1. a) This is the closed form for an arithmetic sequence with first term $-3$ and common difference 2. Thus a recursive definition for the sequence is 

$$a_{n+1} = a_n + 2 \quad (n = 0, 1, 2, \ldots) \quad a_0 = -3.$$ 

b) This is the closed form for a geometric sequence with first term $\frac{1}{3}$ and common ratio 2. Thus a recursive definition for the sequence is 

$$b_{n+1} = 2b_n \quad (n = 1, 2, 3, \ldots) \quad b_1 = \frac{1}{3}.$$ 

2. The (arithmetic) sequence $\{a_n\}$ starts with $n = 0$ so $a_{n-1}$ is the $n$th term.

a) A closed form for $\{a_n\}$ is $a_n = -14 + 5n \quad (n = 0, 1, 2, \ldots)$. 

b) The 10th term of the sequence is $a_{10} = -14 + 5 \times 9 = 31$. 

c) We need to find the smallest integer $n$ such that $a_n > 1000$. We need 

$$-14 + 5n > 1000, \quad \text{so} \quad n > 202.8.$$ 

Thus, the 204th term, $a_{203}$, is the first term that exceeds 1000.

The (geometric) sequence $\{b_n\}$ starts with $n = 1$, so $b_n$ is the $n$th term.

a) A closed form for $\{b_n\}$ is $b_n = 2 \times 3^{n-1} \quad (n = 1, 2, 3, \ldots)$. 

b) The 10th term of the sequence is $b_{10} = 2(3)^9 = 39366$. 

c) We need to find the smallest $n$ such that $b_n > 1000$. We use logarithms to solve 

$$2 \times 3^{n-1} = 1000.$$ 

Thus $n \approx 6.656$, so the first term to exceed 1000 is the 7th term.

3. a) 

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
</tr>
<tr>
<td>$F_{n-1}F_{n+1} - F_n^2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) It seems reasonable to guess that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

You did not have to prove your formula, but if you are interested in the proof of the above formula, here it is. Let $C_n = F_{n-1}F_{n+1} - F_n^2$. Thus 

$$C_{n+1} = F_nF_{n+2} - F_{n+1}^2 = F_n(F_{n+1} + F_n) - F_{n+1}^2 = F_nF_{n+1} + F_n^2 - F_{n+1}^2 = F_n^2 - F_{n+1}(F_{n+1} - F_n) = F_n^2 - F_{n+1}F_{n-1} = -C_n.$$ 

Combining the two facts that $C_1 = 1$ and $C_{n+1} = -C_n$ for $n = 1, 2, 3, \ldots$, we see that 

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \quad \text{for all integers } n \geq 1.$$
4. Each series is either arithmetic or geometric.
   a) This is an arithmetic series having 50 terms, with first term $a = -2$ and common difference $d = 3$. Thus,
   \[
   \sum_{i=1}^{50} 3n - 5 = \frac{50}{2}(2(-2) + 49(3)) = 3575.
   \]
   b) This is an arithmetic series having 32 terms, with first term $a = 84$ and common difference $d = -4$. Thus,
   \[
   \sum_{i=4}^{35} -4n + 100 = \frac{32}{2}(2(84) + 31(-4)) = 704.
   \]
   c) This is a geometric series having 11 terms, with first term $a = -3$ and common ratio $r = 2$. Thus,
   \[
   \sum_{i=0}^{10} -3 \times 2^i = \frac{-3(2^{11} - 1)}{2 - 1} = -6141.
   \]
   d) This is an infinite geometric series, with first term $a = -4$ and common ratio $r = -4$. Since $|r| > 1$, this series diverges so this sum is not a finite number.
   e) This is an infinite geometric series, with first term $a = \frac{1}{27}$ and common ratio $r = \frac{1}{3}$. Thus,
   \[
   \sum_{i=3}^{\infty} \left(\frac{1}{3}\right)^i = \frac{\frac{1}{27}}{1 - \frac{1}{3}} = \frac{1}{27} \times \frac{3}{2} = \frac{1}{18}.
   \]

5. a)
   \[
   \begin{array}{c|cccccc}
   & 3 & 7 & 13 & 21 & 31 & 43 \\
   1st dif & 4 & 6 & 8 & 10 & 12 & \\
   2nd dif & 2 & 2 & 2 & 2 & \\
   \end{array}
   \]
   Since the second differences are constant, a closed form for a sequence \(\{a_n\}\) with these first six terms is \(a_n = c_2n^2 + c_1n + c_0\) for some real numbers \(c_2, c_1, c_0\). Considering \(n = 1, 2, 3\) we obtain the following three equations
   \[
   \begin{align*}
   3 &= c_2 + c_1 + c_0 & (1) \\
   7 &= 4c_2 + 2c_1 + c_0 & (2) \\
   13 &= 9c_2 + 3c_1 + c_0 & (3)
   \end{align*}
   \]
   From equation (1) we have \(c_0 = 3 - c_2 - c_1\). Substituting this into equations (2) and (3) we obtain
   \[
   \begin{align*}
   7 &= 4c_2 + 2c_1 + (3 - c_2 - c_1) & \text{which gives} & \quad 3c_2 + c_1 = 4 \\
   13 &= 9c_2 + 3c_1 + (3 - c_2 - c_1) & \text{which gives} & \quad 8c_2 + 2c_1 = 10
   \end{align*}
   \]
   Thus \(2c_2 = 2\), so \(c_2 = 1\). Hence \(c_1 = 1\) and \(c_0 = 1\). A closed form for this sequence is
   \[
   a_n = n^2 + n + 1 \quad (n = 1, 2, 3, \ldots).
   \]
5. b) 

<table>
<thead>
<tr>
<th>1st dif</th>
<th>10  3 -4 -11 -18 -25</th>
</tr>
</thead>
</table>

Since the first differences are constant, a closed form for a sequence \( \{a_n\} \) with these first six terms is \( a_n = c_1 n + c_0 \) for some real numbers \( c_1 \) and \( c_0 \). Considering \( n = 1, 2 \) we obtain the following two equations

\[
\begin{align*}
10 &= c_1 + c_0 \\
3 &= 2c_1 + c_0
\end{align*}
\]

Thus \( c_1 = -7 \) and \( c_0 = 17 \). A closed form for this sequence is

\[ a_n = -7n + 17 \quad (n = 1, 2, 3, \ldots). \]

c) 

<table>
<thead>
<tr>
<th>1st dif</th>
<th>-1 -3 1 17 51 109</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd dif</td>
<td>-2 4 16 34 58</td>
</tr>
<tr>
<td>3rd dif</td>
<td>6 12 18 24</td>
</tr>
</tbody>
</table>

Since the third differences are constant, a closed form for a sequence \( \{a_n\} \) with these first six terms is \( a_n = c_3 n^3 + c_2 n^2 + c_1 n + c_0 \) for some real numbers \( c_3, c_2, c_1, c_0 \). Considering \( n = 1, 2, 3, 4 \) we obtain the following four equations

\[
\begin{align*}
-1 &= c_3 + c_2 + c_1 + c_0 \\
-3 &= 8c_3 + 4c_2 + 2c_1 + c_0 \\
1 &= 27c_3 + 9c_2 + 3c_1 + c_0 \\
17 &= 64c_3 + 16c_2 + 4c_1 + c_0
\end{align*}
\]

From equation (1) we have \( c_0 = -1 - c_3 - c_2 - c_1 \). Substituting this into equations (2), (3) and (4) we obtain

\[
\begin{align*}
-3 &= 8c_3 + 4c_2 + 2c_1 + (-1 - c_3 - c_2 - c_1) \\
1 &= 27c_3 + 9c_2 + 3c_1 + (-1 - c_3 - c_2 - c_1) \\
17 &= 64c_3 + 16c_2 + 4c_1 + (-1 - c_3 - c_2 - c_1)
\end{align*}
\]

That is, \( 7c_3 + 3c_2 + c_1 = -2 \) \( (5) \)

\[
\begin{align*}
26c_3 + 8c_2 + 2(-2 - 7c_3 - 3c_2) &= 2 \\
63c_3 + 15c_2 + 3(-2 - 7c_3 - 3c_2) &= 18
\end{align*}
\]

From equation (5) we have \( c_1 = -2 - 7c_3 - 3c_2 \). Substituting this into equations (6) and (7) we obtain

\[
\begin{align*}
26c_3 + 8c_2 + 2(-2 - 7c_3 - 3c_2) &= 2 \\
63c_3 + 15c_2 + 3(-2 - 7c_3 - 3c_2) &= 18
\end{align*}
\]

which gives \( 12c_3 + 2c_2 = 6 \) \( (6) \)

\[
\begin{align*}
42c_3 + 6c_2 &= 24
\end{align*}
\]

Thus \( 6c_3 = 6 \), so \( c_3 = 1 \). Hence \( c_2 = -3 \), \( c_1 = 0 \) and \( c_0 = 1 \). A closed form for this sequence is

\[ a_n = n^3 - 3n^2 + 1 \quad (n = 1, 2, 3, \ldots). \]
6. a) Here \( a_{n+1} - a_n = 12(4^{n-1}) \) \((n = 1, 2, 3, \ldots)\). Thus
\[
\begin{align*}
a_2 - a_1 &= 12 \times 4^0 \\
a_3 - a_2 &= 12 \times 4^1 \\
\vdots & \quad \vdots \\
a_{n-1} - a_{n-2} &= 12 \times 4^{n-3} \\
a_n - a_{n-1} &= 12 \times 4^{n-2}
\end{align*}
\]

Summing these equations gives
\[
a_n - a_1 = 12 \times 4^0 + 12 \times 4^1 + 12 \times 4^2 + \cdots + 12 \times 4^{n-2} = 12(1 + 4 + 16 + \cdots + 4^{n-2}).
\]
Using the formula for a geometric series and the fact that \( a_1 = 1 \), we have
\[
a_n - 1 = 12 \times \frac{1(4^{n-1} - 1)}{4 - 1} = 4(4^{n-1} - 1) = 4^n - 4.
\]
Thus a closed form for this sequence is \( a_n = 4^n - 3 \) \((n = 1, 2, 3, \ldots)\).

b) Here \( b_{n+1} - b_n = 2n \) \((n = 1, 2, 3, \ldots)\). Thus
\[
\begin{align*}
b_2 - b_1 &= 2 \\
b_3 - b_2 &= 4 \\
\vdots & \quad \vdots \\
b_{n-1} - b_{n-2} &= 2(n - 2) \\
b_n - b_{n-1} &= 2(n - 1)
\end{align*}
\]

Summing these equations gives
\[
b_n - b_1 = 2 + 4 + 6 + \cdots + 2(n - 2) + 2(n - 1).
\]
Using the formula for an arithmetic series and the fact that \( b_1 = 4 \), we have
\[
b_n - 4 = \frac{n-1}{2}(2(2) + (n - 2)2) = \frac{n-1}{2}(2n) = n(n - 1) = n^2 - n.
\]
Thus a closed form for this sequence is \( b_n = n^2 - n + 4 \) \((n = 1, 2, 3, \ldots)\).

7. Let \( V_n \) be the value of the deposit after \( n \) years. Then we have
\[
V_5 = 10000(1 + \frac{0.06}{4})^{20} = 13468.55.
\]
Thus the value of your deposit after 5 years is $13,468.55.

8. Let \( V_n \) be the value of the investment after \( n \) years. We want to find \( V_0 \) such that \( V_{10} = 20000 \), that is
\[
20000 = V_0(1 + \frac{0.05}{12})^{120}.
\]
Hence \( V_0 = 12143.22 \). Thus you need to invest $12,143.22 now to have $20,000 in 10 years time.
9. a) Let \( P_0 \) be the squirrel population on 30 June 1980, and let \( P_n \) be the squirrel population on 30 June, \( n \) years after 1980. Thus \( P_0 = 50 \) and \( P_{10} = 60 \). Now let \( k \) be the annual growth rate of the population, and assume that the population follows an exponential model. Then

\[
P_{10} = P_0(1 + k)^{10} \\
60 = 50(1 + k)^{10}
\]

\[
k = \sqrt[10]{\frac{60}{50}} - 1 \\
k \approx 0.0184.
\]

Thus the annual growth rate (to 4 decimal place accuracy) is 0.0184.

b) \( P_5 = P_0(1 + k)^6 = 50(1.0184)^5 \approx 54.78 \). Thus the population is estimated to be 55 squirrels.

10. a) Let \( P(n) \) be the proposition that \( \sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{4} \).

Then \( P(1) \) is \( \sum_{i=1}^{1} i^3 = \frac{1^2(1 + 1)^2}{4} = 1 \). The left-hand side gives \( \sum_{i=1}^{1} i^3 = 1^3 = 1 \) and the right-hand side gives \( \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1 \). Thus the proposition is true for \( n = 1 \).

Now assume that the proposition is true for \( n = k \), so \( \sum_{i=1}^{k} i^3 = \frac{k^2(k + 1)^2}{4} \).

We need to show that \( P(k + 1) \) is true, that is \( \sum_{i=1}^{k+1} i^3 = \frac{(k + 1)^2(k + 2)^2}{4} \).

\[
\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k + 1)^3 \\
= \frac{k^2(k+1)^2}{4} + (k + 1)^3 \\
= (k + 1)^2 \left( \frac{k^2}{4} + k + 1 \right) \\
= (k + 1)^2 \left( \frac{k^2 + 4k + 4}{4} \right) \\
= (k+1)^2(k+2)^2 \\
\]

Therefore, if \( P(k) \) is true, then \( P(k + 1) \) is true.

Hence, by mathematical induction \( \sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{4} \) for all integers \( n \geq 1 \).
b) Let \( P(n) \) be the proposition that \( \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \).

Then \( P(1) \) is \( \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1+1} \). The left-hand side gives \( \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \times 2} = \frac{1}{2} \)
and the right-hand side gives \( \frac{1}{1+1} = \frac{1}{2} \) Thus the proposition is true for \( n = 1 \).

Now assume that the proposition is true for \( n = k \), so \( \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1} \).

We need to show that \( P(k+1) \) is true, that is \( \sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2} \).

\[
\begin{align*}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{k+1(k+2)} \\
&= \frac{1}{k+1} \left( \frac{1}{k} + \frac{1}{k+2} \right) + \frac{1}{(k+1)(k+2)} \\
&= \frac{1}{k+1} \left( \frac{k+1}{k(k+2)} + \frac{1}{(k+1)(k+2)} \right) \\
&= \frac{1}{k+1} \left( \frac{k+1}{k^2 + 2k + 1} + \frac{1}{k+1} \right) \\
&= \frac{1}{k+1} \frac{(k+1)^2}{(k+1)^2} \\
&= \frac{k+1}{k+2}
\end{align*}
\]

Therefore, if \( P(k) \) is true, then \( P(k+1) \) is true.

Hence, by mathematical induction \( \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \) for all integers \( n \geq 1 \).

c) Let \( P(n) \) be the proposition that 5 divides \( 6^n - 1 + 5n \).

Then \( P(1) \) is 5 divides \( 6^1 - 1 + 5(1) \). Since \( 6^1 - 1 + 5(1) = 10 \) and 5 divides 10, the proposition is true for \( n = 1 \).

Now assume that the proposition is true for \( n = k \), so we assume that 5 divides \( 6^k - 1 + 5k \). Written in a more useful form, this means that \( 6^k - 1 + 5k = 5x \) for some integer \( x \).

We need to show that \( P(k+1) \) is true, that is, that 5 divides \( 6^{k+1} - 1 + 5(k+1) \).

\[
\begin{align*}
6^{k+1} - 1 + 5(k+1) &= 6^k \times 6 - 1 + 5k + 5 \\
&= (5x - 5k + 1) \times 6 - 1 + 5k + 5 \quad \text{since } P(k) \text{ is assumed to be true} \\
&= 30x - 30k + 6 - 1 + 5k + 5 \\
&= 30x - 25k + 10 \\
&= 5(6x - 5k + 2)
\end{align*}
\]

Since \( 6x - 5k + 2 \) is an integer, we conclude that 5 divides \( 6^{k+1} - 1 + 5(k+1) \). Thus, if \( P(k) \) is true, then \( P(k+1) \) is true. Hence, by mathematical induction, 5 divides \( 6^n - 1 + 5n \) for all integers \( n \geq 1 \).