|\partial_t z\|^2_{L^2_t L^2_s} \,.$$

□

Next we will derive the governing equations for the quantities $\omega(t,s)$ and $\lambda(t,s)$. Taking the derivative of the evolution equation in (2.6) and explicitly evaluating all the derivatives we infer

$$\begin{aligned}\partial_t z' + \big((\omega'''' + 3(p-2)\omega'^2\omega'')z'^\perp + (\omega''' + (p-2)\omega'^3)(-1)\omega' z' + \\ + (2p-5)(\omega''\omega')' z' + (2p-5)\omega''\omega'\omega' z'^\perp\big) - \\ - (\lambda_p'' z' + (\lambda_p\omega')' z'^\perp + \lambda_p'\omega' z'^\perp + \lambda_p\omega'(-1)\omega' z') = 0 \,.\end{aligned} \tag{2.16}$$

Multiplying by $z'^\perp$ and z' respectively we get

$$\begin{cases} \partial_t\omega + \omega'''' + (5p-11)\,\omega'^2\omega'' - \left(\omega''\lambda_p + 2\omega'\lambda_p'\right) = 0 \,, \\ -(\omega''' + (p-2)\omega'^3)\omega' + (2p-5)(\omega''\omega')' - \left(\lambda_p'' - \omega'^2\lambda_p\right) = 0 \,, \\ \omega', \omega'', \lambda_p\big|_{s=0,1} = 0 \,. \end{cases} \tag{2.17}$$

where $\partial_t\omega = z'^\perp \cdot \partial_t z'$. Observe that $\lambda_p''(0) = \lambda_p''(1) = 0$ as a consequence of the second equation and the boundary conditions. Let us compare the system for some "canonical" choices for p. In the case $p = 1$ we obtain

$$\begin{cases} \partial_t\omega + \omega'''' - 6\,\omega'^2\omega'' - (\omega''\lambda_1 + 2\omega'\lambda_1') = 0 \,, \\ -(\omega''' - \omega'^3)\omega' - 3(\omega''\omega')' - \left(\lambda_1'' - \omega'^2\lambda_1\right) = 0 \,, \\ \omega', \omega'', \lambda_1\big|_{s=0,1} = 0 \,. \end{cases} \tag{2.18}$$

As a consequence of the fact that the variations of $\mathcal{E}$ and $\mathcal{E}_1$ coincide by (B.2), the elliptic equation for λ_1 is the one that was found by Koiso ([Koi96]).

In the case $p = 2$, using the notation (2.9), we obtain the system (1.6) and prove the **Theorem 3**.

Proof. We start with a weak solution of (2.6) satisfying (2.13) for all $v \in \mathcal{C}_c^\infty(\mathbb{R}_+, \mathcal{C}_s^\infty)$ in the case $p = 2$ according to Lemma 7. The boundary integrals imply that $\lambda, \omega', \omega''\big|_{0,1} = 0$ a.e. on $\mathbb{R}_+$.

We choose a regularising sequence η_k with $\operatorname{supp} \eta_k \subset B(0, 1/k) \subset \mathbb{R}^2$ and set $v = \eta_k(\tilde{t}-t, \tilde{s}-s)$ for $(\tilde{t}, \tilde{s}) \in U_k$ with $U_k := [1/k, \infty) \times (1/k, 1 - 1/k)$ obtaining

$$\partial_t z_k + \left(\omega'' z'^{\perp}\right)' * \eta_k - (\lambda z')' * \eta_k = 0 \ ,$$

where $z_k := z * \eta_k$. We omit the tilde and integrate against $-\left(\varphi z_k' + \frac{1}{|z_k'|^2}\psi z_k'^{\perp}\right)'$ with $\varphi, \psi \in \mathcal{D}(\mathbb{R}_+ \times [0, 1])$ and k large enough such that U_k covers the support of both testfunctions,

$$\begin{aligned} &\iint_{U_k} -\partial_t z_k \cdot \left(z_k'^{\perp} \frac{1}{|z_k'|^2}\psi\right)' - \left(\left(\omega'' z'^{\perp}\right)' * \eta_k - (\lambda z')' * \eta_k\right) \cdot \left(z_k'^{\perp} \frac{1}{|z_k'|^2}\psi\right)' ds\, dt = 0 \ , \\ &\iint_{U_k} -\partial_t z_k \cdot (z_k'\varphi)' - \left(\left(\omega'' z'^{\perp}\right)' * \eta_k - (\lambda z')' * \eta_k\right) \cdot \left(z_k'\varphi\right)' ds\, dt = 0 \ . \end{aligned} \tag{2.19}$$

Making use of the regularity results in Corollary 2 it holds that $z_k' \to z'$ and, as a consequence, also $\omega_k \to \omega$, uniformly on compact subsets of $\mathbb{R}_+ \times [0, 1]$, where ω_k is the indicatrix of z_k satisfying $z_k'/|z_k'| = \big(\cos(\omega_k), \sin(\omega_k)\big)$. Using these results we perform integrations by parts with respect to s and t with the expressions in (2.19) that involve $\partial_t z_k$ and pass to the limit, obtaining

$$\iint_{U_k} -\partial_t z_k \cdot (z_k'\varphi)'\, ds\, dt = \iint_{U_k} \partial_t z_k' \cdot z_k'\varphi\, ds\, dt = \iint_{U_k} -\frac{1}{2}|z_k'|^2 \partial_t\varphi\, ds\, dt \to 0 \quad \text{as} \quad k \to \infty$$

and

$$\begin{aligned} \iint_{U_k} -\partial_t z_k \cdot \left(z_k'^{\perp} \frac{1}{|z_k'|^2}\psi\right)' ds\, dt &= \iint_{U_k} \partial_t z_k' \cdot z_k'^{\perp} \frac{1}{|z_k'|^2}\psi\, ds\, dt = \\ &= \iint_{U_k} \partial_t\omega_k\psi\, ds\, dt = \iint_{U_k} -\omega_k\partial_t\psi\, ds\, dt \to \int_0^\infty \int_0^1 -\omega\partial_t\psi\, ds\, dt \quad \text{as} \quad k \to \infty \ , \end{aligned}$$

where we used that $\partial_t\omega_k = \partial_t z_k' \cdot z_k'^{\perp} \frac{1}{|z_k'|^2}$. Making use of the the fact that $z'' \in L_t^\infty L_s^2$ by Corollary 2 and ω''', λ' in $L_t^2 L_s^2$ by Lemma 8 we now pass to the limit $k \to \infty$ in the remaining expressions of (2.19) and conclude

$$\begin{aligned} &\int_0^\infty \int_0^1 -\omega\partial_t\psi - \left(\omega''' z'^{\perp} - \lambda' z' - \omega''\omega' z' - \lambda\omega' z'^{\perp}\right) \cdot \left(-z'\omega'\psi + z'^{\perp}\psi'\right) ds\, dt = 0 \ , \\ &\int_0^\infty \int_0^1 -\left(\omega''' z'^{\perp} - \lambda' z' - \omega''\omega' z' - \lambda\omega' z'^{\perp}\right) \cdot \left(z'^{\perp}\omega'\varphi + z'\varphi'\right) ds\, dt = 0 \ . \end{aligned}$$

This implies (1.9), (1.10) where every term is well defined by Lemma 8 and where we allow for ψ, ϕ in function spaces in which testfunctions are densily contained. ☐

An interesting structural observation concerning the problem (1.6) is the following.

Remark 1. Let (ω, λ) be a solution of (1.6), then, at least on a formal level, the evolution, represented by $\partial_t\omega$, and the Langrange mulitplier satisfy the orthogonality condition

$$\int_0^1 \partial_t\omega\lambda\, ds = 0 \ ,$$

which we obtain integrating the first equation in (1.6) against λ,

(2.20)
$$\int_0^1 \partial_t\omega\lambda\, ds = \int_0^1 \left(\omega'''\lambda' + \omega''\omega'^2\lambda + \lambda'\omega'\lambda - \lambda\omega'\lambda'\right) ds\, dt = \int_0^\infty \int_0^1 \left(\omega'''\lambda' + \omega''\omega'^2\lambda\right) ds\ ,$$

and integrating the second equation in (1.6) against ω'',

$$\int_0^1 \left(\omega'''\lambda' + \omega'^2\omega''\lambda\right) ds = \int_0^1 \left(-\omega'\omega''\omega''' + \omega'\omega'''\omega''\right) ds = 0\ .$$

This observation seems to be related to the special structure of forces resulting from curvature and from pressure/tension (cp (2.8)) and to the fact that we are dealing with a constrained gradient flow, i.e. the evolution is tangential to the constrained set.

3. Energy dissipation and long time convergence

Testing the first equation in the system (1.6) formally against ω'' and the second one against λ and taking their difference implies that the energy dissipation is given by

$$\frac{d}{dt}\mathcal{E}[\omega(t)] = \frac{d}{dt}\frac{1}{2}\int_0^1 \omega'^2\, ds = -\int_0^1 \left[\left(\omega'\omega'' + \lambda'\right)^2 + \left(\omega''' - \omega'\lambda\right)^2\right] ds \le 0\ . \tag{3.1}$$

This can be made rigorous stating that the energy dissipation equality (3.1) holds weakly in time (**Theorem 4**).

Proof. Here the problem is that we cannot directly set ψ in (1.9) equal to ω'', since its time derivative cannot necessarily be interpreted as a function. Therefore we regularise using a sequence of mollifiers $(\eta_k)_{k=1,2,...}$ with supp $\eta_k \subset [-1/k, 1/k]$. For $\tilde{t} \ge 1/k$ we denote by $\omega_k(\tilde{t}) := (\omega * \eta_k)(\tilde{t}) = \int_0^\infty \omega(t)\eta_k(\tilde{t} - t)\, dt$ the regularised version of ω and evaluate (1.9) using $\psi = \eta_k(\tilde{t} - t)\, \omega_k''(\tilde{t})$. We find

$$\int_0^1 \left[\partial_t\omega_k\, \omega_k'' - \omega_k'''\omega_k''' - (\omega''(\omega')^2) * \eta_k\, \omega_k'' - (\lambda'\omega') * \eta_k\, \omega_k'' + (\lambda\, \omega') * \eta_k\, \omega_k'''\right] ds = 0\ .$$

We omit the tildes and integrate against a test function in time $\vartheta \in \mathcal{D}(\mathbb{R}_+)$ obtaining

$$\int_{1/k}^\infty \left[\partial_t\vartheta \int_0^1 \omega_k'^2/2\, ds + \vartheta \int_0^1 \left[-\omega_k'''^2 - (\omega''(\omega')^2) * \eta_k\, \omega_k'' - (\lambda'\omega') * \eta_k\, \omega_k'' + (\lambda\, \omega') * \eta_k\, \omega_k'''\right] ds\right] dt = 0\ ,$$

with k large enough such that supp $\vartheta \subset [1/k, \infty)$. Passing to the limit $k \to \infty$ we conclude

$$\int_0^\infty \left[\partial_t\vartheta \int_0^1 \omega'^2/2\, ds + \vartheta \int_0^1 \left[-\omega'''^2 - (\omega''\omega')^2 - \lambda'\omega'\omega'' + \lambda\, \omega'\omega'''\right] ds\right] dt = 0\ . \tag{3.2}$$

Setting $\phi = \lambda\vartheta(t)$ in (1.10)

$$\int_0^\infty \vartheta(t) \int_0^1 \left[-\omega'''\omega'\lambda + \omega''\omega'\lambda' + \lambda'^2 + \lambda^2\, (\omega')^2\right] ds\, dt = 0\ . \tag{3.3}$$

The difference of (3.2) and (3.3) finally yields a weak in time formulation of (3.1). □

As an immediate consequence we obtain

Corollary 9. *Let z be a solution of* (1.2) *according to Corollary 2 with initial energy $\mathcal{E}_I$, then it holds that*

$$\|\partial_t z\|^2_{L^2_t L^2_s} = \int_0^\infty \mathcal{D}\, dt = \mathcal{E}_I \ . \tag{3.4}$$

Furthermore it holds that

$$\begin{aligned} \|\omega''\|_{L^2_t L^\infty_s}\,,\ \|\lambda\|_{L^2_t L^\infty_s} &\leq \sqrt{\mathcal{E}_I}\ , \\ \|\omega'\lambda\|_{L^2_t L^2_s}\,,\ \|\omega'\omega''\|_{L^2_t L^2_s} &\leq \sqrt{2}\mathcal{E}_I \quad \textit{and} \\ \|\lambda'\|_{L^2_t L^2_s}\,,\ \|\omega'''\|_{L^2_t L^2_s} &\leq \sqrt{2}\mathcal{E}_I + \sqrt{\mathcal{E}_I}\ . \end{aligned}$$

Proof. Integrate (3.1) with respect to time and combine it with (2.15). Using this result we go again through the estimates in the proof of Lemma 8. ☐

The energy dissipation equality (3.1) motivates the definition of $[\omega]$ in (1.12). The equivalent expression in terms of z,$\mathcal{D}[z]$, is then a consequence of

Lemma 10. *Let (z, λ_1) be a solution of* (1.2) *and let ω be the indicatrix of z and $\lambda = \lambda_1 + \omega'^2$ such that (ω, λ) satisfy* (1.6)*, then it holds that*

$$\mathcal{D}[\omega(t,.)] = \int_0^1 \left|z''''(t,s) - (\lambda_1(t,s) z'(t,s))'\right|^2\, ds \quad \textit{a.e. in}\, \mathbb{R}_+\ .$$

Proof. We compute iterated derivatives of the constraint (1.3),

$$\begin{aligned} 0 &= z' \cdot z''\ , \\ 0 &= z' \cdot z''' + z'' \cdot z'' = z' \cdot z''' + \omega'^2\ , \\ 0 &= z' \cdot z'''' + z'' \cdot z''' + 2\omega'\omega'' = z' \cdot z'''' + \omega' z'^\perp \cdot z''' + 2\omega'\omega'' = z' \cdot z'''' + 3\omega'\omega''\ , \end{aligned}$$

making use of (2.2) and iterated derivatives of (2.1),

$$\begin{aligned} \omega'' &= z'^\perp \cdot z'''\ , \\ \omega''' &= z''^\perp \cdot z''' + z'^\perp \cdot z'''' = -\omega' z' \cdot z''' + z'^\perp \cdot z'''' = \omega'^3 + z'^\perp \cdot z''''\ . \end{aligned} \tag{3.5}$$

We obtain $z'''' \cdot z' = -3\omega'\omega''$ and $z'''' \cdot z'^\perp = \omega''' - \omega'^3$, which imply together with $\lambda_1 = \lambda - \omega'^2$ (cp. (2.7)) that

$$\int_0^1 \left[\left(\omega'\omega'' + \lambda'\right)^2 + \left(\omega''' - \omega'\lambda\right)^2\right] ds = \int_0^1 \left[(z'''' \cdot z' - \lambda_1')^2 + (z'''' \cdot z'^\perp - \lambda_1\omega')^2\right] ds =$$

$$= \int_0^1 \left|z'''' \cdot z'z' - \lambda_1' z' + z'''' \cdot z'^\perp z'^\perp - \lambda_1\omega' z'^\perp\right|^2\, ds = \int_0^1 \left|z'''' - (\lambda_1 z')'\right|^2\, ds\ ,$$

where we again used (2.2). ☐

We finally deal the convergence to equilibrium. First we obtain

Lemma 11. *Let ω be a function such that $\mathcal{D} = \mathcal{D}[\omega] = 0$, then $\omega' \equiv 0$, i.e. stationary points of the energy are straight lines.*

Proof. We integrate $\omega'\omega'' + \lambda' = 0$ and make use of the boundary conditions on λ to conclude that $\lambda = -\omega'^2/2$. Furthermore together with $(\omega''' - \omega'\lambda)^2 = 0$ this implies that $\omega''' + \omega'^3/2 = 0$ and the boundary conditions on ω' imply that $\omega' \equiv 0$ on $[0, 1]$. ☐

Furthermore we find the exponential decay of the energy formulated in **Theorem 5**.

Proof. The proof uses the fact that the best constant in the Poincaré-type inequality

$$\int_0^1 v^2 \, ds \le C \int_0^1 v'^2 \, ds \tag{3.6}$$

for $v \in H_0^1((0,1))$ is given by the reciprocal value of the first eigenvalue of the differential operator v'' in that space, $C = 1/\pi^2$. Furthermore we use (3.5) to obtain

$$\begin{aligned} 2\pi^2\mathcal{E} &= \pi^2 \int_0^1 \omega'^2 \, ds \le \int_0^1 \omega''^2 \, ds = \\ &= \int_0^1 (z''' \cdot z'^{\perp})^2 \, ds = \int_0^1 ((z''' - \lambda_1 z') \cdot z'^{\perp})^2 \, ds \le \\ &\le \int_0^1 \left| z''' - \lambda_1 z' \right|^2 \, ds \le \\ &\le \frac{1}{\pi^2} \int_0^1 \left| \left(z''' - \lambda_1 z' \right)' \right|^2 \, ds = \frac{1}{\pi^2} \mathcal{D} \, , \end{aligned} \tag{3.7}$$

where we used (1.12). As a consequence, the energy dissipation $\mathcal{D}$ is coercive with respect to the energy and and the following Poincaré-type inequality holds

$$\mathcal{D}/\mathcal{E} \ge 2\pi^4 \, . \tag{3.8}$$

Observe that the estimate (3.8) is not sharp, since in the estimations (3.7) the inequality (3.6) is used twice with the optimal constant C, but with different functions and because the step from line 2 to line 3 involves adding $((z''' - \lambda_1 z') \cdot z')^2 = ((|z''|^2 + \lambda_1))^2 = \lambda^2$ to the integrand. □

The plot in figure 1 show the numerically found ground state of the exponential dissipation rate, i.e. the minimiser of the functional $\mathcal{D}/\mathcal{E}$. Its rate of convergence is $\mathcal{D}/\mathcal{E} \approx 505.917$, which still leaves a gap to the constant we found in (3.8), $2\pi^4 \approx 200$.

We finally are able to give the proof of **Theorem 6**.

Proof. Let $\overline{\omega}(t) := \int_0^1 \omega(t,s) \, ds$ be the mean-value of the indicatrix, then Theorem 5 implies

$$\|\omega(t,.) - \overline{\omega}(t)\|_{L^2}^2 \le \frac{2}{\pi^2} \mathcal{E}[\omega(t,.)] \tag{3.9}$$

due to a Poincaré-inequality which is analogous to (3.6). Let $\mu(s) := 6s(1-s)$ and let (η_k) be a regularizing sequence such that $\operatorname{supp} \eta_k \subset [-1/k, 1/k]$. In (1.9) we set $\psi(t,s) = \eta_k(\tilde{t} - t)\mu(s)$ with $\tilde{t} > 1/k$ obtaining

$$\frac{d}{d\tilde{t}} \int_0^1 \omega_k \mu \, ds + \int_0^1 \left[-\omega_k''' \mu' - (\omega''(\omega')^2) * \eta_k \mu - (\lambda' \omega') * \eta_k \mu + (\lambda \, \omega') * \eta_k \mu' \right] ds = 0 \, .$$

Let $0 < t_1 < t_2$ and integrate with respect to time, omitting the tilde. We pass to the limit as $k \to \infty$ to get

$$\hat{\omega}(t_2) - \hat{\omega}(t_1) + \int_{t_1}^{t_2} \int_0^1 \left[-\omega''(\omega')^2 \mu - \lambda' \omega' \mu + \lambda \, \omega' \mu' \right] ds \, dt = 0 \, ,$$

where

$$\hat{\omega}(t) := \int_0^1 \omega(t,s) \mu(s) \, ds$$

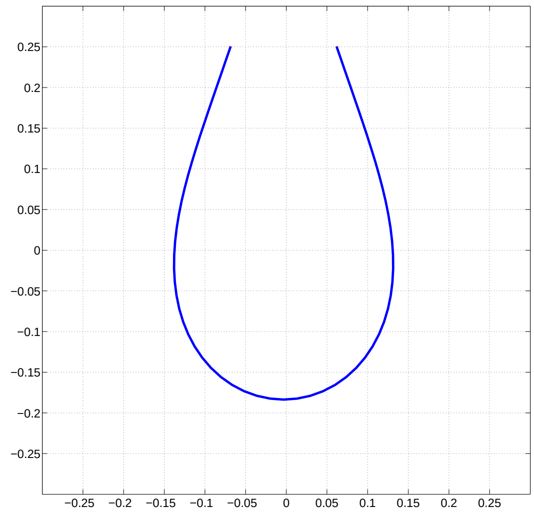

FIGURE 1. Plot of the minimiser of the functional $\mathcal{D}/\mathcal{E}$ with $\mathcal{D}/\mathcal{E} = 505.917$

is a weighted mean of the function $\omega(t,.)$. Observe that $\int_0^1 -\omega'''\mu'\,ds = \int_0^1 \omega''\mu''\,ds = -12(\omega'(1)-\omega'(0)) = 0$ has canceled due to the boundary values of ω'. We infer

$$|\hat\omega(t_2)-\hat\omega(t_1)| = \left|\int_{t_1}^{t_2}\int_0^1 \left[-\omega''(\omega')^2\mu - 2\lambda'\omega'\mu - \lambda\,\omega''\mu\right] ds\,dt\right| \le$$
$$\le \frac{3}{2}\left[\|\omega''\omega'\|_{L_t^2L_s^2}\|\omega'\|_{L_t^2L_s^2} + 2\|\lambda'\|_{L_t^2L_s^2}\|\omega'\|_{L_t^2L_s^2} + \|\lambda\|_{L_t^2L_s^\infty}\|\omega''\|_{L_t^2L_s^\infty}\right] .,$$

where norms in the dimension t are evaluated on the interval (t_1,t_2). We now interprete the data at $t = t_1 < t_2$ as initial datum. Doing so Theorem 5 implies that $\|\omega'\|_{L^2((t_1,t_2);L_s^2} \le \sqrt{\mathcal{E}[\omega(t_1)]}/(\pi^2)$ and together with the results of Corollary 9 this implies

$$|\hat\omega(t_2)-\hat\omega(t_1)| \le \frac{3}{2}\left[\frac{3}{\pi^2}\sqrt{2}\mathcal{E}[\omega(t_1,.)]^{3/2} + \left(\frac{2}{\pi^2}+1\right)\mathcal{E}[\omega(t_1,.)]\right] . \tag{3.10}$$

The decay of the energy by Theorem 5 implies that for a given sequence of times (t_n) the sequence $\hat\omega(t_n)$ is Cauchy and it yields a limit value which we call $\omega_\infty \in \mathbb{R}$. Setting $t_1 = t$ and $t_2 = t_n$ in (3.10) and passing to the limit as $n \to \infty$ we obtain

$$|\hat\omega(t)-\omega_\infty| \le \frac{3}{2}\left[\frac{3}{\pi^2}\sqrt{2}\mathcal{E}[\omega(t,.)]^{3/2} + \left(\frac{2}{\pi^2}+1\right)\mathcal{E}[\omega(t,.)]\right] . \tag{3.11}$$

Furthermore, since $\int_0^1 \mu(s)\,ds = 1$, it holds that

(3.12)
$$|\overline\omega(t)-\hat\omega(t)| = \left|\int_0^1 \omega(t,s)(1-\mu(s))\,ds\right| = \left|\int_0^1\left(\omega(t,0)+\int_0^s \omega'(t,\tilde s)\,d\tilde s\right)(1-\mu(s))\,ds\right| =$$
$$= \left|\int_0^1\left(\int_0^s \omega'(t,\tilde s)\,d\tilde s\right)(1-\mu(s))\,ds\right| \le \int_0^1 |\omega'(t,\tilde s)|\,d\tilde s \le \|\omega'(t,\tilde s)\|_{L_s^2} = \sqrt{2\mathcal{E}(t)} .$$

Together, (3.9), (3.11) and (3.12) imply

$$\|\omega(t,.)-\omega_\infty\|_{L^2_s} \le \|\omega(t,.)-\overline{\omega}(t)\|_{L^2_s} + |\overline{\omega}(t)-\hat{\omega}(t)| + |\hat{\omega}(t)-\omega_\infty| \le$$
$$\le \sqrt{2}\left(\frac{1}{\pi}+1\right)\sqrt{\mathcal{E}[\omega(t,.)]} + \frac{3}{2}\left[\frac{3}{\pi^2}\sqrt{2}\mathcal{E}[\omega(t,.)]^{3/2} + \left(\frac{2}{\pi^2}+1\right)\mathcal{E}[\omega(t,.)]\right] .$$

We finally use Theorem 5 and control the last term by an interpolation between the smallest and the largest powers of $\exp(-\pi^4 t)$ using $x^2 \le (x+x^3)/2$ for $x \ge 0$, which yields the result. □

4. Numerics

In the sequence of figures 2 we visualise the numerical solution of the recursive scheme (A.1) for $\tau = 10^{-6}$ at various points in time. Information about time, energy and the dissipation rate are printed in the title of each frame. Observe the decay of energy as compared to the changes in the dissipation rate, which seems to evolve according to the following scheme. Initially it drops rapidly, then, when the energy is around $\mathcal{E} \approx 7$, it takes its minimum at $\mathcal{D}/\mathcal{E} \approx 510$ and the profile of the curve results to be U-shaped and similar to the minimiser of the exponential dissipation rate we found numerically (figure 1). Finally, the exponential rate increases again up to values $\mathcal{D}/\mathcal{E} \approx 1000$. This can be explained by claiming that at low energy, due to the bounds of Corollary 9, the linear components of the model (1.6) dominate. Stripping (1.6) from its nonlinearities we obtain $\partial_t\tilde{\omega} + \tilde{\omega}'''' = 0$ with $\tilde{\omega}'(0) = \tilde{\omega}'(1) = \tilde{\omega}''(0) = \tilde{\omega}''(1) = 0$. The exponential dissipation rate of this model in the sense of (3.8) would be $\mathcal{D}/\mathcal{E} \approx 1001.13$, namely twice the first eigenvalue of the respective fourth order operator. We remark that a future study will be devoted to investigating this asymptotic behaviour in more detail.

Appendix A. Construction of solutions by the steepest descent flow

The results Theorem 1 and Corollary 2 respectively are obtained from the usual construction in the theory of gradient flows and steepest descent flows (cp. [DGMT80], [AGS05]). This result indeed is a special case of the existence proof formulated in [OS09a]. It is based on defining the recursive scheme

$$Z^0_\tau = z_I \quad \text{and} \quad Z^n_\tau \in \operatorname{argmin}_{w\in\mathcal{A}} \left\{ \int_0^1 \left[\frac{1}{2}|w''|^2 + \frac{|w(s)-Z^{n-1}_\tau|^2}{2\tau}\right] ds \right\} , \tag{A.1}$$

where $(Z^n_\tau)_{n=0,1,\dots}$ is a stepwise in time approximation of z with stepsize τ. The passage to the limit $\tau \to 0$ is based on the estimate which we obtain by summing up for $n = 1, 2, \dots$ the fact that

$$\int_0^1 \left[\frac{1}{2\tau}\left(Z^n_\tau - Z^{n-1}_\tau\right)^2 + \frac{1}{2}((Z^n_\tau)'')^2\right] ds \le \int_0^1 \frac{1}{2}((Z^{n-1}_\tau)'')^2\, ds . \tag{A.2}$$

Taking the sum and the passage to the limit with respect to τ with interpolations of the sequence $(Z^n_\tau)_n$ implies the $L^2_t L^2_s$ a priori bound on $\partial_t z$,

$$\int_0^\infty \int_0^1 |\partial_t z|^2\, ds\, dt \le 2\,\mathcal{E}_I = \int_0^1 |z_I''|^2\, ds . \tag{A.3}$$

Furthermore the variational equation of (A.1) after letting $\tau \to 0$ becomes (1.5) for variations $\delta z \in H^2([0,1])$ and the Lagrange multiplier $\lambda_1 = \lambda_1(t,s)$ to enforce the constraint $|z'| \equiv 1$. The system (1.2) is then the strong formulation of (1.5) coupled with the constraint. We remark that the "natural" boundary conditions $z'', z''' - \lambda_1 z'|_{s=0,1} = 0$ are equivalent to those in the

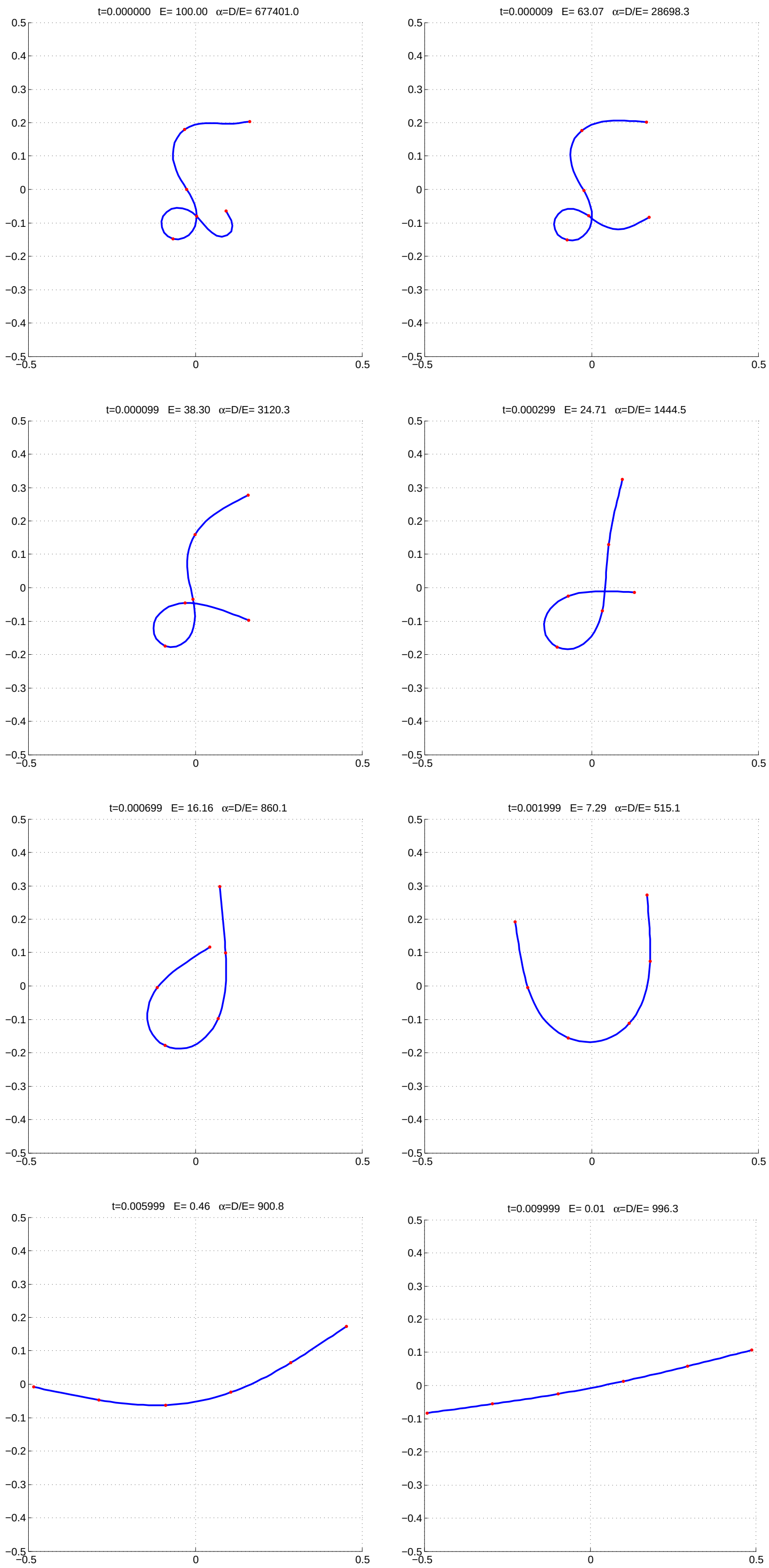

Figure 2. Evolution computed numerically.

formulation (1.2), since taking the second derivative of the constraint (1.3) implies $|z''|^2+z'\cdot z'''=0$ and therefore it holds that $0=z'\cdot z'''-\lambda_1|_{s=0,1}=-|z''|^2-\lambda_1|_{s=0,1}=-\lambda_1|_{s=0,1}$.

Appendix B. Variation of the Kirchhoff bending energy

In this appendix we gather some of the variational formulas needed in Section 2. The energies defined in (1.1) and (2.3) coincide, if the constraint (1.3) is satisfied. This holds too for the variations of these energies with respect to variations $\delta z\in H^2([0,1],\mathbb{R}^2)$ which are admissible to the constraint (1.3), i.e. which satisfy $z'\cdot\delta z'\equiv 0$. Variations in a general direction $\delta z\in H^2([0,1],\mathbb{R}^2)$ might differ, we compute

$$\delta\mathcal{E}_p[z]\delta z=\int_0^1\left(\frac{\omega'}{|z'|^{p-2}}\right)\left(\frac{(\delta\omega)'}{|z'|^{p-2}}-(p-2)\frac{\omega'}{|z'|^p}z'\cdot\delta z'\right)\,ds=$$
$$=\int_0^1\left(\frac{\omega'}{|z'|^{2(p-2)}}\right)\left(\frac{z'^{\perp}\cdot\delta z'}{|z'|^2}\right)'-(p-2)\left(\frac{\omega'^2}{|z'|^{2(p-1)}}\right)z'\cdot\delta z'\,ds$$

making use of the fact that we can exchange variation and differentiation in the case of $\delta\omega'=(\delta\omega)'$ and that the variation of the indicatrix is given by $\delta\omega[z]\delta z=z'^{\perp}\cdot\delta z'/|z'|^2$.

Observe that for $p=5/2$ the bending energy corresponds to the square curvature functional, $\mathcal{E}_{5/2}[z]=\int_0^1(z'^{\perp}\cdot z''/|z'|^3)^2dz$ and its variation is invariant with respect to variations that only act as reparametrisations, i.e. $\delta z=z'\phi$ for test functions $\phi\in\mathcal{D}([0,1])$. In fact it holds that $\delta\mathcal{E}_{5/2}[z]z'\phi=0$.

Continuing this computation using the assumption that the constraint $|z'|\equiv 1$ holds, the variation of the bending energy reads

$$\text{(B.1)}\quad \delta\mathcal{E}_p[z]\delta z=\int_0^1\omega'\left(\left(z'^{\perp}\cdot\delta z'\right)'-(p-2)\,\omega' z'\cdot\delta z'\right)\,ds=$$
$$=\Big[\omega' z'^{\perp}\cdot\delta z'\Big]_0^1+\int_0^1-\omega'' z'^{\perp}\cdot\delta z'-(p-2)\,\omega'^2 z'\cdot\delta z'\,ds=$$
$$=\Big[\omega' z'^{\perp}\cdot\delta z'-\left(\omega'' z'^{\perp}+(p-2)\,\omega'^2\,z'\right)\cdot\delta z\Big]_0^1+\int_0^1\left(\omega'' z'^{\perp}+(p-2)\,\omega'^2 z'\right)'\cdot\delta z\,ds$$

On the other hand the variation of (1.1) in a general direction δz but evaluated at a curve which satisfies the constraint (1.3) reads

$$\text{(B.2)}\quad \delta\mathcal{E}[z]\delta z=\int_0^1 z''\cdot\delta z''\,ds=\int_0^1\underbrace{z''\cdot z'}_{=0}\,z'\cdot\delta z''+z''\cdot z'^{\perp}\,z'^{\perp}\cdot\delta z''\,ds=$$
$$=\int_0^1\omega'\,z'^{\perp}\cdot\delta z''\,ds=\Big[\omega'\,z'^{\perp}\cdot\delta z'-\left(\omega'\,z'^{\perp}\right)'\cdot\delta z\Big]_0^1+\int_0^1\left(\omega'\,z'^{\perp}\right)''\cdot\delta z\,ds=$$
$$=\Big[\omega'\,z'^{\perp}\cdot\delta z'-\left(\omega''\,z'^{\perp}-\omega'^2\,z'\right)\cdot\delta z\Big]_0^1+\int_0^1\left(\omega''\,z'^{\perp}-\omega'^2\,z'\right)'\cdot\delta z\,ds=\delta\mathcal{E}_1[z]\delta z$$

and apparently coincides with the variation of $\mathcal{E}_1$.

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