The surprising robustness of dynamic Mean-Variance portfolio optimization to model misspecification errors

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November 12, 2021

Abstract

In single-period portfolio optimization settings, Mean-Variance (MV) optimization can result in notoriously unstable asset allocations due to small changes in the underlying asset parameters. This has resulted in the widespread questioning of whether and how MV optimization should be implemented in practice, and has also resulted in a number of alternatives being proposed to the MV objective for asset allocation purposes. In contrast, in dynamic or multi-period MV portfolio optimization settings, preliminary numerical results show that MV investment outcomes can be remarkably robust to model misspecification errors, which arise when the investor derives an optimal investment strategy based on some chosen model for the underlying asset dynamics (the investor model), but implements this strategy in a market driven by potentially completely different dynamics (the true model). In this paper, we systematically investigate the causes of this surprising robustness of dynamic MV portfolio optimization to model misspecification errors under both the pre-commitment MV (PCMV) and time-consistent MV (TCMV) approaches. We identify particular combinations of parameters that play a key role in explaining the observed model misspecification errors. We investigate the impact of the chosen dynamic MV approach, underlying model formulation, portfolio rebalancing frequency and the application of multiple realistic investment constraints on the robustness of investment outcomes, as well as the implications for model calibration.

Keywords: Asset allocation, constrained optimal control, time-consistent, mean-variance

AMS Subject Classification: 91G, 65N06, 65N12, 35Q93

1 Introduction

Mean-variance (MV) portfolio optimization, originating with Markowitz (1952), has become the foundation of modern portfolio theory (Elton et al. (2014)). MV investment strategies are appealing due to their intuitive nature, since they clearly illustrate the trade-off between reward (expected return) and risk (variance of returns).

In single-period settings, MV optimization can provide notoriously unstable asset allocations arising from small changes in the underlying asset parameters (Michaud and Michaud (2008)). This issue has become especially pressing in recent machine learning applications (see for example Sato (2019)), where the use of "robo-advisors" for automatic portfolio allocation may in fact require significant human intervention to compensate for this sensitivity of MV portfolio allocations (Bourgeron et al. (2018); Perrin and Roncalli (2019)).

In order to increase the robustness of single-period MV asset allocations to changes in the underlying parameters, two fundamental approaches can be distinguished in the literature: (i) adjusting

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the MV objective by incorporating for example a penalty function that regularizes the asset allocation weights (see Bailey and Lopez de Prado (2012); Bourgeron et al. (2018); Brodie et al. (2009); Bruder et al. (2013); Carrasco and Noumon (2010); De Jong (2018); DeMiguel et al. (2009); Lopez de Prado (2016); Perrin and Roncalli (2019), among many others), or (ii) abandoning the MV objective altogether in favor of other objective functions. This includes the adoption of variants of the so-called "smart beta" portfolios, such as equal risk contribution or "risk parity" portfolios (Bruder et al. (2016); Lee (2011); Maillard et al. (2010); Qian (2005)), risk budgeting portfolios (Richard and Roncalli (2015); Roncalli (2013, 2015); Scherer (2007)) or "most diversified" portfolios (Choueifaty and Coignard (2008)), among other approaches.

In contrast, in the case of multi-period or dynamic MV optimization (Zhou and Li (2000)), the issue of robustness (or lack thereof) of MV investment outcomes to the underlying stochastic process parameters has to our knowledge not been systematically studied (an overview of the available literature is provided below). However, preliminary numerical results appear to suggest that in the case of dynamic MV optimization, the investment outcomes appear to be surprisingly stable in spite of changes in the underlying process specification and parameters (Dang and Forsyth (2016); Forsyth and Vetzal (2017a)). If true, this has potentially far-reaching consequences for practical asset allocation problems. For example, machine learning applications based on dynamic MV optimization (Li and Forsyth (2019); Wang and Zhou (2019)) might be able to achieve stable asset allocations in practical applications without requiring the adjustments described above that are often necessary in the context of single-period MV optimization problems.

Before discussing the robustness of dynamic MV results in more precise terms, we give a brief overview of the necessary background information regarding dynamic MV portfolio optimization. We start by observing that in dynamic settings, since the variance component of the MV objective is not separable in the sense of dynamic programming, two main approaches to perform dynamic MV optimization can be identified. The first approach, referred to as pre-commitment MV (PCMV) optimization, usually results in time-inconsistent optimal strategies (Basak and Chabakauri (2010)). However, since the PCMV problem is solved using the embedding approach of Li and Ng (2000); Zhou and Li (2000), the resulting optimal controls are time-consistent from the perspective of the quadratic objective function with a fixed target used in the corresponding embedding problem (see Vigna (2014, 2016)). This induced time-consistent objective function (see Strub et al. (2019)) is therefore feasible to implement as a trading strategy.

The second approach, referred to as time-consistent MV (TCMV) optimization, is based on a game-theoretic approach (Bjork and Murgoci (2014)). By optimizing only over a subset of controls which are time-consistent from the perspective of the original MV problem, or equivalently, by imposing a time-consistency constraint, the resulting TCMV-optimal strategies are guaranteed to be time-consistent (Basak and Chabakauri (2010); Bjork and Murgoci (2014); Wang and Forsyth (2011)).

Regardless of approach, dynamic MV optimization in a parametric setting requires the dynamics of the underlying assets in the market to be specified by the investor. The MV problem is then solved under the implicit assumption that the specified dynamics provide an accurate description of reality. However, given the sensitivity of single-period MV optimization results to the underlying parameters described above, together with the fact that inaccurate parameters can lead to substantial investment losses (Best and Grauer (1991); Britten-Jones (1999)), the sensitivity of dynamic MV optimization results to the underlying process assumptions poses a potential problem.

To address this potential problem, a number of approaches has been proposed in the literature. Perhaps the most common approach consists of implicitly acknowledging the possibility of using incorrect model parameters, and then performing a parameter sensitivity analysis of the optimization results (Li et al. (2015a, 2012); Lin and Qian (2016); Sun et al. (2016); Zhang and Chen (2016)). Another approach is to consider the MV optimization problem under partial information, where the specified dynamics for the risky asset might incorporate, for example, a random drift component which is not observable in the market, with only the asset prices being observable (Li et al. (2015b); Liang and Song (2015); Zhang et al. (2016)). A third approach consists of explicitly incorporating concerns regarding model parameters in some way in the objective of the portfolio optimization problem, thereby constructing a "robust" variation of the original problem - see, for example, Cong and Oosterlee (2017); Garlappi et al. (2007); Gulpinar and Rustem (2007); Kim et al. (2014); Kuhn et al. (2009); Tütüncü and Koenig (2004). However, it appears that all of the above-mentioned approaches consider a scenario which could perhaps best be described as *parameter* misspecification, where the concerns are associated with the model parameters of a *fixed* assumed underlying model type.

A more general, and perhaps more realistic, situation than parameter misspecification is *model* misspecification. Specifically, model misspecification describes the scenario where an optimal investment strategy (i) is obtained by solving the MV optimization problem based on some chosen model for the underlying asset dynamics, hereinafter referred to as the "investor model", but (ii) is then implemented in a market driven by potentially completely different dynamics, unknown to the investor, hereinafter referred to as the "true model". The MV outcome in the model misspecification scenario is potentially different from the MV outcome associated with the investor model-implied optimal strategy obtained in (i). We define the difference between these two quantities as a model misspecification error.

In the context of PCMV optimization, Dang and Forsyth (2016); Forsyth and Vetzal (2017a) numerically assess the impact of model misspecification. As observed above, preliminary findings show that, in the particular case of PCMV optimization with discrete rebalancing, the MV outcomes of terminal wealth can be surprisingly robust to such model misspecification errors. By robustness to model misspecification errors, we mean that these errors are surprisingly small even in cases where there are fundamental differences between the investor and true models.

Motivated by the above interesting preliminary findings, the main objective of this paper is a systematic investigation of the robustness of dynamic MV portfolio optimization to model misspecification. Our main contributions are as follows.

- We rigorously define and analyze the model misspecification problem in the context of PCMV and TCMV optimization, where the risky asset dynamics are allowed to follow pure-diffusion dynamics (e.g. GBM) or any of the standard finite-activity jump-diffusion models commonly encountered in financial settings.

- Under certain assumptions, we derive analytical solutions which enable us to quantify the impact of the MV approach (PCMV or TCMV) and rebalancing frequency (continuous or discrete rebalancing) on the resulting model misspecification error in MV outcomes. This allows us to provide a rigorous and intuitive explanation of the robustness of dynamic MV optimization results.

- Numerical tests are performed to (i) assess the practical implications of the analytical solutions using realistic investment data, and (ii) to compare the conclusions with numerical results for the case where multiple investment constraints (liquidation in the event of bankruptcy, leverage constraint) are applied simultaneously. To draw realistic conclusions from the numerical experiments, we consider multiple models and different calibration choices, with calibration data being inflation-adjusted, long-term US market data (89 years). We also discuss the implications of our results for model calibration choices.

- As an additional check on robustness, we also carry out tests using bootstrap resampling of historical data.

The remainder of the paper is organized as follows. Section 2 describes the underlying dynamics, the rebalancing of the portfolio, as well as the PCMV and TCMV optimization approaches. The robustness of MV optimization to model misspecification is rigorously defined in Section 3, where new analytical results are derived and discussed. Numerical results are presented in Section 4, while Section 5concludes the paper and outlines possible future work.

2 Formulation

Let $T > 0$ denote the fixed investment time horizon or maturity. We consider portfolios consisting of a well-diversified stock index (the risky asset) and a risk-free asset, which allows us to focus on the primary investment question of the risky vs. risk-free mix of the portfolio under the different model specifications, instead of secondary questions such as risky asset basket compositions[1]. Furthermore, since in practical applications investors are mostly concerned with inflation-adjusted outcomes (see, for example, Forsyth and Vetzal (2017b)), we introduce the following assumption.

Assumption 2.1. *(Inflation-adjusted parameters) Both the risky and risk-free asset dynamics are assumed to model inflation-adjusted (i.e. real) asset returns, so that all parameter values (including the risk-free interest rate) are assumed to reflect the appropriate real values.*

As a result, we make the following assumption throughout this paper.

Assumption 2.2. *(Correct real risk-free rate) We assume that the investor correctly specifies the underlying real dynamics of the risk-free asset. In particular, we assume that the constant, continuously compounded real risk-free rate, denoted by r, used by the investor is equal to the true real risk-free rate, which is also assumed to be constant and continuously compounded.*

We argue that Assumption 2.2 is reasonable given (i) the long time horizon under consideration (for example $T = 20$ years), together with (ii) the mean-reverting nature of interest rates, and (iii) Assumption 2.1, which typically results in an inflation-adjusted (real) risk-free rate of approximately zero[2], as expected. Nonetheless, in Appendix A, we include numerical tests of our conclusions using resampled historical interest rates to validate our results.

In contrast, we consider the realistic scenario where the investor might make an incorrect assumption regarding the underlying dynamics of the *risky* asset, which is formalized in Definition 2.1.

Definition 2.1. ("investor model" and "true model") An *investor model* is a model specified by the investor for the (inflation-adjusted) risky asset dynamics of the MV portfolio optimization problem which is to be solved to obtain the optimal control. The *true model* is the model that the (inflation-adjusted) risky asset dynamics follow in reality, which may or may not correspond to the investor model.

Our distinction between the investor model and true model in Definition 2.1 leads to the following definition.

Definition 2.2. (Model misspecification) *Model misspecification* is defined as the scenario where the investor model does not correspond to the true model, either in terms of the model parameters or in terms of the fundamental model types (e.g. pure diffusion vs. jump-diffusion).

The following definition distinguishes between two different categories of model misspecification.

Definition 2.3. (Category I and Category II model misspecification) A *Category I model misspecification* is defined as the scenario where the investor makes an incorrect assumption regarding the fundamental type of model (e.g. GBM vs the Merton jump-diffusion). A *Category II model misspecification*, or parameter misspecification, is defined as the scenario where the investor model and the true model refer to the same fundamental type of model, but the investor model's parameters differ from the true model's parameters.

[1] The available analytical solutions for multi-asset PCMV and TCMV problems (see, for example, Li and Ng (2000) and Zeng and Li (2011)) show that the overall composition of the risky asset basket remains relatively stable over time, indicating that the overall risky asset basket vs. risk-free asset composition of the portfolio is indeed the primary investment question.

[2] See Section 4 for a concrete example using US T-bill rates, where the risk-free rate of $r = 0.00623$ is obtained.

It is implicitly assumed in Definition 2.2 and Definition 2.3 that the investor does not update the investor model (either the model type or the calibrated parameters) over $[0,T]$. Not only is this assumption justified given our aim of quantifying the impact of model misspecification, but it can also be argued that it is a reasonable assumption if the model and parameter choice is based on a very long historical time series of data together with a significantly shorter (though still comparatively large) maturity T - see, for example, Section 4.

We also distinguish between discrete and continuous (portfolio) rebalancing. Discrete rebalancing refers to the case where the investor adjusts the wealth allocation between the risky and risk-free assets (portfolio rebalancing) only at fixed, pre-specified, discrete time intervals separated by a time interval of length $\Delta t > 0$. In contrast, in the case of continuous rebalancing the relative portfolio wealth allocations are adjusted continuously. In the limit as $\Delta t \downarrow 0$, discrete rebalancing and continuous rebalancing results should agree, as we will show subsequently.

For simplicity and clarity, we introduce the following notational conventions. Quantities applicable to discrete rebalancing are identified by the subscript Δt to distinguish them from their continuous rebalancing counterparts. Additionally, a subscript $j \in \{\text{iv}, \text{tr}\}$ is used to distinguish the investor model, denoted by the case of $j = \text{iv}$, from the true model where $j = \text{tr}$.

We will occasionally use the term "investor model-implied" to identify quantities associated with the investor model. For example, an "investor model implied optimal control" is an optimal control obtained by solving an MV optimization problem under the investor model $j = \text{iv}$.

In analyzing the model misspecification error, the subscript $(\text{iv} \to \text{tr})$ is reserved for quantities related to the case where the investor model-implied optimal control (i.e. using $j = \text{iv}$) is implemented in a market evolving according to the true model (i.e. $j = \text{tr}$).

Finally, a superscript "p" is used to identify quantities related to PCMV optimization, while quantities related to TCMV optimization will be denoted using a superscript "c".

We now describe the model and portfolio rebalancing assumptions in more detail, starting with the case of discrete rebalancing.

2.1 Discrete rebalancing

Let $S_j(t)$ and $B(t)$ denote the *amounts* invested in the risky and risk-free asset[3], respectively, at time $t \in [0,T]$, where $j \in \{\text{iv}, \text{tr}\}$. Let $X_j(t) = (S_j(t), B(t)), t \in [0,T]$ denote the multi-dimensional controlled underlying process, and $x = (s,b)$ the state of the system. The controlled portfolio wealth $W_{j,\Delta t}(t)$ in the case of discrete rebalancing is simply given by

$$W_{j,\Delta t}(t) = W(S_j(t), B(t)) \quad = \quad S_j(t) + B(t), \qquad t \in [0,T], \quad j \in \{\text{iv}, \text{tr}\}. \tag{2.1}$$

We define $\mathcal{T}_m$ as the set of m discrete, predetermined, equally spaced rebalancing times in $[0,T]$,

$$\mathcal{T}_m \quad = \quad \{t_n | \, t_n = (n-1)\Delta t, \; n = 1, \ldots, m\}, \quad \Delta t = T/m. \tag{2.2}$$

For any functional f, let $f(t^-) \coloneqq \lim_{\epsilon \to 0^+} f(t-\epsilon)$ and $f(t^+) \coloneqq \lim_{\epsilon \to 0^+} f(t+\epsilon)$. Informally, t^- (resp. t^+) denotes the instant of time immediately before (resp. after) the forward time $t \in [0,T]$. Fix two consecutive rebalancing times $t_n, t_{n+1} \in T_m$. Since there is no rebalancing by the investor according to some control strategy over $\left[t_n^+, t_{n+1}^-\right]$, the dynamics of the amount $B(t)$ in the absence of control is assumed to be given by

$$dB(t) \quad = \quad rB(t)\,dt, \quad t \in \left[t_n^+, t_{n+1}^-\right], \tag{2.3}$$

with $r > 0$ denoting the real risk-free rate. Observe that we do not make use of a stochastic interest rate model, partly due to the inflation-adjusted risk-free rates being approximately zero (see Assumptions 2.1 and 2.2). However, we include a bootstrap resampling test using historical real interest rates to

[3]As observed in Dang et al. (2017), in the case of the discrete rebalancing of the portfolio, it is simpler to model the dollar amounts invested in the risky and risk-free asset directly.

validate our results (see Appendix A), confirming that explicitly modelling stochastic interest rates are not particularly important in this setting.

For the purposes of modelling the amount invested in the risky asset, it is reasonable to consider incorporating (i) jumps and (ii) stochastic volatility in the process dynamics. However, the results from Ma and Forsyth (2016) show that the effects of stochastic volatility, with realistic mean-reverting dynamics, are not important for long-term MV investors with time horizons greater than 10 years. As a result, we incorporate jump-diffusion and pure diffusion models for the risky asset in our analysis, as highlighted in the following assumption, leaving alternative model specifications for our future work.

Assumption 2.3. *(Types of models for the risky asset) We assume that any risky asset model under consideration, whether the investor model or the true model, can be classified into one of the following two fundamental model types: (i) pure diffusion (geometric Brownian motion / GBM), or (ii) any of the finite-activity jump-diffusion models commonly encountered in financial settings (such as the Merton (1976) and Kou (2002) models).*

For defining the jump-diffusion model dynamics, let ξ_j be a random variable denoting the jump multiplier with probability density function (pdf) $p_j(\xi)$, where $j \in \{\text{iv}, \text{tr}\}$. For subsequent reference, we define $\kappa_{j,1} = \mathbb{E}[\xi_j - 1]$ and $\kappa_{j,2} = \mathbb{E}\left[(\xi_j - 1)^2\right]$. Between any two consecutive rebalancing times $t_n, t_{n+1} \in T_m$, we assume the following dynamics for the amount S_j in the absence of control,

$$\frac{dS_j(t)}{S_j(t^-)} = (\mu_j - \lambda_j \kappa_{j,1})\, dt + \sigma_j dZ_j + d\left(\sum_{i=1}^{\pi_j(t)} \left(\xi_j^i - 1\right)\right), \; t \in \left[t_n^+, t_{n+1}^-\right], \; j \in \{\text{iv}, \text{tr}\}, \quad (2.4)$$

where μ_j and σ_j are drift and volatility respectively, Z_j denotes a standard Brownian motion, $\pi_j(t)$ is a Poisson process with intensity $\lambda_j \geq 0$, and ξ_j^i are i.i.d. random variables with the same distribution as ξ_j. It is futhermore assumed that ξ_j^i, $\pi_j(t)$ and Z_j for $j \in \{\text{iv}, \text{tr}\}$ are all mutually independent. Note that pure diffusion (GBM) dynamics for $S_j(t)$ can be recovered from (2.4) by setting the intensity parameter λ_j to zero. For subsequent reference, we use $\Delta t > 0$ as in (2.2) to define

$$\alpha_j = e^{\mu_j \Delta t} - e^{r\Delta t}, \; \psi_j = \left[e^{\left(2\mu_j + \sigma_j^2 + \lambda_j \kappa_{j,2}\right)\Delta t} - e^{2\mu_j \Delta t}\right]^{1/2}, \; A_{j,\Delta t} = \left(\frac{\alpha_j^2}{\psi_j^2} \cdot \frac{1}{\Delta t}\right), \; j \in \{\text{iv}, \text{tr}\}. \quad (2.5)$$

Discrete portfolio rebalancing is modelled using the impulse control formulation as discussed in for example Dang and Forsyth (2014); Van Staden et al. (2018, 2019), which we now briefly summarize. Suppose that the system is in state $x = (s, b) = (S(t_n^-), B(t_n^-))$ for some $t_n \in \mathcal{T}_m$. Let $u_{\Delta t}(t_n)$ denote the impulse value or amount invested in the risky asset after rebalancing the portfolio at time t_n, and let $\mathcal{Z}$ denote the set of admissible impulse values. If $(S_j(t_n), B(t_n))$ denotes the state of the system immediately after the application of the impulse $u_{\Delta t}(t_n)$, we define

$$S_j(t_n) = u_{\Delta t}(t_n), \qquad B(t_n) = (s + b) - u_{\Delta t}(t_n), \quad j \in \{\text{iv}, \text{tr}\}. \quad (2.6)$$

Let $\mathcal{A}_{\Delta t}$ denote the set of admissible discretized impulse controls in the case of discrete rebalancing, defined as

$$\mathcal{A}_{\Delta t} = \left\{u_{\Delta t} = \{u_{\Delta t}(t_n)\}_{n=1,\ldots,m} : \; t_n \in \mathcal{T}_m \text{ and } u_{\Delta t}(t_n) \in \mathcal{Z}, \text{ for } n = 1, \ldots, m\right\}. \quad (2.7)$$

Let $E_{u_{\Delta t}}^{x,t_n}[W_{j,\Delta t}(T)]$ and $Var_{u_{\Delta t}}^{x,t_n}[W_{j,\Delta t}(T)]$ denote the mean and variance of the terminal wealth as per model $j \in \{\text{iv}, \text{tr}\}$, respectively, given that we are in state $x = (s, b) = (S(t_n^-), B(t_n^-))$ for some $t_n \in \mathcal{T}_m$, and using impulse control $u_{\Delta t} \in \mathcal{A}_{\Delta t}$ over $[t_n, T]$. Using the standard scalarization method for multi-criteria optimization problems (Yu (1971)), the MV objective using investor model dynamics ($j = \text{iv}$) is given by

$$\sup_{u_{\Delta t} \in \mathcal{A}_{\Delta t}} \left(E_{u_{\Delta t}}^{x,t_n}[W_{\text{iv},\Delta t}(T)] - \rho \cdot Var_{u_{\Delta t}}^{x,t_n}[W_{\text{iv},\Delta t}(T)]\right), \quad (2.8)$$

where the scalarization (or risk-aversion) parameter $\rho > 0$ reflects the investor's level of risk aversion. Dynamic programming cannot be applied directly to (2.8), since variance does not satisfy the smoothing property of conditional expectation. Instead, the technique of Li and Ng (2000); Zhou and Li (2000) embeds (2.8) in a new optimization problem, often referred to as the embedding problem, which is amenable to dynamic programming techniques.

We follow the convention in literature (see, for example, Cong and Oosterlee (2017); Dang and Forsyth (2014)) of defining the PCMV optimization problem as the associated embedding MV problem[4]. Specifically, in the case of discrete rebalancing, $PCMV_{\Delta t}\left(t_n;\gamma\right)$ denotes the PCMV problem at time t_n using embedding parameter $\gamma \in \mathbb{R}$ under the assumption that the investor model is used,

$$\left(PCMV_{\Delta t}\left(t_n;\gamma\right)\right): \qquad V_{\Delta t}^{p}\left(s,b,t_n\right) = \inf_{u_{\Delta t}\in\mathcal{A}_{\Delta t}} E_{u_{\Delta t}}^{x,t_n}\left[\left(W_{\mathrm{iv},\Delta t}\left(T\right)-\frac{\gamma}{2}\right)^2\right], \qquad \gamma\in\mathbb{R}, \quad (2.9)$$

where the risk-free and risky asset dynamics between rebalancing events are respectively given by (2.3) and (2.4) with $j = \mathrm{iv}$. The optimal control which solves $\left(PCMV_{\Delta t}\left(t_n;\gamma\right)\right)$ will be denoted by $u_{\mathrm{iv},\Delta t}^{p*} = \left\{u_{\mathrm{iv},\Delta t}^{p*}\left(t_k\right) : k = n,\ldots,m\right\}$.

For any fixed value of $\gamma \in \mathbb{R}$, we note that the optimal control $u_{\mathrm{iv},\Delta t}^{p*}$ is a time-consistent control for the corresponding quadratic shortfall objective function in (2.9), and is therefore feasible to implement as a trading strategy (see Strub et al. (2019)).

The TCMV formulation involves maximizing the objective (2.8) subject to a time-consistency constraint (see, for example, Wang and Forsyth (2011)), so that the resulting optimal control is time-consistent from the perspective of the original MV objective. In the case of discrete rebalancing, given that the portfolio is in state $x = (s,b) = \left(S\left(t_n^-\right), B\left(t_n^-\right)\right)$ for some $t_n \in \mathcal{T}_m$, the TCMV problem is defined for $\rho > 0$ by

$$\begin{aligned}\left(TCMV_{\Delta t}\left(t_n;\rho\right)\right): \; V_{\Delta t}^{c}\left(s,b,t_n\right) &:= \sup_{u_{\Delta t}\in\mathcal{A}_{\Delta t}}\left(E_{u_{\Delta t}}^{x,t_n}\left[W_{\mathrm{iv},\Delta t}\left(T\right)\right] - \rho\cdot Var_{u_{\Delta t}}^{x,t_n}\left[W_{\mathrm{iv},\Delta t}\left(T\right)\right]\right), \quad (2.10)\\ \text{s.t. } u_{\Delta t} &= \left\{u_{\Delta t}\left(t_n\right), u_{\mathrm{iv},\Delta t}^{c*}\left(t_{n+1}\right),\ldots,u_{\mathrm{iv},\Delta t}^{c*}\left(t_m\right)\right\}, \quad (2.11)\end{aligned}$$

where $u_{\mathrm{iv},\Delta t}^{c*} = \left\{u_{\mathrm{iv},\Delta t}^{c*}\left(t_k\right) : k = n,\ldots,m\right\}$ is the optimal control[5] for problem $TCMV_{\Delta t}\left(t_n;\rho\right)$.

Remark 2.4. (Portfolio optimization and model misspecification) The PCMV and TCMV problems, and associated optimal controls, have been defined using only the investor model ($j = \mathrm{iv}$). While the formulation and analytical results presented in this section also hold for $j = \mathrm{tr}$, this seemingly additional generality obscures the fact that by practical necessity, these problems are defined and solved by the investor only under the investor model dynamics (which the investor believes to be correct), which may of course agree with the true model dynamics in the special case where $j = \mathrm{iv} = \mathrm{tr}$.

The following lemma gives the analytical solutions for the PCMV and TCMV problems in the case of discrete rebalancing with no investment constraints.

Lemma 2.5. *(Discrete rebalancing: investor model, no investment constraints) Assume the discrete rebalancing of the portfolio, with given state* $x = (s,b) = \left(S\left(t_n^-\right), B\left(t_n^-\right)\right)$ *and wealth* $w = s+b$ *for some* $t_n \in \mathcal{T}_m$, $n \in \{1,\ldots,m\}$*, investor model wealth dynamics (2.1) with* $j = \mathrm{iv}$*, and that no investment*

[4]For a discussion of the elimination of spurious optimization results when using the embedding formulation, see Dang et al. (2016). Note that it might be optimal under some conditions to withdraw cash from the portfolio (see Cui et al. (2012); Dang and Forsyth (2016)), but in order to ensure a like-for-like comparison with the TCMV results, we do not consider the withdrawal of cash. While this treatment potentially penalizes large gains, the robustness of the PCMV problem incorporating free cash flow is numerically investigated in great detail in Forsyth and Vetzal (2017a), and it is clear from their results that the fundamental conclusions of this paper are not affected by excluding the withdrawal of cash.

[5]$u_{\mathrm{iv},\Delta t}^{c*}$ satisfies the conditions of a subgame perfect Nash equilibrium control, so that the terminology "equilibrium" control is sometimes used (see e.g. Bjork et al. (2014)). We follow for example of Basak and Chabakauri (2010); Cong and Oosterlee (2016); Wang and Forsyth (2011) and retain the terminology "optimal" control for simplicity.

constraints are applicable ($\mathcal{Z} = \mathbb{R}$). Solutions to problem $PCMV_{\Delta t}(t_n;\gamma)$ in (2.9) are given by

$$u^{p*}_{iv,\Delta t}(t_n) = \frac{A_{iv,\Delta t}\cdot\Delta t}{(1+A_{iv,\Delta t}\cdot\Delta t)}\cdot\frac{e^{r\Delta t}}{\alpha_{iv}}\cdot e^{-r(T-t_n)}\left[\frac{\gamma}{2} - we^{r(T-t_n)}\right],\ n=1,\ldots,m, \quad (2.12)$$

$$E^{x,t_n}_{u^{p*}_{iv,\Delta t}}\left[W_{iv,\Delta t}(T)\right] = we^{r(T-t_n)} + \left[1-\left(1-\frac{A_{iv,\Delta t}\cdot\Delta t}{(1+A_{iv,\Delta t}\cdot\Delta t)}\right)^{m-n+1}\right]\left[\frac{\gamma}{2} - we^{r(T-t_n)}\right], \quad (2.13)$$

$$Stdev^{x,t_n}_{u^{p*}_{iv,\Delta t}}\left[W_{iv,\Delta t}(T)\right] = \left(1-\frac{A_{iv,\Delta t}\cdot\Delta t}{(1+A_{iv,\Delta t}\cdot\Delta t)}\right)^{m-n+1}\left[\left(1-\frac{A_{iv,\Delta t}\cdot\Delta t}{(1+A_{iv,\Delta t}\cdot\Delta t)}\right)^{-(m-n+1)}-1\right]^{\frac{1}{2}} \times\left[\frac{\gamma}{2} - we^{r(T-t_n)}\right]. \quad (2.14)$$

Solutions to problem $TCMV_{\Delta t}(t_n;\rho)$ in (2.10)-(2.11) are given by

$$u^{c*}_{iv,\Delta t}(t_n) = \frac{1}{2\rho}\cdot(A_{iv,\Delta t}\cdot\Delta t)\cdot\frac{e^{r\Delta t}}{\alpha_{iv}}\cdot e^{-r(T-t_n)}, \qquad n=1,\ldots,m, \quad (2.15)$$

$$E^{x,t_n}_{u^{c*}_{iv,\Delta t}}\left[W_{iv,\Delta t}(T)\right] = we^{r(T-t_n)} + \frac{1}{2\rho}A_{iv,\Delta t}(T-t_n), \quad (2.16)$$

$$Stdev^{x,t_n}_{u^{c*}_{iv,\Delta t}}\left[W_{iv,\Delta t}(T)\right] = \frac{1}{2\rho}\sqrt{A_{iv,\Delta t}\cdot(T-t_n)}. \quad (2.17)$$

Proof. The PCMV results (2.12)-(2.14) can be obtained by applying the results of Li and Ng (2000) to our formulation, while TCMV results (2.15)-(2.17) using the impulse control formulation can be found in Van Staden et al. (2019). □

2.2 Continuous rebalancing

In the case of continuous rebalancing, we specify the controlled wealth dynamics of the self-financing portfolio in terms of a single stochastic differential equation by (implicitly) modelling the value of a unit investment in each asset (see, for example, Bjork et al. (2014); Zeng et al. (2013)).

Let $W_j(t)$ also denote the controlled wealth process in the case of continuous rebalancing, where we again distinguish the dynamics of the investor model and true model using $j \in \{\text{iv}, \text{tr}\}$. Let $u : (W_j(t), t) \mapsto u(t) = u(W_j(t), t),\ t \in [0,T]$ be the adapted feedback control representing the amount invested in the risky asset at time t given wealth $W_j(t)$, and let $\mathcal{A} = \{\, u(t) = u(w,t) |\, u : \mathbb{R}\times[0,T] \to \mathbb{U}\}$ denote the set of admissible controls in the case of continuous rebalancing, where $\mathbb{U} \subseteq \mathbb{R}$ is the admissible control space.

If the unit value of the risky asset has the same dynamics as (2.4), then the dynamics of $W_j(t)$, for $j \in \{\text{iv}, \text{tr}\}$, is given by

$$dW_j(t) = \left[rW_j(t) + (\mu_j - \lambda_j\kappa_{j,1} - r)\,u(t)\right]dt + \sigma_j u(t)\,dZ_j + u(t)\,d\left(\sum_{i=1}^{\pi_j(t)}\left(\xi^i_j - 1\right)\right). \quad (2.18)$$

For subsequent reference, we define the following combination of parameters associated with (2.18),

$$A_j = \frac{(\mu_j - r)^2}{\sigma_j^2 + \lambda_j\kappa_{j,2}}, \qquad j \in \{\text{iv}, \text{tr}\}. \quad (2.19)$$

Given state $x = (s,b)$ at time $t \in [0,T]$ and $w = s+b$, we denote the mean and variance of terminal wealth $W_j(T)$ under control u, respectively, by $E^{w,t}_u[W_j(T)]$ and $Var^{w,t}_u[W_j(T)]$. In the case of continuous rebalancing, the PCMV optimization problem $PCMV(t;\gamma)$ is given by

$$(PCMV(t;\gamma)): \qquad V^p(w,t) = \inf_{u\in\mathcal{A}} E^{w,t}_u\left[\left(W_{\text{iv}}(T) - \frac{\gamma}{2}\right)^2\right], \qquad \gamma\in\mathbb{R}, \quad (2.20)$$

where the controlled wealth W_{iv} has dynamics given by (2.18) with $j = \text{iv}$. We denote by u_{iv}^{p*} the optimal control which solves $(PCMV\,(t;\gamma))$ using the investor model dynamics.

We follow Wang and Forsyth (2011) in defining the TCMV problem in the case of continuous rebalancing, $TCMV\,(t;\rho)$, as

$$
\begin{aligned}
(TCMV\,(t;\rho)):\ \ V^c(w,t) &:= \sup_{u\in\mathcal{A}} \left(E_u^{w,t}\left[W_{\text{iv}}(T)\right] - \rho\cdot Var_u^{w,t}\left[W_{\text{iv}}(T)\right]\right), \quad \rho>0, \qquad (2.21)\\
\text{s.t. } u_{\text{iv}}^{c*}(t;y,v) &= u_{\text{iv}}^{c*}\left(t';y,v\right), \quad \text{for } v \geq t', t'\in[t,T], \qquad (2.22)
\end{aligned}
$$

where $u_{\text{iv}}^{c*}(t;y,v)$ denotes the optimal control for problem $TCMV\,(t;\rho)$ calculated at time t and to be applied at some future time $v \geq t' \geq t$ given future state $W_{\text{iv}}(v) = y$, while $u_{\text{iv}}^{c*}(t';v,y)$ denotes the optimal control calculated at some future time $t' \in [t,T]$ for problem $TCMV\,(t';\rho)$, also to be applied at the same later time $v \geq t'$ given the same future state $W_{\text{iv}}(v) = y$. To lighten notation, we will simply use the notation $u_{\text{iv}}^{c*}(t)$ to denote the optimal control for problem (2.21)-(2.22).

We have the following analytical solutions for the PCMV and TCMV problems in the case of continuous rebalancing with no investment constraints.

Lemma 2.6. *(Continuous rebalancing: investor model, no investment constraints) Assume the continuous rebalancing of the portfolio, with wealth w at time $t\in[0,T]$, investor model wealth dynamics (2.18) with $j = \text{iv}$, and that no investment constraints are applicable $(\mathbb{U}=\mathbb{R})$. Solutions to problem $PCMV\,(t;\gamma)$ in (2.20) are given by*

$$
\begin{aligned}
u_{iv}^{p*}(t) &= \frac{A_{iv}}{(\mu_{iv}-r)}e^{-r(T-t)}\left[\frac{\gamma}{2} - we^{r(T-t)}\right], \qquad (2.23)\\
E_{u_{iv}^{p*}}^{w,t}\left[W_{iv}(T)\right] &= we^{r(T-t)} + \left(1-e^{-A_{iv}(T-t)}\right)\left[\frac{\gamma}{2} - we^{r(T-t)}\right], \qquad (2.24)\\
Stdev_{u_{iv}^{p*}}^{w,t}\left[W_{iv}(T)\right] &= e^{-A_{iv}(T-t)}\left[e^{A_{iv}(T-t)}-1\right]^{\frac{1}{2}}\left[\frac{\gamma}{2} - we^{r(T-t)}\right]. \qquad (2.25)
\end{aligned}
$$

Solutions to problem $TCMV\,(t;\rho)$ in (2.21)-(2.22) are given by

$$
\begin{aligned}
u_{iv}^{c*}(t) &= \frac{1}{2\rho}\cdot\frac{A_{iv}}{(\mu_{iv}-r)}e^{-r(T-t)}, \qquad (2.26)\\
E_{u_{iv}^{c*}}^{w,t}\left[W_{iv}(T)\right] &= we^{r(T-t)} + \frac{1}{2\rho}A_{iv}(T-t), \qquad (2.27)\\
Stdev_{u_{iv}^{c*}}^{w,t}\left[W_{iv}(T)\right] &= \frac{1}{2\rho}\sqrt{A_{iv}(T-t)}. \qquad (2.28)
\end{aligned}
$$

Furthermore, taking the limit as $\Delta t \downarrow 0$ in the discrete rebalancing results (2.12)-(2.14) and (2.15)-(2.17) recovers the continuous rebalancing results (2.23)-(2.25) and (2.26)-(2.28), respectively.

Proof. The PCMV results (2.23)-(2.25) can be found in Zhou and Li (2000); Zweng and Li (2011), while the TCMV results (2.26)-(2.28) are given in Basak and Chabakauri (2010); Zeng et al. (2013). The convergence results as $\Delta t \downarrow 0$ using our impulse control formulation can be found in Van Staden et al. (2019). Here we simply observe that (2.2) implies $(m-n+1)\Delta t = T - t_n$, and we note the following limits which are useful for proving subsequent results: $\lim_{\Delta t\downarrow 0} A_{\text{iv},\Delta t} = A_{\text{iv}}$ and

$$
\lim_{\Delta t\downarrow 0}\frac{A_{\text{iv},\Delta t}\cdot\Delta t}{\alpha_{\text{iv}}\left(1+A_{\text{iv},\Delta t}\cdot\Delta t\right)} = \lim_{\Delta t\downarrow 0}\frac{A_{\text{iv},\Delta t}\cdot\Delta t}{\alpha_{\text{iv}}} = \frac{A_{\text{iv}}}{(\mu_{\text{iv}}-r)}, \quad \lim_{\Delta t\downarrow 0}\left(1-\frac{A_{\text{iv},\Delta t}\cdot\Delta t}{\left(1+A_{\text{iv},\Delta t}\cdot\Delta t\right)}\right)^{1/\Delta t} = e^{-A_{\text{iv}}}. \qquad (2.29)
$$

□

2.3 MV efficient points under the investor model

The following definition of MV efficient point and MV efficient frontier is standard in the literature (see, for example, Dang et al. (2016)).

Definition 2.7. (MV efficient point, MV efficient frontier) Assume a given initial state $x_0 = (s_0, b_0)$ with initial wealth $w_0 = s_0 + b_0 > 0$, at time $t_0 \equiv t_1 = 0$, and investor model wealth dynamics (2.1) with $j = \text{iv}$. For a fixed value of the scalarization parameter $\rho > 0$ and the embedding parameter $\gamma \in \mathbb{R}$, an MV efficient point in $\mathbb{R}^2$ is defined as follows:

$$(\mathcal{S}, \mathcal{E}) = \begin{cases} (\mathcal{S}, \mathcal{E})^p_\gamma & := \left(Stdev^{w_0,t_0}_{u^{p*}_{\text{iv}}} \left[W_{\text{iv}}(T) \right], E^{w_0,t_0}_{u^{p*}_{\text{iv}}} \left[W_{\text{iv}}(T) \right] \right), \quad \text{for } PCMV\left(t_0; \gamma\right), \\ (\mathcal{S}, \mathcal{E})^c_\rho & := \left(Stdev^{w_0,t_0}_{u^{c*}_{\text{iv}}} \left[W_{\text{iv}}(T) \right], E^{w_0,t_0}_{u^{c*}_{\text{iv}}} \left[W_{\text{iv}}(T) \right] \right), \quad \text{for } TCMV\left(t_0; \rho\right), \\ (\mathcal{S}, \mathcal{E})^p_{\gamma,\Delta t} & := \left(Stdev^{x_0,t_0}_{u^{p*}_{\text{iv},\Delta t}} \left[W_{\text{iv},\Delta t}(T) \right], E^{x_0,t_0}_{u^{p*}_{\text{iv},\Delta t}} \left[W_{\text{iv},\Delta t}(T) \right] \right), \quad \text{for } PCMV_{\Delta t}\left(t_0; \gamma\right), \\ (\mathcal{S}, \mathcal{E})^c_{\rho,\Delta t} & := \left(Stdev^{x_0,t_0}_{u^{c*}_{\text{iv},\Delta t}} \left[W_{\text{iv},\Delta t}(T) \right], E^{x_0,t_0}_{u^{c*}_{\text{iv},\Delta t}} \left[W_{\text{iv},\Delta t}(T) \right] \right), \quad \text{for } TCMV_{\Delta t}\left(t_0; \rho\right). \end{cases} \tag{2.30}$$

The MV efficient frontiers traced out in $\mathbb{R}^2$ using (2.30) are respectively given by $\mathcal{Y}^p = \bigcup_{\gamma \in \mathbb{R}} (\mathcal{S}, \mathcal{E})^p_\gamma$, $\mathcal{Y}^c = \bigcup_{\rho > 0} (\mathcal{S}, \mathcal{E})^c_\rho$, $\mathcal{Y}^p_{\Delta t} = \bigcup_{\gamma \in \mathbb{R}} (\mathcal{S}, \mathcal{E})^p_{\gamma,\Delta t}$, and $\mathcal{Y}^c_{\Delta t} = \bigcup_{\rho > 0} (\mathcal{S}, \mathcal{E})^c_{\rho,\Delta t}$.

It is well-known that the coordinates of the MV efficient point in Definition 2.7 exhibit a linear relationship if no investment constraints are applicable. This is given by the following lemma.

Lemma 2.8. *(MV efficient point linear relationship, no investment constraints) If no investment constraints are applicable, the relationship between the coordinates $(\mathcal{S}, \mathcal{E})$ of an MV efficient point in Definition 2.7 is given by*

$$\mathcal{E} = w_0 e^{rT} + \Gamma_{iv} \cdot \mathcal{S}, \tag{2.31}$$

where Γ_{iv}, the slope of the associated efficient frontier, is given by

$$\Gamma_{iv} = \begin{cases} \Gamma^p_{iv} & = \left(e^{A_{iv} T} - 1 \right)^{\frac{1}{2}}, \quad \text{for } PCMV\left(t_0; \gamma\right), \\ \Gamma^c_{iv} & = \sqrt{A_{iv} T}, \quad \text{for } TCMV\left(t_0; \rho\right), \\ \Gamma^p_{iv,\Delta t} & = \left[\left(1 + A_{iv,\Delta t} \cdot \Delta t \right)^m - 1 \right]^{\frac{1}{2}}, \quad \text{for } PCMV_{\Delta t}\left(t_0; \gamma\right), \\ \Gamma^c_{iv,\Delta t} & = \sqrt{A_{iv,\Delta t} T}, \quad \text{for } TCMV_{\Delta t}\left(t_0; \rho\right). \end{cases} \tag{2.32}$$

Here, $A_{iv,\Delta t}$ and A_{iv} are respectively defined in (2.5) *and* (2.19).

Proof. Follows from rearranging the results of Lemmas 2.5 and 2.6. □

2.4 Investor efficient point

After considering the MV efficient frontier (Definition 2.30), by necessity, the investor has to choose a particular reference MV efficient point according to their risk appetite/preferences. We make the practical assumption that the investor chooses some target value of the investor model-implied standard deviation of terminal wealth, $\mathcal{S}_{\text{iv}}$, with the intention of implementing the corresponding optimal strategy over $[0, T]$. Associated with the fixed target $\mathcal{S}_{\text{iv}}$ is a particular expected value of terminal wealth $W_{\text{iv}}(T)$, denoted by $\mathcal{E}_{\text{iv}}$, for which the pair $(\mathcal{S}_{\text{iv}}, \mathcal{E}_{\text{iv}})$ is an MV efficient point as per Definition 2.7. In subsequent discussion, we refer to the point $(\mathcal{S}_{\text{iv}}, \mathcal{E}_{\text{iv}})$ as an investor efficient point.

Naturally, in this case, fixing the target $\mathcal{S}_{\text{iv}} > 0$ is equivalent to fixing particular values of the parameter $\rho \in \{\rho_{\text{iv}}, \rho_{\text{iv},\Delta t}\}$ and $\gamma \in \{\gamma_{\text{iv}}, \gamma_{\text{iv},\Delta t}\}$. That is, with these fixed values, the optimal controls of $PCMV\left(t_0; \gamma_{\text{iv}}\right)$, $TCMV\left(t_0; \rho_{\text{iv}}\right)$, $PCMV_{\Delta t}\left(t_0; \gamma_{\text{iv},\Delta t}\right)$ and $TCMV_{\Delta t}\left(t_0; \rho_{\text{iv},\Delta t}\right)$ all achieve a standard deviation of terminal wealth $W_{\text{iv}}(T)$ equal to $\mathcal{S}_{\text{iv}}$. These values can be obtained by (numerically) solving for $\rho \in \{\rho_{\mathrm{iv}}, \rho_{\mathrm{iv},\Delta t}\}$ and $\gamma \in \{\gamma_{\mathrm{iv}}, \gamma_{\mathrm{iv},\Delta t}\}$ in the (non-linear) equations

$$\mathcal{S}_{\mathrm{iv}} = \left\{(\mathcal{S})^c_\rho, (\mathcal{S})^c_{\rho,\Delta t}\right\} \quad \text{and} \quad \mathcal{S}_{\mathrm{iv}} = \left\{(\mathcal{S})^p_\gamma, (\mathcal{S})^p_{\gamma,\Delta t}\right\}, \tag{2.33}$$

where $(\mathcal{S})^c_\rho$, $(\mathcal{S})^c_{\rho,\Delta t}$, $(\mathcal{S})^p_\gamma$, and $(\mathcal{S})^p_{\gamma,\Delta t}$ are defined in Definition 2.7. When investment constraints are not applicable, the values of $\gamma_{\mathrm{iv}}, \rho_{\mathrm{iv}}, \gamma_{\mathrm{iv},\Delta t}$, and $\rho_{\mathrm{iv},\Delta t}$ can be obtained in closed-form as follows:

$$\begin{aligned}
\gamma_{\mathrm{iv}} &= 2w_0 e^{rT} + 2\mathcal{S}_{\mathrm{iv}} \cdot e^{A_{\mathrm{iv}}T}\left[e^{A_{\mathrm{iv}}T} - 1\right]^{-\frac{1}{2}}, && (2.34)\\
\rho_{\mathrm{iv}} &= \sqrt{A_{\mathrm{iv}}T}/\left(2\cdot\mathcal{S}_{\mathrm{iv}}\right), && (2.35)\\
\gamma_{\mathrm{iv},\Delta t} &= 2w_0 e^{rT} + 2\mathcal{S}_{\mathrm{iv}} \cdot \left(1 - \frac{A_{\mathrm{iv},\Delta t}\cdot\Delta t}{(1 + A_{\mathrm{iv},\Delta t}\cdot\Delta t)}\right)^{-m}\left[\left(1 - \frac{A_{\mathrm{iv},\Delta t}\cdot\Delta t}{(1 + A_{\mathrm{iv},\Delta t}\cdot\Delta t)}\right)^{-m} - 1\right]^{-\frac{1}{2}}, && (2.36)\\
\rho_{\mathrm{iv},\Delta t} &= \sqrt{A_{\mathrm{iv},\Delta t}T}/\left(2\cdot\mathcal{S}_{\mathrm{iv}}\right), && (2.37)
\end{aligned}$$

with $A_{\mathrm{iv},\Delta t}$ and A_{iv} respectively given in (2.5) and (2.19). We now formally define an investor efficient point.

Definition 2.9. (Investor efficient point) For a fixed target $\mathcal{S}_{\mathrm{iv}} > 0$ for the investor model-implied standard deviation of terminal wealth, an investor efficient point, denoted by $(\mathcal{S}_{\mathrm{iv}}, \mathcal{E}_{\mathrm{iv}})$, is defined as

$$(\mathcal{S}_{\mathrm{iv}}, \mathcal{E}_{\mathrm{iv}}) := \begin{cases}
\left(Stdev^{w_0,t_0}_{u^{p*}_{\mathrm{iv}}}\left[W_{\mathrm{iv}}(T)\right], E^{w_0,t_0}_{u^{p*}_{\mathrm{iv}}}\left[W_{\mathrm{iv}}(T)\right]\right) & \text{for } PCMV\left(t_0;\gamma_{\mathrm{iv}}\right),\\
\left(Stdev^{w_0,t_0}_{u^{c*}_{\mathrm{iv}}}\left[W_{\mathrm{iv}}(T)\right], E^{w_0,t_0}_{u^{c*}_{\mathrm{iv}}}\left[W_{\mathrm{iv}}(T)\right]\right) & \text{for } TCMV\left(t_0;\rho_{\mathrm{iv}}\right),\\
\left(Stdev^{x_0,t_0}_{u^{p*}_{\mathrm{iv},\Delta t}}\left[W_{\mathrm{iv},\Delta t}(T)\right], E^{x_0,t_0}_{u^{p*}_{\mathrm{iv},\Delta t}}\left[W_{\mathrm{iv},\Delta t}(T)\right]\right) & \text{for } PCMV_{\Delta t}\left(t_0;\gamma_{\mathrm{iv},\Delta t}\right),\\
\left(Stdev^{x_0,t_0}_{u^{c*}_{\mathrm{iv},\Delta t}}\left[W_{\mathrm{iv},\Delta t}(T)\right], E^{x_0,t_0}_{u^{c*}_{\mathrm{iv},\Delta t}}\left[W_{\mathrm{iv},\Delta t}(T)\right]\right) & \text{for } TCMV_{\Delta t}\left(t_0;\rho_{\mathrm{iv},\Delta t}\right).
\end{cases} \tag{2.38}$$

Here, γ_{iv}, ρ_{iv}, $\gamma_{\mathrm{iv},\Delta t}$ and $\rho_{\mathrm{iv},\Delta t}$ are obtained by solving (2.33). When investment constraints are not applicable, these values are given in (2.34)-(2.37), respectively.

We conclude by noting that, without investment constraints, $(\mathcal{S}_{\mathrm{iv}}, \mathcal{E}_{\mathrm{iv}})$ also satisfies Lemma 2.8.

3 Analysis of robustness

3.1 True efficient points and efficient point errors

We take the perspective of an investor who believes that the investor model provides a sufficiently accurate representation of reality. The investor has fixed an investor efficient point $(\mathcal{S}_{\mathrm{iv}}, \mathcal{E}_{\mathrm{iv}})$ (Definition 2.9). Associated with this efficient point is an investor model-implied optimal control $u^*_{\mathrm{iv}} \in \left\{u^{p*}_{\mathrm{iv},\Delta t}, u^{c*}_{\mathrm{iv},\Delta t}, u^{p*}_{\mathrm{iv}}, u^{c*}_{\mathrm{iv}}\right\}$. This control is obtained by solving the respective MV optimization problem under the investor model with $\gamma \in \{\gamma_{\mathrm{iv}}, \gamma_{\mathrm{iv},\Delta t}\}$ or $\rho \in \{\rho_{\mathrm{iv}}, \rho_{\mathrm{iv},\Delta t}\}$ being solution to (2.33). When no investment constraints are applicable, these γ and ρ values are given by (2.34)-(2.37), and the closed-form of u^*_{iv} are given in Lemma 2.5 or Lemma 2.6.

The optimal control u^*_{iv} is then implemented under the true model (Definition 2.1) over the investment time horizon $[0,T]$ in a market where the risky asset evolves according to the dynamics (2.4) given by the true model $j = \mathrm{tr}$. The resulting mean and standard deviation of the true terminal wealth under the control u^*_{iv} are respectively denoted by $E^{x,t_n}_{u^{q*}_{\mathrm{iv},\Delta t}}\left[W_{\mathrm{tr},\Delta t}(T)\right]$ and $Stdev^{x,t_n}_{u^{q*}_{\mathrm{iv},\Delta t}}\left[W_{\mathrm{tr},\Delta t}(T)\right]$ in the case of discrete rebalancing, where $q \in \{p, c\}$ (pre-commitment or time-consistency). Similarly, for the case of continuous rebalancing, we have the notation $E^{w,t}_{u^{q*}_{\mathrm{iv}}}\left[W_{\mathrm{tr}}(T)\right]$ and $Stdev^{w,t}_{u^{q*}_{\mathrm{iv}}}\left[W_{\mathrm{tr}}(T)\right]$. These MV outcomes are collectively referred to as the "true efficient point", and are denoted by $\left(\mathcal{S}_{(\mathrm{iv}\to\mathrm{tr})}, \mathcal{E}_{(\mathrm{iv}\to\mathrm{tr})}\right)$. We formally define the true efficient point $\left(\mathcal{S}_{(\mathrm{iv}\to\mathrm{tr})}, \mathcal{E}_{(\mathrm{iv}\to\mathrm{tr})}\right)$ in Definition 3.1.

Definition 3.1. (True efficient point) Associated with each investor efficient point $(\mathcal{S}_{\text{iv}}, \mathcal{E}_{\text{iv}})$ defined in Definition 2.9 is the true efficient point $\left(\mathcal{S}_{(\text{iv}\to\text{tr})}, \mathcal{E}_{(\text{iv}\to\text{tr})}\right)$, defined by

$$\left(\mathcal{S}_{(\text{iv}\to\text{tr})}, \mathcal{E}_{(\text{iv}\to\text{tr})}\right) = \begin{cases} \left(Stdev^{w_0,t_0}_{u^{p*}_{\text{iv}}}\left[W_{\text{tr}}\left(T\right)\right], E^{w_0,t_0}_{u^{p*}_{\text{iv}}}\left[W_{\text{tr}}\left(T\right)\right]\right) & \text{a.w. } PCMV\left(t_0;\gamma_{\text{iv}}\right), \\ \left(Stdev^{w_0,t_0}_{u^{c*}_{\text{iv}}}\left[W_{\text{tr}}\left(T\right)\right], E^{w_0,t_0}_{u^{c*}_{\text{iv}}}\left[W_{\text{tr}}\left(T\right)\right]\right) & \text{a.w. } TCMV\left(t_0;\rho_{\text{iv}}\right), \\ \left(Stdev^{x_0,t_0}_{u^{p*}_{\text{iv},\Delta t}}\left[W_{\text{tr},\Delta t}\left(T\right)\right], E^{x_0,t_0}_{u^{p*}_{\text{iv},\Delta t}}\left[W_{\text{tr},\Delta t}\left(T\right)\right]\right) & \text{a.w. } PCMV_{\Delta t}\left(t_0;\gamma_{\text{iv},\Delta t}\right), \\ \left(Stdev^{x_0,t_0}_{u^{c*}_{\text{iv},\Delta t}}\left[W_{\text{tr},\Delta t}\left(T\right)\right], E^{x_0,t_0}_{u^{c*}_{\text{iv},\Delta t}}\left[W_{\text{tr},\Delta t}\left(T\right)\right]\right) & \text{a.w. } TCMV_{\Delta t}\left(t_0;\rho_{\text{iv},\Delta t}\right). \end{cases} \tag{3.1}$$

Here, γ_{iv}, ρ_{iv}, $\gamma_{\text{iv},\Delta t}$ and $\rho_{\text{iv},\Delta t}$ are obtained by solving (2.33). When investment constraints are not applicable, these values are given in (2.34)-(2.37), respectively. Note that "a.w." abbreviates "associated with" for purposes of clarity.

In a model misspecification scenario, the true efficient point $\left(\mathcal{S}_{(\text{iv}\to\text{tr})}, \mathcal{E}_{(\text{iv}\to\text{tr})}\right)$ does not necessarily coincide with the investor efficient point $(\mathcal{S}_{\text{iv}}, \mathcal{E}_{\text{iv}})$. In Definition 3.2, we formally define three different measures of the resulting error or difference between the above-mentioned points, each measure being associated with certain advantages and disadvantages.

Definition 3.2. (Efficient point error, relative efficient point error, error norm) The efficient point error is defined as $(\Delta\mathcal{S}, \Delta\mathcal{E}) = \left(\mathcal{S}_{(\text{iv}\to\text{tr})} - \mathcal{S}_{\text{iv}}, \mathcal{E}_{(\text{iv}\to\text{tr})} - \mathcal{E}_{\text{iv}}\right)$. The relative efficient point error is defined as

$$(\%\Delta\mathcal{S}, \%\Delta\mathcal{E}) \quad = \quad \left(\frac{\mathcal{S}_{(\text{iv}\to\text{tr})} - \mathcal{S}_{\text{iv}}}{\mathcal{S}_{\text{iv}}}, \frac{\mathcal{E}_{(\text{iv}\to\text{tr})} - \mathcal{E}_{\text{iv}}}{\mathcal{E}_{\text{iv}}}\right) \times 100. \tag{3.2}$$

The (relative) error norm is defined as the Euclidean norm of $(\%\Delta\mathcal{S}, \%\Delta\mathcal{E})$, namely

$$\mathcal{R}_{(\text{iv}\to\text{tr})} \quad = \quad \sqrt{(\%\Delta\mathcal{S})^2 + (\%\Delta\mathcal{E})^2}. \tag{3.3}$$

We observe that (3.2) enables the investor to distinguish the sign and contribution of the standard deviation and expected value components to the error. For example, all else being equal, the investor is likely to prefer an outcome of $(-\%\Delta\mathcal{S}, +\%\Delta\mathcal{E})$ to an outcome of $(+\%\Delta\mathcal{S}, -\%\Delta\mathcal{E})$. In contrast, (3.3) reduces the relative efficient point error to a single number, so that all else being equal, a smaller value of $\mathcal{R}_{(\text{iv}\to\text{tr})}$ would imply that the MV results for that particular choice of (iv $\to$ tr) are more robust to model misspecification errors. In this sense, the relative efficient point error (3.2) and error norm (3.3) are complementary measures of the extent to which the MV outcomes are robust to a model misspecification error.

While the investor does not have access to the true wealth dynamics, for analysis purposes, we assume the true model belongs to a certain class of dynamics (see Assumption 2.3). This assumption allows the computation of the mean and variance outcomes of the above-mentioned implementation of u^*_{iv}. When investment constraints are not applied, these outcomes can be computed in closed form (Subsection 3.2 below), enabling the derivation of some interesting results. When investment constraints are applicable, the computation of u^*_{iv} and its implementation under the true model must be achieved by a numerical method. More details for this case are given in Subsection 3.3.

3.2 No investment constraints

We introduce below ratios involving combinations of model parameters which play a key role in the subsequent analysis.

$$M = \frac{\mu_{\text{tr}} - r}{\mu_{\text{iv}} - r}, \qquad M_{\Delta t} = \frac{\alpha_{\text{tr}}}{\alpha_{\text{iv}}}, \qquad L = \frac{\sigma^2_{\text{tr}} + \lambda_{\text{tr}}\kappa_{\text{tr},2}}{\sigma^2_{\text{iv}} + \lambda_{\text{iv}}\kappa_{\text{iv},2}}, \qquad L_{\Delta t} = \frac{\psi^2_{\text{tr}}}{\psi^2_{\text{iv}}}. \tag{3.4}$$

Note that $\lim_{\Delta t\downarrow 0} M_{\Delta t} = M$ and $\lim_{\Delta t\downarrow 0} L_{\Delta t} = L$. The ratios (3.4) capture the degree to which the investor model ($j = \text{iv}$) and true model ($j = \text{tr}$) agree in terms of the expected excess returns and variance of returns of the risky asset[6]. Perfect correspondence between the investor model and true model obviously implies that the ratios (3.4) are equal to one, but the converse does not necessarily hold.

Starting with the case of discrete rebalancing, we have the following analytical result.

Theorem 3.3. *(Discrete rebalancing - MV of true terminal wealth, no investment constraints) Assume the discrete rebalancing of the portfolio, a given state $x = (s,b) = \left(S\left(t_n^-\right), B\left(t_n^-\right)\right)$ and wealth $w = s+b$ for some $t_n \in \mathcal{T}_m$, $n \in \{1,\ldots,m\}$, and that no investment constraints are applicable ($\mathcal{Z} = \mathbb{R}$). Implementing the investor model PCMV-optimal control $u^{p*}_{iv,\Delta t}$ given by (2.12) in the true model wealth dynamics (2.1) with $j = \text{tr}$, results in the mean and standard deviation of the true terminal wealth respectively given by*

$$E^{x,t_n}_{u^{p*}_{iv,\Delta t}}\left[W_{tr,\Delta t}(T)\right] = we^{r(T-t_n)} + \left[1 - \left(1 - \frac{M_{\Delta t}A_{iv,\Delta t}\cdot\Delta t}{1+A_{iv,\Delta t}\cdot\Delta t}\right)^{m-n+1}\right]\left[\frac{\gamma}{2} - we^{r(T-t_n)}\right], \tag{3.5}$$

$$\begin{aligned} Stdev^{x,t_n}_{u^{p*}_{iv,\Delta t}}\left[W_{tr,\Delta t}(T)\right] = &\left(1 - \frac{M_{\Delta t}A_{iv,\Delta t}\cdot\Delta t}{1+A_{iv,\Delta t}\cdot\Delta t}\right)^{m-n+1}\left[\left(1 + \frac{L_{\Delta t}A_{iv,\Delta t}\cdot\Delta t}{\left[1+(1-M_{\Delta t})A_{iv,\Delta t}\cdot\Delta t\right]^2}\right)^{m-n+1} - 1\right]^{1/2} \\ &\times\left[\frac{\gamma}{2} - we^{r(T-t_n)}\right]. \end{aligned} \tag{3.6}$$

Similarly, implementing the investor model TCMV-optimal control $u^{c*}_{iv,\Delta t}$ given by (2.12) in the true model wealth dynamics (2.1) with $j = \text{tr}$, gives

$$E^{x,t_n}_{u^{c*}_{iv,\Delta t}}\left[W_{tr,\Delta t}(T)\right] = we^{r(T-t_n)} + \frac{1}{2\rho}\cdot M_{\Delta t}A_{iv,\Delta t}(T-t_n), \tag{3.7}$$

$$Stdev^{x,t_n}_{u^{c*}_{iv,\Delta t}}\left[W_{tr,\Delta t}(T)\right] = \frac{1}{2\rho}\sqrt{L_{\Delta t}A_{iv,\Delta t}(T-t_n)}. \tag{3.8}$$

Proof. We summarize the proof of (3.5)-(3.6), since the results (3.7)-(3.8) are obtained in a similar way. Using the auxiliary functions and recursive relations $g^p_{\Delta t}(x,t_n) := E^{x,t_n}_{u^{p*}_{\text{iv},\Delta t}}\left[W_{\text{tr},\Delta t}(T)\right] = E^{x,t_n}_{u^{p*}_{\text{iv},\Delta t}}\left[g^p_{\Delta t}\left(X\left(t^-_{n+1}\right), t_{n+1}\right)\right]$ and $h^p_{\Delta t}(x,t_n) := E^{x,t_n}_{u^{p*}_{\text{iv},\Delta t}}\left[W^2_{\text{tr},\Delta t}(T)\right] = E^{x,t_n}_{u^{p*}_{\text{iv},\Delta t}}\left[h^p_{\Delta t}\left(X\left(t^-_{n+1}\right), t_{n+1}\right)\right]$, where $X\left(t^-_{n+1}\right) = \left(S_{\text{tr}}\left(t^-_{n+1}\right), B\left(t^-_{n+1}\right)\right)$ and S_{tr} has dynamics (2.4) with $j = \text{tr}$, we solve problem $PCMV_{\Delta t}(t_n;\gamma)$ recursively backwards from $n = m$ using terminal conditions $g^p_{\Delta t}(x,t_{m+1}) = (s+b) = w$ and $h^p_{\Delta t}(x,t_{m+1}) = w^2$. Using backward induction on n, it follows that the function $g^p_{\Delta t}$ satisfies (3.5), while the function $h^p_{\Delta t}$ is given by

$$\begin{aligned} h^p_{\Delta t}(x,t_n) = &\left[\left(1 - \frac{M_{\Delta t}A_{\text{iv},\Delta t}\cdot\Delta t}{1+A_{\text{iv},\Delta t}\cdot\Delta t}\right)^2 + \frac{L_{\Delta t}A_{\text{iv},\Delta t}\cdot\Delta t}{\left(1+A_{\text{iv},\Delta t}\cdot\Delta t\right)^2}\right]^{(m-n+1)}\left[\frac{\gamma}{2} - we^{r(T-t_n)}\right]^2 \\ &-2\left(\frac{\gamma}{2}\right)\left(1 - \frac{M_{\Delta t}A_{\text{iv},\Delta t}\cdot\Delta t}{1+A_{\text{iv},\Delta t}\cdot\Delta t}\right)^{(m-n+1)}\left[\frac{\gamma}{2} - we^{r(T-t_n)}\right] + \left(\frac{\gamma}{2}\right)^2. \end{aligned} \tag{3.9}$$

Taking the square root of $h^p_{\Delta t}(x,t_n) - \left[g^p_{\Delta t}(x,t_n)\right]^2$ gives (3.6). □

In the case of continuous rebalancing, the corresponding analytical results are given below.

Theorem 3.4. *(Continuous rebalancing - MV of true terminal wealth, no investment constraints) Assume the continuous rebalancing of the portfolio, with given wealth w at time $t \in [0,T]$, and that no investment constraints are applicable ($\mathbb{U} = \mathbb{R}$). Implementing the investor model PCMV-optimal control u^{p*}_{iv} given by (2.23) in the true model wealth dynamics (2.18) with $j = \text{tr}$, results in the mean*

[6]This follows since we can write, informally, $\mathbb{E}\left[dS_j(t)/S_j\left(t^-\right)\right] = \mu_j dt$ and $Var\left[dS_j(t)/S_j\left(t^-\right)\right] = \left(\sigma_j^2 + \lambda_j\kappa_{j,2}\right)dt$.

and standard deviation of the true terminal wealth respectively given by

$$E^{w,t}_{u^{p*}_{iv}}\left[W_{tr}\left(T\right)\right] \;=\; we^{r(T-t)}+\left[1-e^{-MA_{iv}(T-t)}\right]\left[\frac{\gamma}{2}-we^{r(T-t)}\right], \tag{3.10}$$

$$Stdev^{w,t}_{u^{p*}_{iv}}\left[W_{tr}\left(T\right)\right] \;=\; e^{-MA_{iv}(T-t)}\left[e^{LA_{iv}(T-t)}-1\right]^{\frac{1}{2}}\cdot\left[\frac{\gamma}{2}-we^{r(T-t)}\right]. \tag{3.11}$$

Implementing the investor model TCMV-optimal control u^{c*}_{iv} given by (2.26) in the true model wealth dynamics (2.18) with $j=\mathrm{tr}$, gives

$$E^{w,t}_{u^{c*}_{iv}}\left[W_{tr}\left(T\right)\right] \;=\; we^{r(T-t)}+\frac{1}{2\rho}\cdot MA_{iv}\left(T-t\right), \tag{3.12}$$

$$Stdev^{w,t}_{u^{c*}_{iv}}\left[W_{tr}\left(T\right)\right] \;=\; \frac{1}{2\rho}\sqrt{LA_{iv}\left(T-t\right)}. \tag{3.13}$$

Proof. We summarize the proof of (3.10)-(3.11), since the proof of (3.12)-(3.13) proceeds similarly. Implementing control $u^{p*}_{\mathrm{iv}}\left(t\right)$ as per (2.23) in the true wealth dynamics ((2.18)) for the case of continuous rebalancing, we establish that the auxiliary function $g^{p}\left(\tau\right)=g^{p}\left(\tau;w,t\right):=E^{w,t}_{u^{p*}_{\mathrm{iv}}}\left[W_{\mathrm{tr}}\left(\tau\right)\right],\tau\in\left[t,T\right]$ satisfies the following ODE,

$$\begin{aligned}\frac{dg^{p}\left(\tau\right)}{d\tau} &= \left(r-MA_{\mathrm{iv}}\right)g^{p}\left(\tau\right)+MA_{\mathrm{iv}}\frac{\gamma}{2}e^{-r(T-\tau)}, \qquad \tau\in\left(t,T\right], \\ g^{p}\left(t\right) &= w,\end{aligned} \tag{3.14}$$

which is solved to obtain $g^{p}\left(T\right)=E^{w,t}_{u^{p*}_{\mathrm{iv}}}\left[W_{\mathrm{tr}}\left(T\right)\right]$ given by (3.10). Using Ito's lemma to obtain the dynamics of the squared true wealth W^{2}_{tr} using control $u^{p*}_{\mathrm{iv}}\left(t\right)$, the auxiliary function $h^{p}\left(\tau\right)=h^{p}\left(\tau;w,t\right)=E^{w,t}_{u^{p*}_{\mathrm{iv}}}\left[W^{2}_{\mathrm{tr}}\left(\tau\right)\right],\tau\in\left[t,T\right]$ satisfies the ODE

$$\begin{aligned}\frac{dh^{p}\left(\tau\right)}{d\tau} &= 2\left(M-L\right)A_{\mathrm{iv}}\left(\frac{\gamma}{2}\right)\left[we^{(r-MA_{\mathrm{iv}})(\tau-t)-r(T-\tau)}+\frac{\gamma}{2}e^{-2r(T-\tau)}\left(1-e^{-MA_{\mathrm{iv}}(\tau-t)}\right)\right] \\ &\quad +\left[2r+\left(L-2M\right)A_{\mathrm{iv}}\right]h^{p}\left(\tau\right)+LA_{\mathrm{iv}}\left(\frac{\gamma}{2}\right)^{2}e^{-2r(T-\tau)}, \qquad \tau\in\left(t,T\right], \\ h^{p}\left(t\right) &= w^{2},\end{aligned} \tag{3.15}$$

which is solved to obtain $h^{p}\left(T\right)=E^{w,t}_{u^{p*}_{\mathrm{iv}}}\left[W^{2}_{\mathrm{tr}}\left(T\right)\right]$. Together with (3.10), this gives (3.11). □

Although the discrete and continuous rebalancing formulation is structurally different, Lemma 3.5 establishes the expected convergence result in the limit as $\Delta t\downarrow 0$ in (2.2).

Lemma 3.5. *(Convergence, no investment constraints) Fix a rebalancing time $t_{n}\in\mathcal{T}_{m}$ and state $x=\left(s,b\right)=\left(S\left(t^{-}_{n}\right),B\left(t^{-}_{n}\right)\right)$. Set time $t=t_{n}$ and wealth $w=s+b$. Taking the limit as $\Delta t\downarrow 0$ in the discrete rebalancing results (3.5)-(3.8), we have*

$$\lim_{\Delta t\downarrow 0}E^{x,t_{n}}_{u^{p*}_{iv,\Delta t}}\left[W_{tr,\Delta t}\left(T\right)\right]=E^{w,t}_{u^{p*}_{iv}}\left[W_{tr}\left(T\right)\right],\quad \lim_{\Delta t\downarrow 0}Stdev^{x,t_{n}}_{u^{p*}_{iv,\Delta t}}\left[W_{tr,\Delta t}\left(T\right)\right]=Stdev^{w,t}_{u^{p*}_{iv}}\left[W_{tr}\left(T\right)\right], \tag{3.16}$$

$$\lim_{\Delta t\downarrow 0}E^{x,t_{n}}_{u^{c*}_{iv,\Delta t}}\left[W_{tr,\Delta t}\left(T\right)\right]=E^{w,t}_{u^{c*}_{iv}}\left[W_{tr}\left(T\right)\right],\quad \lim_{\Delta t\downarrow 0}Stdev^{x,t_{n}}_{u^{c*}_{iv,\Delta t}}\left[W_{tr,\Delta t}\left(T\right)\right]=Stdev^{w,t}_{u^{c*}_{iv}}\left[W_{tr}\left(T\right)\right]. \tag{3.17}$$

Proof. This follows from the limits (2.29), as well as

$$\lim_{\Delta t\downarrow 0}\left(1-\frac{M_{\Delta t}A_{\mathrm{iv},\Delta t}\cdot\Delta t}{1+A_{\mathrm{iv},\Delta t}\cdot\Delta t}\right)^{1/\Delta t}=e^{-MA_{\mathrm{iv}}},\ \lim_{\Delta t\downarrow 0}\left(1+\frac{L_{\Delta t}A_{\mathrm{iv},\Delta t}\cdot\Delta t}{\left[1+\left(1-M_{\Delta t}\right)A_{\mathrm{iv},\Delta t}\cdot\Delta t\right]^{2}}\right)^{1/\Delta t}=e^{LA_{\mathrm{iv}}}. \tag{3.18}$$

□

3.2.1 Quantifying robustness

As a first step toward quantifying the MV robustness with respect to an efficient point error, we show that, when no investment constraints are applicable, the efficient point error can be expressed elegantly in terms of $\mathcal{S}_{\mathrm{iv}}$ using the notion of error multipliers.

Lemma 3.6. *(Efficient point error in terms of error multipliers, no investment constraints) Assume that no investment constraints are applicable. We have*

$$\mathcal{E}_{(iv\to tr)} - \mathcal{E}_{iv} = \Theta_{(iv\to tr)} \cdot \Gamma_{iv} \cdot \mathcal{S}_{iv}, \tag{3.19}$$

$$\mathcal{S}_{(iv\to tr)} - \mathcal{S}_{iv} = \Psi_{(iv\to tr)} \cdot \mathcal{S}_{iv}. \tag{3.20}$$

Here, the appropriate slope Γ_{iv} of the investor MV frontier defined in (2.32). *The error multiplier $\Theta_{(iv\to tr)}$ associated with the expected value error (3.19) is given by*

$$\Theta_{(iv\to tr)} = \begin{cases} \Theta^p_{(iv\to tr)} = \left[\left(1-e^{-MA_{iv}T}\right)/\left(1-e^{-A_{iv}T}\right)\right]-1 & a.w.\ PCMV\left(t_0;\gamma_{iv}\right), \\ \Theta^c_{(iv\to tr)} = M-1, & a.w.\ TCMV\left(t_0;\rho_{iv}\right), \\ \Theta^p_{(iv\to tr),\Delta t} = \left[\frac{1-\left(1+(1-M_{\Delta t})A_{iv,\Delta t}\cdot\Delta t\right)^m\left(1+A_{iv,\Delta t}\cdot\Delta t\right)^{-m}}{1-\left(1+A_{iv,\Delta t}\cdot\Delta t\right)^{-m}}\right]-1, & a.w.\ PCMV_{\Delta t}\left(t_0;\gamma_{iv,\Delta t}\right), \\ \Theta^c_{(iv\to tr),\Delta t} = M_{\Delta t}-1. & a.w.\ TCMV_{\Delta t}\left(t_0;\rho_{iv,\Delta t}\right), \end{cases} \tag{3.21}$$

The error multiplier $\Psi_{(iv\to tr)}$ associated with the standard deviation error (3.20) is given by

$$\Psi_{(iv\to tr)} = \begin{cases} \Psi^p_{(iv\to tr)} = e^{(1-M)A_{iv}T}\cdot\left[\left(e^{LA_{iv}T}-1\right)/\left(e^{A_{iv}T}-1\right)\right]^{\frac{1}{2}}-1, & a.w.\ PCMV\left(t_0;\gamma_{iv}\right), \\ \Psi^c_{(iv\to tr)} = \sqrt{L}-1, & a.w.\ TCMV\left(t_0;\rho_{iv}\right), \\ \Psi^p_{(iv\to tr),\Delta t}, & a.w.\ PCMV_{\Delta t}\left(t_0;\gamma_{iv,\Delta t}\right), \\ \Psi^c_{(iv\to tr),\Delta t} = \sqrt{L_{\Delta t}}-1, & a.w.\ TCMV_{\Delta t}\left(t_0;\rho_{iv,\Delta t}\right), \end{cases} \tag{3.22}$$

$$\text{where}\quad \Psi^p_{(iv\to tr),\Delta t} = \frac{\left[1+(1-M_{\Delta t})A_{iv,\Delta t}\cdot\Delta t\right]^m}{\left[(1+A_{iv,\Delta t}\cdot\Delta t)^m-1\right]^{1/2}}\cdot\left[\left(1+\frac{L_{\Delta t}A_{iv,\Delta t}\cdot\Delta t}{\left[1+(1-M_{\Delta t})A_{iv,\Delta t}\cdot\Delta t\right]^2}\right)^m-1\right]^{\frac{1}{2}}-1.$$

In the above, $A_{iv,\Delta t}$ and A_{iv} are respectively defined in (2.5) *and* (2.19).

Proof. The results (3.19)-(3.22) follow from combining and rearranging the results from Theorem 3.4, Theorem 3.3, and Lemma 2.8. □

The analytical results of Lemma 3.6 allow us to draw several interesting conclusions about MV robustness to model misspecification errors. Specifically, consider a fixed T, and, for discrete rebalancing, a fixed Δt. Examination of (3.19)-(3.20) indicates that the efficient point errors depend on (i) the investor target $\mathcal{S}_{\mathrm{iv}}$, (ii) the ratios M, $M_{\Delta t}$, L, and $L_{\Delta t}$, defined in (3.4), as well as (iii) $A_{\mathrm{iv},\Delta t}$ and A_{iv}. Note that, once selected, the target $\mathcal{S}_{\mathrm{iv}}$ remains fixed. For a chosen investor model, $A_{\mathrm{iv},\Delta t}$ and A_{iv} are also fixed, since they depend only on the parameters of the investor model. The ratios M, $M_{\Delta t}$, L, and $L_{\Delta t}$, defined in (3.4), depend on certain combinations of parameters of both the investor and true models, not individual parameter values. These ratios play a key role in quantifying efficient point errors, implying that individual parameter values only play a secondary role. Specifically, the closer the ratios M, $M_{\Delta t}$, L, and $L_{\Delta t}$ are to one, the smaller the model misspecification errors, hence the more robust MV outcomes, regardless of differences in fundamental types or individual parameter values between the investor and true models.

Finally, the impact of a model misspecification error on the tradeoff between mean and variance of terminal wealth is worth highlighting. In particular, the slope $\Gamma_{\mathrm{iv}} = \left(\mathcal{E}_{\mathrm{iv}} - w_0 e^{rT}\right)/\mathcal{S}_{\mathrm{iv}}$ (see Lemma 2.8) can be interpreted as the *price of risk* (Zhou and Li (2000)) as per the investor model. All else being equal, the investor would prefer a larger slope, since for a fixed level of risk as measured by $\mathcal{S}_{\mathrm{iv}}$, a larger slope would imply a larger value of $\mathcal{E}_{\mathrm{iv}}$. However, the true efficient point $\left(\mathcal{S}_{(\mathrm{iv}\to\mathrm{tr})}, \mathcal{E}_{(\mathrm{iv}\to\mathrm{tr})}\right)$ is associated with a different (true) price of risk, $\Gamma_{(\mathrm{iv}\to\mathrm{tr})}$, which is quantified by the following lemma.

Lemma 3.7. *(True price of risk, no investment constraints). If no investment constraints are applicable, the true price of risk $\Gamma_{(iv\to tr)}$ is related to the price of risk according to the investor model, Γ_{iv}, as follows:*

$$\Gamma_{(iv\to tr)} \quad := \quad \frac{\mathcal{E}_{(iv\to tr)} - w_0 e^{rT}}{\mathcal{S}_{(iv\to tr)}} = \left[\frac{1+\Theta_{(iv\to tr)}}{1+\Psi_{(iv\to tr)}}\right] \cdot \Gamma_{iv}, \tag{3.23}$$

with the values of $\Theta_{(iv\to tr)}$, $\Psi_{(iv\to tr)}$ and $\Gamma_{(iv\to tr)}$ given by (3.21), (3.22) and (2.32) respectively, all consistent with the chosen investment objective and rebalancing frequency. In particular, $\Gamma_{(iv\to tr)}$ is given by

$$\Gamma_{(iv\to tr)} \quad = \quad \begin{cases} \Gamma^{p}_{(iv\to tr)} = \left[e^{MA_{iv}T}-1\right]\left[e^{LA_{iv}T}-1\right]^{-1/2}, & \text{a.w. } PCMV\left(t_0;\gamma_{iv}\right), \\ \Gamma^{c}_{(iv\to tr)} = \left[MA_{iv}T\right]\left[LA_{iv}T\right]^{-1/2} = \sqrt{A_{tr}T}, & \text{a.w. } TCMV\left(t_0;\rho_{iv}\right), \\ \Gamma^{p}_{(iv\to tr),\Delta t}, & \text{a.w. } PCMV_{\Delta t}\left(t_0;\gamma_{iv,\Delta t}\right), \\ \Gamma^{c}_{(iv\to tr),\Delta t} = \left[M_{\Delta t}A_{iv,\Delta t}T\right]\left[L_{\Delta t}A_{iv,\Delta t}T\right]^{-1/2} = \sqrt{A_{tr,\Delta t}T}, & \text{a.w. } TCMV_{\Delta t}\left(t_0;\rho_{iv,\Delta t}\right), \end{cases} \tag{3.24}$$

$$\textit{where} \quad \Gamma^{p}_{(iv\to tr),\Delta t} = \left[\left(1-\frac{M_{\Delta t}A_{iv,\Delta t}\cdot\Delta t}{1+A_{iv,\Delta t}\cdot\Delta t}\right)^{-m}-1\right]\left[\left(1+\frac{L_{\Delta t}A_{iv,\Delta t}\cdot\Delta t}{\left[1+\left(1-M_{\Delta t}\right)A_{iv,\Delta t}\cdot\Delta t\right]^2}\right)^{m}-1\right]^{-1/2}.$$

Proof. The results follow from Lemma 2.8, Definition 3.1 and Lemma 3.6. □

Considering the definition (3.23) of the true price of risk $\Gamma_{(\mathrm{iv}\to\mathrm{tr})}$, the practical relevance of Lemma 3.7 follows from the observation that $\Gamma_{(\mathrm{iv}\to\mathrm{tr})}$ can be viewed as an indicator of the MV-efficiency of the investment strategy in the presence of model misspecification. In other words, the true price of risk can be used as a measure of robustness that is complementary to the quantities introduced in Definition 3.2, since $\Gamma_{(\mathrm{iv}\to\mathrm{tr})}$ gives the robustness to model misspecification of the MV-tradeoff of the investor's terminal wealth.

In addition, Lemma 3.7 has some interesting theoretical consequences, which we illustrate using the case of continuous rebalancing. According to the investor model, Lemma 2.8 implies that $\Gamma^{p}_{\mathrm{iv}}/\Gamma^{c}_{\mathrm{iv}} > 1$; in other words, all else being equal, the PCMV strategy should result in a better trade-off between mean and variance of terminal wealth than the TCMV strategy as measured by the corresponding price of risk. However, when a model misspecification error occurs, Lemma 3.7 shows that the ratio $\Gamma^{p}_{(\mathrm{iv}\to\mathrm{tr})}/\Gamma^{c}_{(\mathrm{iv}\to\mathrm{tr})}$ is given by

$$\frac{\Gamma^{p}_{(\mathrm{iv}\to\mathrm{tr})}}{\Gamma^{c}_{(\mathrm{iv}\to\mathrm{tr})}} = \underbrace{\left[\frac{LA_{\mathrm{iv}}T}{e^{LA_{\mathrm{iv}}T}-1}\right]^{\frac{1}{2}}}_{<1} \cdot \underbrace{\left[\frac{e^{MA_{\mathrm{iv}}T}-1}{MA_{\mathrm{iv}}T}\right]}_{>1}. \tag{3.25}$$

Given fixed values of A_{iv} and T, the first component of (3.25) depends on L while the second component depends on M. As such, it is possible that a situation might arise where $\Gamma^{p}_{(\mathrm{iv}\to\mathrm{tr})}/\Gamma^{c}_{(\mathrm{iv}\to\mathrm{tr})} < 1$; in other words, it is possible that the TCMV strategy might outperform the PCMV strategy on the basis of the corresponding true price of risk[7]. However, as illustrated in Subsection 4.2, this particular scenario does not arise in the numerical results presented in Section 4.

3.2.2 A robustness comparison between PCMV and TCMV

We further explore and compare the robustness of PCMV and TCMV with respect to model misspecification when no investment constraints are applicable. From Lemma 3.6, assuming fixed values

[7]Interestingly, a similar observation is made in Cong and Oosterlee (2017), where an entirely different formulation of the robustness problem is used.

of A_{iv} and T, we observe that the expected value error $(\mathcal{E}_{(\text{iv}\to\text{tr})} - \mathcal{E}_{\text{iv}})$ depends only on M (PCMV and TCMV, continuous rebalancing) or $M_{\Delta t}$ (PCMV and TCMV, discrete rebalancing). We have the following theorem.

Theorem 3.8. *(Comparison of expected value error multipliers, no investment constraints) Assume that no investment constraints are applicable, and that $\mu_j > r$ and $\sigma_j > 0$ for $j \in \{iv, tr\}$. In the case of continuous rebalancing, we have*

$$\left|\Theta^p_{(iv\to tr)}\right| \leq \left|\Theta^c_{(iv\to tr)}\right|, \qquad \forall M > 0, \tag{3.26}$$

with strict inequality except when $M = 1$. In the case of discrete rebalancing, for any $\Delta t > 0$ there exists a unique value $M_{\Theta\Delta t} > 1 + \frac{2}{A_{iv,\Delta t}\cdot\Delta t}$ such that

$$\left|\Theta^p_{(iv\to tr),\Delta t}\right| \leq \left|\Theta^c_{(iv\to tr),\Delta t}\right|, \qquad \forall M_{\Delta t} \in (0, M_{\Theta\Delta t}], \Delta t > 0, \tag{3.27}$$

$$\left|\Theta^p_{(iv\to tr),\Delta t}\right| > \left|\Theta^c_{(iv\to tr),\Delta t}\right|, \qquad \forall M_{\Delta t} > M_{\Theta\Delta t}, \Delta t > 0, \tag{3.28}$$

with the inequality (3.27) strict except when $M_{\Delta t} = 1$ or $M_{\Delta t} = M_{\Theta\Delta t}$. Furthermore, comparing continuous and discrete rebalancing, we also have

$$\left|\Theta^c_{(iv\to tr)}\right| \quad \leq \quad \left|\Theta^c_{(iv\to tr),\Delta t}\right|, \qquad \forall M > 0, \Delta t > 0, \tag{3.29}$$

with strict inequality except when $M = 1$.

Proof. The results follow from the error multiplier definitions in Lemma 3.6. The exact value of $M_{\Theta\Delta t}$ in (3.27)-(3.28) can be determined numerically as the unique root of the function $M_{\Delta t} \to f_{\Theta,\Delta t}(M_{\Delta t}) := \left|\Theta^p_{(\text{iv}\to\text{tr}),\Delta t}\right| - \left|\Theta^c_{(\text{iv}\to\text{tr}),\Delta t}\right|$ in the domain $M_{\Delta t} \in \left(1 + \frac{2}{A_{\text{iv},\Delta t}\cdot\Delta t}, \infty\right)$. Note that the existence and uniqueness of the root $M_{\Theta\Delta t}$ can be established through a detailed analysis of the properties of the function $f_{\Theta,\Delta t}(M_{\Delta t})$ for various cases and ranges of $M_{\Delta t}$. This proof is long and tedious, we will not try the reader's patience by including the details. ☐

Theorem 3.8 shows that, when no constraints are applicable, the expected value error multipliers for PCMV is expected to be smaller than for TCMV. This is always the case for continuous rebalancing, but since $M_{\Theta\Delta t} \gg 1$ in typical applications (for example, the results of Section 4), this is also expected to be true for discrete rebalancing as a result of (3.27). Furthermore, (3.29) shows that the magnitude of $\Theta^c_{(\text{iv}\to\text{tr}),\Delta t}$ for discrete rebalancing is always bounded below by the magnitude of $\Theta^c_{(\text{iv}\to\text{tr})}$ for continuous rebalancing. However, without any further reference to the particular underlying process parameters, such a general statement is not possible in the case of the corresponding PCMV error multipliers.

Lemma 3.6 also indicates that with fixed investor model and investment parameters (i.e. fixed values of A_{iv} and T), the standard deviation error $(\mathcal{S}_{(\text{iv}\to\text{tr})} - \mathcal{S}_{\text{iv}})$ depends on (i) both M and L (PCMV, continuous rebalancing); (ii) only L (TCMV, continuous rebalancing); (iii) both $M_{\Delta t}$ and $L_{\Delta t}$ (PCMV, discrete rebalancing); and (iv) only $L_{\Delta t}$ (TCMV, discrete rebalancing). As a result, the following theorem illustrates that comparing the standard deviation error multipliers is not as simple as comparing expected value error multipliers.

Theorem 3.9. *(Comparison of standard deviation error multipliers $\Psi^p_{(iv\to tr)}$ and $\Psi^c_{(iv\to tr)}$ no investment constraints) Assume that no investment constraints are applicable, and that $\mu_j > r$ and $\sigma_j > 0$ for $j \in \{iv, tr\}$. Define M_Ψ as the following quantity,*

$$M_\Psi = 1 - \frac{1}{2A_{iv}T}\log\left[\frac{\left(e^{A_{iv}T} - 1\right)}{A_{iv}T}\right]. \tag{3.30}$$

For any fixed value of $M > M_\Psi$, define $L_\Psi(M) > 0$ as the unique root in $(0,\infty)$ of the function $L \to g_\Psi(L;M)$, where

$$g_\Psi(L;M) \;=\; e^{2(1-M)A_{iv}T}\left(e^{LA_{iv}T}-1\right) - L\left(e^{A_{iv}T}-1\right), \qquad L>0, M>M_\Psi. \tag{3.31}$$

Then depending on the values of the ratios M and L, we have the following relationship between multipliers $\Psi^p_{(iv\to tr)}$ and $\Psi^c_{(iv\to tr)}$:

$$\begin{aligned}
\Psi^p_{(iv\to tr)} &> \Psi^c_{(iv\to tr)}, && \forall M \le M_\Psi \text{ and } L>0,\\
\Psi^p_{(iv\to tr)} &< \Psi^c_{(iv\to tr)}, && \forall M > M_\Psi \text{ and } 0<L<L_\Psi(M),\\
\Psi^p_{(iv\to tr)} &= \Psi^c_{(iv\to tr)}, && \forall M > M_\Psi \text{ and } L=L_\Psi(M),\\
\Psi^p_{(iv\to tr)} &> \Psi^c_{(iv\to tr)}, && \forall M > M_\Psi \text{ and } L>L_\Psi(M).
\end{aligned} \tag{3.32}$$

Proof. It is straightforward to show that $M_\Psi \in \left(\frac{1}{2},\frac{3}{4}\right)$, since $A_{\text{iv}}T>0$. Fix $M>0$, and consider the auxiliary function $L\to f_\Psi(L;M)$ defined by

$$f_\Psi(L;M) \;=\; e^{2(1-M)A_{\text{iv}}T}\cdot\frac{\left(e^{LA_{\text{iv}}T}-1\right)}{\left(e^{A_{\text{iv}}T}-1\right)} - L, \qquad L>0, M>0. \tag{3.33}$$

Observe that $L\to f_\Psi(L;M)$ is strictly convex, with $\lim_{L\downarrow 0} f_\Psi(L;M)=0$. As a result, $L\to f_\Psi(L;M)$ attains a global minimum in $[0,\infty)$ at L^*_Ψ, where

$$L^*_\Psi \;=\; \begin{cases} 0 & \text{if } M\le M_\Psi,\\ \frac{1}{A_{\text{iv}}T}\log\left[\frac{\left(e^{A_{\text{iv}}T}-1\right)}{A_{\text{iv}}T}\right]-2(1-M) & \text{if } M>M_\Psi. \end{cases} \tag{3.34}$$

Comparing f_Ψ with the function g_Ψ defined in (3.31), we see that g_Ψ has a unique root $L_\Psi(M)>0$ in the case where $M>M_\Psi$. Furthermore, $M\in(M_\Psi,1)$ implies $0<L_\Psi(M)<1$, while $M\ge 1$ implies that $L_\Psi(M)\ge 1$. The result (3.32) then follows from the properties of the function $f_\Psi(L;M)$. □

Note that the results of Theorem 3.9 can be extended to compare the magnitude of the corresponding multipliers, namely $\left|\Psi^p_{(\text{iv}\to\text{tr})}\right|$ and $\left|\Psi^c_{(\text{iv}\to\text{tr})}\right|$. In addition, similar results as in Theorem 3.9 can also be derived for the other standard deviation error multiplier pairs. Unfortunately, the resulting set of comparison results relies heavily on particular choices of the underlying investor model and investment parameters, which makes general statements of comparable simplicity to those of Theorem 3.8 impossible. However, in the numerical results presented in Section 4 below, we see that when a fairly large set of reasonably calibrated inflation-adjusted model parameters are compared, it is typical to observe values of $M\simeq 1$ but a much larger range is observed for the values of L.

As a result, the following theorem presents a comparison of the standard deviation error multipliers for the important special case where $M\equiv 1$, since this turns out to be very useful for explaining and interpreting the numerical results in Section 4.

Theorem 3.10. *(Comparison of standard deviation error multipliers when $M\equiv 1$, no investment constraints) Assume that no investment constraints are applicable, and that $\mu_j>r$ and $\sigma_j>0$ for $j\in\{iv,tr\}$. In the special case where $M=M_{\Delta t}=1$, we have the following relationships between standard deviation error multipliers:*

$$\left|\Psi^c_{(iv\to tr)}\right| \;\le\; \left|\Psi^p_{(iv\to tr)}\right|, \qquad \forall L>0, \quad \Delta t>0, M=1, \tag{3.35}$$

$$\left|\Psi^c_{(iv\to tr),\Delta t}\right| \;\le\; \left|\Psi^p_{(iv\to tr),\Delta t}\right|, \qquad \forall L_{\Delta t}>0, \Delta t>0, M_{\Delta t}=1, \tag{3.36}$$

with strict inequality in both cases except when $L=1$ or $L_{\Delta t}=1$, respectively. Furthermore, comparing *discrete and continuous rebalancing, we also have*

$$\left|\Psi^{c}_{(iv\to tr)}\right| \quad \leq \quad \left|\Psi^{c}_{(iv\to tr),\Delta t}\right|, \qquad \forall L>0, \Delta t>0, M=1. \tag{3.37}$$

Proof. The proof proceeds along similar lines as the proof of Theorem 3.9, except that the analysis is limited to the case where $M = M_{\Delta t} = 1$. ☐

Theorem 3.10 is key to providing an explanation of the numerical results presented in Section 4, since in this special case, the relative error norm is given by (see (3.3))

$$\mathcal{R}_{(\text{iv}\to\text{tr})} \quad = \sqrt{\left(\%\Delta\mathcal{S}\right)^2} = \quad \left|\Psi_{(\text{iv}\to\text{tr})}\right|, \qquad \text{if } M = M_{\Delta t} = 1. \tag{3.38}$$

Recall from the definition (3.4) that $M = M_{\Delta t} = 1$ when $\mu_{\text{iv}} = \mu_{\text{tr}}$, in other words the drift coefficients of the investor and true models agree. Theorem 3.10 shows that in the special case where $M = M_{\Delta t} = 1$ and no investment constraints are applicable, PCMV is expected to be less robust than TCMV to a model misspecification error, in the sense that the corresponding error norm $\mathcal{R}_{(\text{iv}\to\text{tr})}$ for PCMV is larger than that of TCMV, regardless of rebalancing frequency (see (3.35)-(3.36)).

However, despite a larger error norm $\mathcal{R}_{(\text{iv}\to\text{tr})}$, the PCMV error is not necessarily worse from the perspective of the investor. For example, the results of Lemma 3.6, Theorem 3.9 and Theorem 3.10 can be combined to show that $\Psi^{p}_{(\text{iv}\to\text{tr})} < \Psi^{c}_{(\text{iv}\to\text{tr})} < 0$ in the particular case where $M = 1$ and $0 < L < 1$. This implies that in this case, $\%\Delta\mathcal{S}$ for PCMV is a larger negative value than the corresponding value for TCMV (i.e. the investor's risk is much lower than anticipated for PCMV compared to TCMV). This illustrates the importance of considering the efficient point error $(\%\Delta\mathcal{S}, \%\Delta\mathcal{E})$, which gives the signs of the error components, in conjunction with the error norm $\mathcal{R}_{(\text{iv}\to\text{tr})}$, as noted in the discussion following Definition 3.2.

Furthermore, (3.37) indicates that for TCMV in this special case, discrete rebalancing results in a larger error compared to the case of continuous rebalancing. However, a general statement of comparable simplicity to (3.37) is not available in the case of the corresponding PCMV error, since in the case of PCMV discrete rebalancing may in fact *reduce* the error depending on the particular set of parameters under consideration - see for example the results in Section 4. Therefore, in the case of PCMV, Lemma 3.6 is used to calculate the error norm (3.3) directly for a chosen set of model and investment parameters.

3.3 Investment constraints

The analytical results presented up to this point assumed that no investment constraints are applicable. In order to assess the effect of realistic investment constraints on the robustness to model misspecification errors, we consider both a solvency constraint and a maximum leverage constraint in the numerical results presented in Section 4. These constraints will only be applied in the context of discrete rebalancing.

Fix an arbitrary rebalancing time $t_n \in \mathcal{T}_m$, and assume that the system is in state $x = (s,b) = \left(S\left(t_n^-\right), B\left(t_n^-\right)\right) \in \Omega^\infty$, where $\Omega^\infty = [0,\infty) \times (-\infty,\infty)$ denotes the spatial domain. We define insolvency or bankruptcy as the event that $W_{j,\Delta t}\left(s,b\right) \leq 0$, $j \in \{\text{iv}, \text{tr}\}$, and define the associated bankruptcy region as $\mathcal{B} = \{(s,b) \in \Omega^\infty : W_{j,\Delta t}\left(s,b\right) \leq 0\}$. The solvency constraint is defined as the requirement that if $(s,b) \in \mathcal{B}$, the investment in the risky asset has to be liquidated, the total wealth is to be placed in the risk-free asset, and all subsequent trading activities much cease. The maximum leverage constraint specifies that after rebalancing at time t_n according to (2.6), the leverage ratio defined as $S_j\left(t_n\right) / \left[S_j\left(t_n\right) + B\left(t_n\right)\right], j \in \{\text{iv}, \text{tr}\}$ should not exceed some given maximum leverage value q_{max} typically in the range $[1.0, 2.0]$, for $n = 1, \ldots, m$.

Since no analytical solutions are known for cases where these investment constraints are applied simultaneously, we solve the problems numerically. For details regarding the numerical algorithms for solving the problems to obtain a target standard deviation $\mathcal{S}_{\text{iv}}$ and the associated investor efficient point (2.38), as well as more detail on the application of the solvency and leverage constraints, we refer the reader to Dang and Forsyth (2014); Van Staden et al. (2018).

To calculate the efficient point error as per Definition 3.2, we first solve the relevant problem numerically to obtain $(\mathcal{S}_{\text{iv}}, \mathcal{E}_{\text{iv}})$, and store the associated investor model-implied optimal strategy for each discrete state value. We then carry out 10 million Monte Carlo simulations of the portfolio value over $[0, T]$ using true model parameters, starting from an initial wealth w_0, while rebalancing the portfolio at each rebalancing time in accordance with the stored investor model-optimal strategy. For each simulation, the resulting true terminal wealth value is stored, which allows us to calculate the corresponding true efficient point, and calculate the relative efficient point error using (3.2).

4 Numerical results

In this section, we numerically investigate the MV efficient point errors using different model and calibration assumptions. We illustrate the implications of the analytical results presented in Section 3, and make use of the distinction of Definition 2.3 in terms of Category I and Category II error. In addition, we investigate the impact of the investment constraints discussed in Subsection 3.3 on the results.

All numerical results in this section is based on an initial wealth of $w_0 = 100$ and maturity $T = 20$ years, and the problems are viewed from the perspective of $t_0 \equiv t_1 = 0$. In the case of discrete rebalancing, we assume $\Delta t = 1$ (annual rebalancing), which is not only realistic for a long-term investor, but also provides a clear contrast with the case of continuous rebalancing. For illustrative purposes, wherever a target standard deviation of terminal wealth is required, a value of $\mathcal{S}_{\text{iv}} = 400$ is assumed, which ensures that a material investment in the risky asset is required[8] at least at some point during $[0, T]$.

4.1 Empirical data and calibration

For concreteness, in the case of the risky asset we consider two jump-diffusion models, namely the Kou (2002) and the Merton (1976) models, and one pure diffusion model (GBM). In the case of the Merton model, the pdf $p_j(\xi), j \in \{\text{iv}, \text{tr}\}$ defined in Section 2 is the lognormal density with parameters $\left(m_j, \gamma_j^2\right)$, while in the case of the Kou model $p_j(\xi)$ is given by the asymmetric double-exponential density

$$p_j(\xi) = \nu_j \zeta_{j,1} \xi^{-\zeta_{j,1}-1} \mathbb{I}_{[1,\infty)}(\xi) + (1 - \nu_j) \zeta_{j,2} \xi^{\zeta_{j,2}-1} \mathbb{I}_{[0,1)}(\xi), \nu_j \in [0, 1] \text{ and } \zeta_{j,1} > 1, \zeta_{j,2} > 0, \quad (4.1)$$

where $\mathbb{I}_{[A]}$ denotes the indicator function of the event A.

In order to parameterize the underlying asset dynamics, the same calibration data and techniques are used as in Dang and Forsyth (2016); Forsyth and Vetzal (2017a). The empirical risky asset data is based on daily total return data (including dividends and other distributions) for the period 1926-2014 from the CRSP's VWD index[9], which is a capitalization-weighted index of all domestic stocks on major US exchanges. The risk-free rate is based on 3-month US T-bill rates[10] over the period 1934-2014, and has been augmented with the NBER's short-term government bond yield data[11] for 1926-1933 to incorporate the impact of the 1929 stock market crash. Prior to calculations, all time series (for both the risky and risk-free asset) were inflation-adjusted using data from the US Bureau

[8] In Lemma 3.6, as $\mathcal{S}_{\text{iv}} \downarrow 0$, the efficient point errors also vanish, since an extremely risk averse investor would simply avoid investing in the risky asset altogther.

[9] Calculations were based on data from the Historical Indexes 2015©, Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third party suppliers.

[10] Data has been obtained from `See http://research.stlouisfed.org/fred2/series/TB3MS`.

[11] Obtained from the National Bureau of Economic Research (NBER) website, `http://www.nber.org/databases/macrohistory/contents/chapter13.html`.

Table 4.1: Calibrated risky asset parameters

	No jumps	Jump models					
Parameters	$Gbm0$	$Mer2$	$Mer3$	$Mer4$	$Kou2$	$Kou3$	$Kou4$
μ_j	0.0816	0.0822	0.0817	0.0820	0.0896	0.0874	0.0866
σ_j	0.1863	0.0972	0.1453	0.1584	0.0970	0.1452	0.1584
λ_j	n/a	2.3483	0.3483	0.1461	2.3483	0.3483	0.1461
m_j	n/a	-0.0192	-0.0700	-0.0521	n/a	n/a	n/a
γ_j	n/a	0.1058	0.1924	0.2659	n/a	n/a	n/a
ν_j	n/a	n/a	n/a	n/a	0.4258	0.2903	0.3846
$\zeta_{j,1}$	n/a	n/a	n/a	n/a	11.2321	4.7941	3.7721
$\zeta_{j,2}$	n/a	n/a	n/a	n/a	10.1256	5.4349	3.9943

of Labor Statistics[12], resulting in a risk-free rate of $r = 0.00623$.

The calibration of the jump-diffusion models is based on the thresholding technique of Cont and Mancini (2011); Cont and Tankov (2004) using the approach of Dang and Forsyth (2016); Forsyth and Vetzal (2017a) which, in contrast to maximum likelihood estimation of jump model parameters, avoids problems such as ill-posedness and multiple local maxima. If $\Delta\chi_i$ denotes the ith inflation-adjusted, detrended log return in the historical risky asset index time series, a jump is identified in period i if $|\Delta\chi_i| > \mathcal{J}\sigma_j\sqrt{\Delta\tau}$, where σ_j is an estimate of the diffusive volatility, $\Delta\tau$ is the time period over which the log return has been calculated, and $\mathcal{J}$ is a threshold parameter used to identify a jump[13]. In the case of GBM, standard maximum likelihood techniques are used.

The calibrated parameters for the risky asset dynamics are provided in Table 4.1, where we also introduce the convention of referring to GBM as $Gbm0$, and the Merton and Kou models respectively as $Mer\mathcal{J}$ and $Kou\mathcal{J}$, where $\mathcal{J} \in \{2, 3, 4\}$ is the chosen value of the threshold parameter.

4.2 No investment constraints

As rigorously shown in the analysis in Section 3, when no investment constraints are applicable, the efficient point errors depend critically on the ratios M, $M_{\Delta t}$, L, and $L_{\Delta t}$ defined in (3.4). The closer these ratios to one, the more robust the MV outcomes to model misspecification, i.e. the smaller the resulting error measures (Definition 3.2). Using the parameters from Table 4.1, these ratios for each (iv, tr) model combination are displayed in Table 4.2.

We make the following observations regarding this set of calibrated parameters. Firstly, we observe that $|M - 1| \simeq 0$, which by (3.38) implies that $\mathcal{R}_{(\text{iv}\to\text{tr})} \simeq \left|\Psi_{(\text{iv}\to\text{tr})}\right|$, regardless of (iv, tr) model combination or threshold. As a result, Theorem 3.10 provides the theoretical basis for an explanation of the errors due to model misspecification in this data set (discussed in detail below). Secondly, $|L - 1| \simeq 0$ for all (iv, tr) model combinations and/or thresholds, except those based on the Kou model ($Kou\mathcal{J}$) and any other model of a different fundamental type, namely $Gbm0$ or $Mer\mathcal{J}$. For example, (iv, tr) = ($Gbm0$, $Mer4$) gives $|L - 1| = 1.02 - 1 = 0.02 \simeq 0$; however, (iv, tr) = ($Mer3$, $Kou4$) results in $|L - 1| = |1.6 - 1| = 0.6 \gg 0$; or (iv, tr) = ($Gbm0$, $Kou4$) gives $|L - 1| = 1.56 - 1 = 0.56 \gg 0$. The same observation holds for $M_{\Delta t}$ (resp. $L_{\Delta t}$), since the values of M (resp. L) and $M_{\Delta t}$ (resp. $L_{\Delta t}$) are very similar. These observations, when considered in conjunction with the results of Lemma 3.6, assist in explaining the Category I and Category II model misspecification errors discussed below.

4.2.1 General MV robustness

For this set of calibrated parameters, we now calculate the different measures of the efficient point error (Definition 3.2), and consider the results in conjunction with the analytical results of Lemma 3.6 and Theorem 3.10. First, consider the definition of the relative efficient point error $(\%\Delta\mathcal{S}, \%\Delta\mathcal{E})$

[12]The annual average CPI-U index, which is based on inflation data for urban consumers, were used - see `http://www.bls.gov.cpi` .

[13]This means that a jump is only identified in the historical time series if the absolute value of the inflation-adjusted, detrended log return in that period exceeds $\mathcal{J}$ standard deviations of the "geometric Brownian motion change".

Table 4.2: Key ratios $M, M_{\Delta t}, L$ and $L_{\Delta t}$ as per (3.4) for each combination of (iv, tr) model, $\Delta t = 1$.

True model	Ratios	Investor model						
		$Gbm0$	$Mer2$	$Mer3$	$Mer4$	$Kou2$	$Kou3$	$Kou4$
$Gbm0$	M , $M_{\Delta t}$	1.00 , 1.00	0.99 , 0.99	1.00 , 1.00	0.99 , 0.99	0.90 , 0.90	0.93 , 0.93	0.94 , 0.94
	L , $L_{\Delta t}$	1.00 , 1.00	0.97 , 0.97	1.03 , 1.03	0.98 , 0.98	0.69 , 0.67	0.69 , 0.67	0.64 , 0.63
$Mer2$	M , $M_{\Delta t}$	1.01 , 1.01	1.00 , 1.00	1.01 , 1.01	1.00 , 1.00	0.91 , 0.91	0.94 , 0.93	0.95 , 0.94
	L , $L_{\Delta t}$	1.03 , 1.03	1.00 , 1.00	1.05 , 1.05	1.00 , 1.00	0.70 , 0.69	0.71 , 0.69	0.66 , 0.65
$Mer3$	M , $M_{\Delta t}$	1.00 , 1.00	0.99 , 0.99	1.00 , 1.00	1.00 , 1.00	0.91 , 0.90	0.93 , 0.93	0.94 , 0.94
	L , $L_{\Delta t}$	0.97 , 0.97	0.95 , 0.95	1.00 , 1.00	0.95 , 0.95	0.67 , 0.65	0.67 , 0.66	0.63 , 0.61
$Mer4$	M , $M_{\Delta t}$	1.01 , 1.01	1.00 , 1.00	1.00 , 1.00	1.00 , 1.00	0.91 , 0.91	0.93 , 0.93	0.94 , 0.94
	L , $L_{\Delta t}$	1.02 , 1.02	1.00 , 1.00	1.05 , 1.05	1.00 , 1.00	0.70 , 0.69	0.70 , 0.69	0.66 , 0.65
$Kou2$	M , $M_{\Delta t}$	1.11 , 1.11	1.10 , 1.10	1.10 , 1.11	1.10 , 1.10	1.00 , 1.00	1.03 , 1.03	1.04 , 1.04
	L , $L_{\Delta t}$	1.46 , 1.49	1.42 , 1.45	1.49 , 1.53	1.43 , 1.46	1.00 , 1.00	1.00 , 1.01	0.94 , 0.94
$Kou3$	M , $M_{\Delta t}$	1.08 , 1.08	1.07 , 1.07	1.08 , 1.08	1.07 , 1.07	0.97 , 0.97	1.00 , 1.00	1.01 , 1.01
	L , $L_{\Delta t}$	1.45 , 1.48	1.42 , 1.44	1.49 , 1.52	1.42 , 1.45	1.00 , 0.99	1.00 , 1.00	0.94 , 0.94
$Kou4$	M , $M_{\Delta t}$	1.07 , 1.07	1.06 , 1.06	1.06 , 1.07	1.06 , 1.06	0.96 , 0.96	0.99 , 0.99	1.00 , 1.00
	L , $L_{\Delta t}$	1.56 , 1.59	1.52 , 1.54	1.60 , 1.63	1.52 , 1.55	1.07 , 1.06	1.07 , 1.07	1.00 , 1.00

defined in (3.2). As shown in Lemma 3.6, given fixed investor model and investment parameters, $\%\Delta\mathcal{E}$ depends on M or $M_{\Delta t}$. Since $|M-1| \simeq 0$ and $|M_{\Delta t}-1| \simeq 0$, $\%\Delta\mathcal{E}$ is fairly negligible for all (iv, tr) model combinations. On the other hand, $\%\Delta\mathcal{S}$ depends on both (M, L) or both $(M_{\Delta t}, L_{\Delta t})$. Table 4.2 shows that for (iv, tr) model combinations based on either $Gbm0$ or $Mer\mathcal{J}$ and $Kou\mathcal{J}$, we have $|L-1| \gg 0$ and $|L_{\Delta t}-1| \gg 0$. It is therefore expected that for these (iv, tr) model combinations, $\%\Delta\mathcal{S}$ will be large (MV results less robust to model misspecification), while it is negligible for the rest of the (iv, tr) model combinations (more robust MV results). That is, $\%\Delta\mathcal{S}$, not $\%\Delta\mathcal{E}$, is the key factor in determining the robustness of the MV optimization results for this data set as measured by $(\%\Delta\mathcal{S}, \%\Delta\mathcal{E})$. Second, considering the error norm (3.3), these observations imply that we would indeed expect $\mathcal{R}_{(\text{iv}\to\text{tr})} \simeq \sqrt{(\%\Delta\mathcal{S})^2} = \left|\Psi_{(\text{iv}\to\text{tr})}\right|$ for this data set, which highlights the relevance of Theorem 3.10 in explaining the results.

To further illustrate this point, Table 4.3 shows $(\%\Delta\mathcal{S}, \%\Delta\mathcal{E})$ for the (iv, tr) model combinations when the true (tr) model is $Mer3$ and $Kou3$, for both discrete and continuous rebalancing. Table 4.4 shows the corresponding results for $\mathcal{R}_{(\text{iv}\to\text{tr})}$ for the same data set.

Table 4.3: $(\%\Delta\mathcal{S}, \%\Delta\mathcal{E})$, defined in (3.2). $T = 20$, $\Delta t = 1$, $\mathcal{S}_{\text{iv}} = 400$, $w_0 = 100$.

True model	Objective	Investor model						
		$Gbm0$	$Mer2$	$Mer3$	$Mer4$	$Kou2$	$Kou3$	$Kou4$
$Mer3$	$PCMV$	(-5% , 0%)	(-6% , 0%)	(0% , 0%)	(-6% , 0%)	(-22%,-2%)	(-26%,-1%)	(-32%,-1%)
	$TCMV$	(-1% , 0%)	(-3% , -1%)	(0% , 0%)	(-2% , 0%)	(-18%,-8%)	(-18%,-6%)	(-21%,-5%)
	$PCMV_{\Delta t}$	(-4% , 0%)	(-5% , 0%)	(0% , 0%)	(-5% , 0%)	(-21%,-3%)	(-25%,-2%)	(-30%,-2%)
	$TCMV_{\Delta t}$	(-1% , 0%)	(-3% , -1%)	(0% , 0%)	(-2% , 0%)	(-19%,-8%)	(-19%,-6%)	(-22%,-5%)
$Kou3$	$PCMV$	(66% , 1%)	(60% , 1%)	(80% , 1%)	(60% , 1%)	(7% , 0%)	(0% , 0%)	(-10%, 0%)
	$TCMV$	(21% , 7%)	(19% , 6%)	(22% , 7%)	(19% , 6%)	(0% , -2%)	(0% , 0%)	(-3% , 1%)
	$PCMV_{\Delta t}$	(55% , 1%)	(50% , 1%)	(65% , 1%)	(50% , 1%)	(5% , -1%)	(0% , 0%)	(-9% , 0%)
	$TCMV_{\Delta t}$	(22% , 7%)	(20% , 6%)	(23% , 7%)	(20% , 6%)	(0% , -2%)	(0% , 0%)	(-3% , 1%)

Based on the preceding analysis, in particular Tables 4.2, 4.3 and 4.4, we reach the following conclusions on the robustness of MV results for this data set.

- MV optimization can be surprisingly robust to Category I errors, where the investor makes an incorrect assumption regarding the fundamental model type, since the resulting efficient point

Table 4.4: $\mathcal{R}_{(\text{iv}\to\text{tr})}$, defined in (3.3). $T = 20$, $\Delta t = 1$, $\mathcal{S}_{\text{iv}} = 400$, $w_0 = 100$.

True model	Objec-tive	Investor model						
		*Gbm*0	*Mer*2	*Mer*3	*Mer*4	*Kou*2	*Kou*3	*Kou*4
*Mer*3	*PCMV*	5%	6%	0%	6%	22%	26%	32%
	TCMV	1%	3%	0%	2%	20%	19%	21%
	$PCMV_{\Delta t}$	4%	5%	0%	5%	21%	25%	30%
	$TCMV_{\Delta t}$	1%	3%	0%	2%	21%	20%	22%
*Kou*3	*PCMV*	66%	60%	80%	60%	7%	0%	10%
	TCMV	22%	20%	23%	20%	2%	0%	3%
	$PCMV_{\Delta t}$	55%	50%	65%	50%	5%	0%	9%
	$TCMV_{\Delta t}$	23%	21%	24%	21%	2%	0%	3%

errors (largely driven by $\%\Delta\mathcal{S}$ in this case) can be very small in this case. However, the extent to which the error remains small when switching fundamental model types depends on certain particular aspects of the models involved, such as the tails of the jump distribution. This is discussed in more detail in Remark 4.1 below.

- MV optimization is generally very robust to Category II errors, since, when (iv, tr) models are within the same fundamental type, both (M, L) or $(M_{\Delta t}, L_{\Delta t})$ are very close to one, resulting in very small efficient point errors regardless of chosen error measure. This is a very encouraging finding from a model calibration perspective, as discussed in Remark 4.2 below.

Remark 4.1. (MV robustness and the modelling of jumps) Tables 4.3 and 4.4 illustrate a surprising consequence of the analytical results presented in Section 3, namely that specifying the fundamental jump-diffusion model type incorrectly can be just as consequential as not incorporating jumps at all. Intuitively, one would expect that jump-diffusion model results ($Mer\mathcal{J}$, $Kou\mathcal{J}$) would be more similar than these results suggest, with larger differences observed if any of these models are combined with a pure-diffusion model. Instead, a (iv, tr) model combination of a pure-diffusion and a jump-diffusion model, such as ($Gbm0$, $Mer\mathcal{J}$) or ($Mer\mathcal{J}$, $Gbm0$), $J = \{2, 3, 4\}$, might not have a large impact on the MV outcomes, in spite of the fatter tails of the return distribution arising in the case of a jump-diffusion model. However, a (iv, tr) model combination that involves $Kou\mathcal{J}$ and a model of different type, namely $Gbm0$ or $Mer\mathcal{J}$, results in significantly larger efficient point errors (since $|L - 1| \gg 0$ or $|L_{\Delta t} - 1| \gg 0$), regardless of error measure.

This is a consequence of the differences between the Merton and the Kou jump models with regards to the modelling of the tails of the jump distribution, and the resulting impact on the factor $\sigma_j^2 + \kappa_{j,2}$. Specifically, recalling that $\kappa_{j,2} = \mathbb{E}\left[(\xi_j - 1)^2\right]$, with ξ_j be a random variable denoting the jump multiplier, the asymmetric double exponential density of the Kou model with its heavy tails results in a value of $\kappa_{j,2}$ that is significantly larger than the corresponding value for the Merton model. All else being equal, the value of $\sigma_j^2 + \kappa_{j,2}$ is therefore expected to be possibly significantly larger for the Kou model than for the Merton model. Furthermore, while the GBM model does not incorporate jumps, the increased value of the diffusive volatility σ_j parameter in this case can assist in bringing the GBM results more in line with the results of the Merton model (see Remark 4.2). Taken together, these observations have the consequence that the key difference in results in not caused by the inclusion or exclusion of jumps in the risky asset process, but by the particular choice of jump model.

Finally, we observe that the modelling of jumps also has an impact on the sign of $\%\Delta\mathcal{S}$, as illustrated in Table 4.3. This can be explained as follows. From Definition 3.2 and Lemma 3.6, we observe that sign of $\%\Delta\mathcal{S}$ is the same as the sign of the error multiplier $\Psi_{(\text{iv}\to\text{tr})}$, which (by Lemma 3.6) depends on a combination of all the process and investment parameters. Changing investment and process parameters will inevitably lead to different results. However, the largest values of $\%\Delta\mathcal{S}$ in Table 4.3 are obtained when the true model is $Mer3$ and the investor model is $Kou\mathcal{J}$ (in which case $\%\Delta\mathcal{S} < 0$), and when the true model is $Kou3$ and the investor model is $Mer\mathcal{J}$ (in which case $\%\Delta\mathcal{S} > 0$). This is again explained with reference to the modelling of the tails of the jump distribution. Suppose the true model is $Mer3$, and the investor model is $Kou\mathcal{J}$. In this case, the investor calculates a standard deviation of terminal wealth $\mathcal{S}_{\text{iv}}$ consistent with the implications of the heavy tails of the asymmetric double exponential distribution, while the true standard deviation $\mathcal{S}_{(\text{iv}\to\text{tr})}$ obtained under the (true) Merton model is much lower. This results in $\mathcal{S}_{(\text{iv}\to\text{tr})} < \mathcal{S}_{\text{iv}}$, so that $\%\Delta\mathcal{S} < 0$. The case of $\%\Delta\mathcal{S} > 0$ can be explained using similar observations.

Closely related to this discussion, Remark 4.2 below provides a discussion of the implications of the MV robustness results for model calibration.

Remark 4.2. (Implications for model calibration) The MV robustness results have some interesting implications for the thresholding calibration methodology used to calibrate the risky asset parameters. Assume that the chosen investor model type matches the true model fundamental type but with potentially different sets of parameters, for example (iv, tr) = ($Mer3$, $Mer2$), or (iv, tr) = ($Mer3$, $Mer4$). The results of Table 4.2 show that the thresholding calibration methodology outlined in Subsection 4.1 is expected to give very robust MV results regardless of the jump threshold $\mathcal{J}$. Specifically, using the analytical results derived in Section 3, the impact of the jump threshold $\mathcal{J}$ on the key ratios (3.4) is relatively straightforward. In particular, since only the *combination* of parameters $\sigma_j^2 + \lambda_j\kappa_{j,2}$ play a role in the ratio L, increasing the jump threshold $\mathcal{J}$ increases the diffusive volatility σ_j (more asset price moves are due to the diffusion component) and also increases the variance of the jump distribution and therefore $\kappa_{j,2}$ (the jumps that occur are larger, regardless of direction), but at the same time fewer jumps occur implying a smaller value of λ_j occur. This robustness of MV results to the choice of jump threshold is encouraging since the threshold can also have somewhat counterintuitive consequences. For example, Tables 4.3 and 4.4 show that if the true model is $Mer2$, then an investor model of $Mer3$ would result in M and L values indicative of larger (though still comparatively immaterial) efficient point errors than if the investor model $Mer4$ was chosen, which is due precisely to the above-mentioned interplay between σ_j, λ_j and $\kappa_{j,2}$ in the thresholding calibration methodology.

4.2.2 PCMV and TCMV

Comparing the efficient point errors in this data set for PCMV and TCMV when no investment constraints are applicable, the analytical results in Section 3 can be used in conjunction with Tables 4.2, 4.3 and 4.4 to reach the following conclusions.

- PCMV is less robust to model misspecification than TCMV, regardless of rebalancing frequency or underlying models. This is clear from the results for $\mathcal{R}_{(\text{iv}\to\text{tr})}$ reported in Table 4.4, and is indeed expected based on the result of Theorem 3.10, since in this data set we observe $|M-1| \simeq 0$ and $|M_{\Delta t}-1| \simeq 0$ (see Table 4.2).

- The differences between PCMV and TCMV efficient point errors increases further when the (iv,tr) models are of different fundamental types, especially when one model is based on the Kou jump-diffusion model formulation. In this particular case, the observation that $|L-1| \gg 0$ or $|L_{\Delta t}-1| \gg 0$, together with the results of Lemma 3.6 and Theorem 3.10 show that these results are expected for this data set.

- Considering the impact of rebalancing frequency on the efficient point error, we observe that for TCMV, discrete rebalancing *increases* the value of $\mathcal{R}_{(\text{iv}\to\text{tr})}$ compared to the corresponding values for continuous rebalancing, which is to be expected given the results of Theorem 3.10 (see (3.37)). However, the overall impact of rebalancing frequency on the error norm in the case of TCMV is actually fairly negligible. In contrast, for PCMV, discrete rebalancing *decreases* the value of $\mathcal{R}_{(\text{iv}\to\text{tr})}$ compared to the corresponding values for continuous rebalancing.

  As noted in the discussion following Theorem 3.10, a simple result comparable to (3.37) cannot be given in the case of PCMV, so for analytical purposes these results can be explained rigorously by Lemma 3.6 for this particular set of investment and model parameters. However, a more intuitive explanation as to why the PCMV and TCMV efficient point errors react so differently to changes in rebalancing frequency is particularly useful when no analytical solutions are available, such as in the case of the results in Subsection 4.3 below.

From the results of Cong and Oosterlee (2016); Van Staden et al. (2018), it is known that the PCMV strategy requires a significantly larger investment in the risky asset in the early years of the investment time horizon than the TCMV strategy. Furthermore, discrete rebalancing reduces this large early investment in the risky asset significantly in the case of PCMV, but has a much smaller impact in the case of TCMV. The relatively larger standard deviation efficient point errors for PCMV in Tables 4.3 and 4.4 can therefore be explained intuitively by noting that the PCMV-optimal strategy places a much heavier reliance on the risky asset during the critical early years of the investment. Therefore, the model misspecification scenario is expected to have a comparatively larger impact on PCMV terminal wealth standard deviation outcomes, which is magnified further if the portfolio is rebalanced continuously.

Table 4.5 illustrates the numerical results for the true price of risk, as defined in Lemma 3.7, in the case of no investment constraints. In particular, we observe that the ratio $\Gamma^{p}_{(\text{iv}\to\text{tr})}/\Gamma^{c}_{(\text{iv}\to\text{tr})}$ always exceeds one for the set of parameters considered in this section. In other words, given this set of model and investment parameters, the PCMV strategy outperforms the TCMV strategy on the basis of the corresponding true price of risk, regardless of true model and investor model combinations. However, as observed in (3.25) and the associated discussion, this is not necessarily guaranteed in all circumstances.

Table 4.5: True price of risk, $\Gamma_{(\text{iv}\to\text{tr})}$, defined in (3.23). $T = 20$, $\Delta t = 1$, $\mathcal{S}_{\text{iv}} = 400$, $w_0 = 100$.

True model	Objective	Investor model						
		$Gbm0$	$Mer2$	$Mer3$	$Mer4$	$Kou2$	$Kou3$	$Kou4$
$Mer3$	$PCMV_{\Delta t}$	4.05	4.04	4.05	4.04	3.74	3.71	3.60
	$TCMV_{\Delta t}$	1.81	1.80	1.83	1.80	1.66	1.62	1.55
$Kou3$	$PCMV_{\Delta t}$	2.55	2.59	2.49	2.59	2.86	2.86	2.85
	$TCMV_{\Delta t}$	1.73	1.72	1.75	1.72	1.57	1.53	1.47

4.3 Impact of investment constraints

Tables 4.6 and 4.7 provide $(\%\Delta\mathcal{S}, \%\Delta\mathcal{E})$ and $\mathcal{R}_{(\text{iv}\to\text{tr})}$, respectively, when investment constraints as outlined in Subsection 3.3 are applied to the results of Tables 4.3 and 4.4. Specifically, in the event of insolvency we require the liquidation of the investment in the risky asset, and allow for a maximum leverage ratio of $q_{max} = 1.5$. In this case, we conclude the following.

- Tables 4.6 and 4.7 show that the PCMV and TCMV results are very robust (i.e. relatively small values of $(\%\Delta\mathcal{S}, \%\Delta\mathcal{E})$ and $\mathcal{R}_{(\text{iv}\to\text{tr})}$) to both Category I and Category II model misspecification errors if investment constraints are applied. Even though for example a $Gbm0$ investor model and a $Kou\mathcal{J}, \mathcal{J} \in \{2, 3, 4\}$ true model represents significantly different perspectives on the underlying asset dynamics, values of for example $\%\Delta\mathcal{S} \simeq 20\%$ and $\%\Delta\mathcal{E} \simeq 5\%$ accumulated over an investment period of 20 years is robust indeed.

- Considering the results in Table 4.7, we observe that in those cases where the largest errors as measured by $\mathcal{R}_{(\text{iv}\to\text{tr})}$ occur, PCMV is associated with smaller errors than TCMV. This stands in contrast to the case where no constraints were applied (see Table 4.4 above). Furthermore, we observe that the TCMV errors are typically somewhat smaller in the case with investment constraints (Table 4.7) than in the case where no constraints are applied (Table 4.4). However, this error reduction effect following the application of investment constraints is significantly more pronounced in the case of PCMV. This phenomenon is discussed in more detail below.

Table 4.6: $(\%\Delta\mathcal{S}, \%\Delta\mathcal{E})$, defined in (3.2). $T = 20$, $\Delta t = 1$, $\mathcal{S}_{\text{iv}} = 400$, $w_0 = 100$, $q_{max} = 1.5$, liquidation in the event of bankruptcy.

True model	Objective	Investor model						
		$Gbm0$	$Mer2$	$Mer3$	$Mer4$	$Kou2$	$Kou3$	$Kou4$
$Mer3$	$PCMV_{\Delta t}$	(1% , 0%)	(-1% , 0%)	(0% , 0%)	(-1% , 0%)	(-12%,-2%)	(-12%,-2%)	(-14%,-3%)
	$TCMV_{\Delta t}$	(0% , 0%)	(-2% , 0%)	(0% , 0%)	(-2% , 0%)	(-20%,-7%)	(-18%,-5%)	(-19%,-4%)
$Kou3$	$PCMV_{\Delta t}$	(14% , 2%)	(13% , 2%)	(13% , 2%)	(13% , 2%)	(1% , 0%)	(0% , 0%)	(0% , 0%)
	$TCMV_{\Delta t}$	(20% , 6%)	(18% , 5%)	(21% , 6%)	(19% , 6%)	(-3% , -2%)	(0% , 0%)	(0% , 1%)

Table 4.7: $\mathcal{R}_{(\text{iv}\to\text{tr})}$, defined in (3.3). $T = 20$, $\Delta t = 1$, $\mathcal{S}_{\text{iv}} = 400$, $w_0 = 100$, $q_{max} = 1.5$, liquidation in the event of bankruptcy.

True model	Objective	Investor model						
		$Gbm0$	$Mer2$	$Mer3$	$Mer4$	$Kou2$	$Kou3$	$Kou4$
$Mer3$	$PCMV_{\Delta t}$	1%	1%	0%	1%	12%	12%	14%
	$TCMV_{\Delta t}$	0%	2%	0%	2%	21%	18%	19%
$Kou3$	$PCMV_{\Delta t}$	15%	13%	14%	13%	1%	0%	0%
	$TCMV_{\Delta t}$	21%	19%	22%	20%	3%	0%	1%

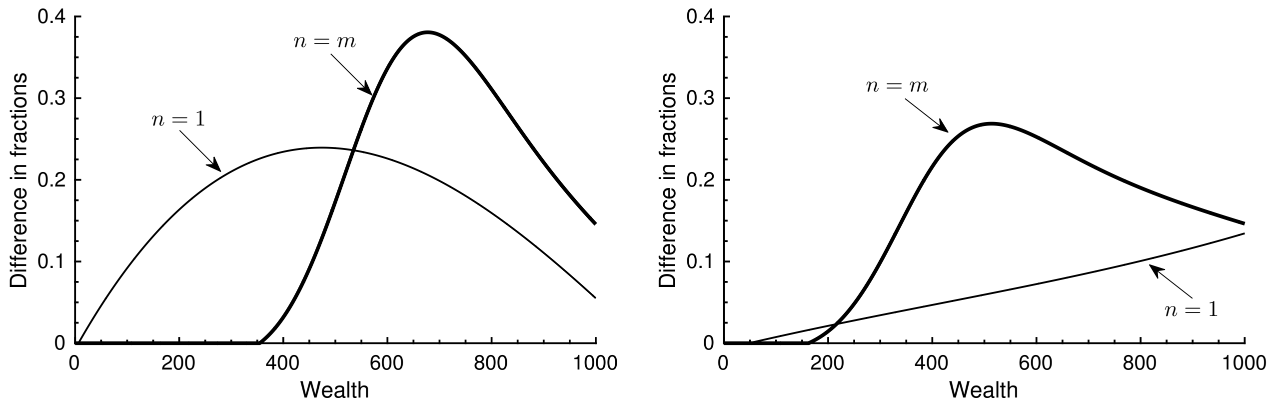
The results of Van Staden et al. (2018) can again be used to provide an intuitive explanation of the relative robustness results for PCMV and TCMV in Tables 4.6 and 4.7.

Specifically, when investment constraints are applied, the smaller errors as measured by $\mathcal{R}_{(\text{iv}\to\text{tr})}$ in the case of PCMV appears to be largely a consequence of the leverage constraint having a much more significant impact on the PCMV results compared to the TCMV results (see Van Staden et al. (2018) for a discussion). Compared to the case of no investment constraints, the maximum leverage ratio leads to a substantial reduction of the amount invested in the risky asset in the case of PCMV during the early years of the investment time horizon. TCMV is of course also impacted by the leverage constraint, but to a significantly smaller degree, with the solvency condition serving as the primary driver of the lower investment in the risky asset in the early years of the investment horizon when constraints are applied. As a result, the error in the TCMV due to model misspecification is not affected to the same extent as the corresponding error for PCMV when investment constraints are applied.

Figure 4.1 shows the difference in optimal controls for the $Mer3$ and $Kou3$ investor models obtained numerically as described in Subsection 3.3. Figure 4.1(a) shows that as wealth increases, the difference in PCMV optimal controls initially increases but then decreases again, behavior which is closely related to the role of the implied terminal wealth target on the PCMV-optimal strategy (see Dang and Forsyth (2016); Vigna (2014) for a discussion). In contrast, this is not the case with TCMV (Figure 4.1(b)), which shows similar behaviour to PCMV in later years as expected[14], while in earlier years we see an increase in the difference in optimal controls as the wealth level increases, but with no associated decrease to the same extent as observed in the case of PCMV. This can be explained by noting that the TCMV investor acts consistently with MV preferences throughout the investment time horizon, with no implicit terminal wealth target being present.

Figure 4.1 therefore assists in providing a numerical explanation of the results of Table 4.6. In particular, in the case of PCMV, the implied target-seeking behavior of the PCMV-optimal strategy implies a reduction in risky asset exposure if prior returns were relatively good, regardless of underlying model, which helps to drive the improved robustness results (smaller errors as measured by $\mathcal{R}_{(\text{iv}\to\text{tr})}$) in the case of PCMV relative to TCMV seen in Table 4.7. In the case of TCMV, while at an individual rebalancing event the maximum error in the control might be smaller than in the case of PCMV (Figure 4.1(a) vs 4.1(b)), the control error for TCMV can overwhelm the corresponding error for PCMV as wealth grows, thus driving the larger overall error for TCMV observed in Table 4.7.

[14] In the extreme case of single-period problems, there is no difference between PCMV and TCMV optimization.



(a) $PCMV_{\Delta t}$ (b) $TCMV_{\Delta t}$

Figure 4.1: Difference in numerically-calculated optimal controls for the $Mer3$ and $Kou3$ models, expressed as the difference in the MV-optimal fractions of wealth invested in the risky asset at the first and last rebalancing events, $n = 1$ and $n = m$, respectively. Specifically, the figures show the difference $\left[u^{q*}_{Mer3,\Delta t}/W_{Mer3,\Delta t}\left(t\right)\right] - \left[u^{q*}_{Kou3,\Delta t}/W_{Kou3,\Delta t}\left(t\right)\right]$ as a function of wealth, for $q \in \{p, c\}$ and $t = t_n$, $n \in \{1, m\}$. $\mathcal{S}_{\text{iv}} = 400$, discrete rebalancing ($\Delta t = 1$), $q_{max} = 1.5$, liquidation in the event of bankruptcy. Note the same scale on the y-axis.

5 Conclusions

In this paper, we investigate the robustness of dynamic MV optimization to model misspecification errors. Under certain assumptions, we derived analytical solutions to quantify the error in MV outcomes when the investment strategy, optimal according to some chosen investor model, is implemented in a market driven by a possibly different true model. The analytical solutions show that the error in MV outcomes is driven by certain combinations of model parameters, so that individual process parameters only play a secondary role, implying that fundamentally different perspectives on the underlying dynamics might still result in very similar MV results for terminal wealth. In the absence of investment constraints, numerical tests show that PCMV results in larger MV errors than TCMV, and continuous rebalancing is associated with larger errors than discrete rebalancing. The analytical results presented show that under certain conditions, this is to be expected. However, in the more realistic scenario of discrete rebalancing together with the simultaneous application of multiple investment constraints, PCMV can be more robust to model misspecification errors than TCMV.

We leave the extension of our results to the recently proposed dynamically optimal MV approach of Pedersen and Peskir (2017), as well as the impact of model misspecification on other percentiles of the terminal wealth distribution, for our future work.

Appendix A: Additional numerical results

Bootstrap resampling test - historical bond and stock returns

To obtain the analytical and numerical results presented in this paper, we have assumed that the underlying asset dynamics can be described in terms of some known diffusion or jump-diffusion models (Assumption 2.3). In addition, we have explicitly not considered stochastic interest rates or stochastic volatility due to the reasons outlined in Section 2. However, as discussed in Forsyth and Vetzal (2017a), for purposes of risk management and validation it is useful to perform historical backtesting of the results using for example a moving block bootstrap resampling method[15], which we perform using the same historical data used for calibration purposes in Subsection 4.1.

Specifically, we assess the MV of true terminal wealth using 5 million resampled historical risky and risk-free asset return paths, rebalancing the portfolio at each rebalancing time according to the stored MV-optimal investment strategies as per the appropriate investment objective and investor

[15]For more information on bootstrapped resampling tests in financial settings, see, for example, Annaert et al. (2009); Bertrand and Prigent (2011); Cogneau and Zakalmouline (2013); Sanfilippo (2003)

model. The resampled paths are constructed by dividing the horizon T into $\tilde{k}$ blocks of size $\tilde{b}$ years (i.e. $T = \tilde{k}\tilde{b}$), where block sizes of $\tilde{b} = 5$ years and $\tilde{b} = 10$ years are considered[16]. Each individual resampled path is constructed by selecting $\tilde{k}$ blocks at random (with replacement) from the historical data, with each block starting at a random quarter and with blocks being wrapped around to avoid end effects in the data, with selected blocks being concatenated to produce the path. In Table A.1, we use the resampled historical paths as the "true" model to report the relative efficient point error exactly as before. We observe that (i) the relative efficient point error is of similar order of magnitude using resampled historical data as in the case of using a model (Table 4.6), and (ii) the qualitative conclusions regarding the relative robustness of PCMV vs. TCMV optimization for the models considered in Table 4.6 appear to hold.

More generally, the results of Table A.1 validate our overall conclusions regarding the robustness of MV optimization to model misspecification errors, as well as Assumption 2.2 regarding interest rates. We leave a detailed discussion of the different performance of PCMV and TCMV-optimal controls in the case of resampled historical data for our future work.

Table A.1: $(\%\Delta\mathcal{S}, \%\Delta\mathcal{E})$ calculated using numerical results based on resampled historical data. $T = 20$, $\Delta t = 1$, $\mathcal{S}_{\text{iv}} = 400$, $w_0 = 100$, $q_{max} = 1.5$, liquidation in the event of bankruptcy.

Block size	Objective	Investor model						
		$Gbm0$	$Mer2$	$Mer3$	$Mer4$	$Kou2$	$Kou3$	$Kou4$
5 years	$PCMV_{\Delta t}$	(6% , 1%)	(5% , 1%)	(5% , 1%)	(5% , 1%)	(-7% , -1%)	(-7% , -1%)	(-7% , 0%)
	$TCMV_{\Delta t}$	(-2% , 0%)	(-4% , -1%)	(-2% , 0%)	(-3% , -1%)	(-11%,-1%)	(-10%,-1%)	(-10%,-1%)
10 years	$PCMV_{\Delta t}$	(7% , 3%)	(5% , 3%)	(6% , 3%)	(5% , 2%)	(-7% , 1%)	(-7% , 1%)	(-7% , -1%)
	$TCMV_{\Delta t}$	(-8% , 0%)	(-10%,-1%)	(-9% , 0%)	(-10%,-1%)	(-14%, 0%)	(-13%,-1%)	(-12%, 0%)

Other attributes of the terminal wealth distribution

The preceding results only focused on the mean and variance of terminal wealth. However, depending on the application, the investor may also be concerned with other aspects of the terminal wealth distribution, especially given the possibility of jumps in the risky asset process. For example, in pension fund applications (see, for example, Forsyth and Vetzal (2017a)) the probability that the terminal wealth $W_{\text{tr},\Delta t}(T)$ falls below some minimum level (for illustrative purposes assumed here to be $w_0 e^{rT}$) may be of interest. Other risk metrics such as the Value-at-Risk (VaR) or Conditional Value-at-Risk (CVaR) might also be considered important - see Rockafellar and Uryasev (2002). Table A.2 uses the Monte Carlo simulations described above to estimate the probability $\mathbb{P}\left[W_{\text{tr},\Delta t}(T) \leq w_0 e^{rT}\right]$, as well as the 95%-VaR and 95%-CVaR[17].

In this case, concluding that MV optimization is robust to model misspecification errors depends not only on some (percentile-based) definition of robustness, but also on the associated implications of estimating some critical value incorrectly. As a result, we leave the broader implications of model misspecification for the terminal wealth distribution for our future work.

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[16]Blocks of historical data of sufficiently large size is required to capture the serial dependence possibly present in the data (see Cogneau and Zakalmouline (2013)), but block sizes that are too large result in unreliable variance estimates. We therefore follow Forsyth and Vetzal (2017a) in considering multiple block sizes.

[17]The α-VaR (resp. α-CVaR) is the VaR (resp. CVaR) corresponding to a confidence level α. In our application, this means that 5% of the simulated values of $W_{\text{tr},\Delta t}(T)$ are equal to or below the reported 95%-VaR value, while the reported 95%-CVaR value is the mean of the simulated values of $W_{\text{tr},\Delta t}(T)$ equal to or less than the 95%-VaR - see Miller and Yang (2017); Rockafellar and Uryasev (2002) for a discussion.

Table A.2: Three quantities associated with the simulated true terminal wealth $W_{\text{tr},\Delta t}(T)$ distribution, discrete rebalancing: $\mathbb{P}\left[W_{\text{tr},\Delta t}(T) \leq w_0 e^{rT}\right]$ ("Probability"), 95%-VaR and 95%-CVaR. $T = 20$, $\Delta t = 1$, $\mathcal{S}_{\text{iv}} = 400$, $w_0 = 100$, $q_{max} = 1.5$, liquidation in the event of bankruptcy.

(a) PCMV

True model	Quantity	Investor model	
		$Mer3$	$Kou3$
$Mer3$	Probability	12.9%	12.6%
	95% -VaR	52.4	49.0
	95%-CVaR	30.3	25.0
$Kou3$	Probability	17.1%	16.8%
	95%-VaR	24.5	16.3
	95%-CVaR	5.0	6.2

(b) TCMV

True model	Quantity	Investor model	
		$Mer3$	$Kou3$
$Mer3$	Probability	11.0%	10.5%
	95%-VaR	63.4	64.8
	95%-CVaR	38.3	36.5
$Kou3$	Probability	14.1%	13.5%
	95%-VaR	40.4	36.2
	95%-CVaR	18.1	16.6

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