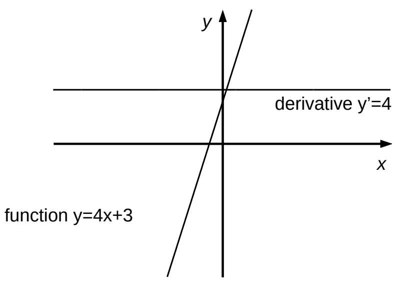
# 10 Derivatives

Derivatives appear in most areas of mathematics, and an understanding of them is vital. When maths is applied to other fields, more often than not the process of finding derivatives is an important component. Fortunately, many derivatives are governed by a small set of simple rules, which can make them quite easy to find.

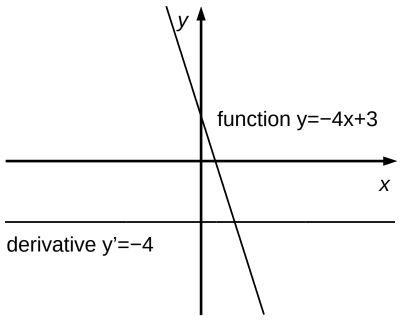
This section contains many references to functions. A *function* is a rule that produces a single output for any given input. Let's make $x$ our input variable and create a function whose output is equal to the input plus 2. We could call this function $f$, and it will be written like this: $f(x) = x + 2$. This means that $f$ is a function which, when $x$ is the input, is equal to $x + 2$. For example, $f(3) = 3 + 2 = 5$. We say the function evaluated at $x = 3$ is equal to 5. Similarly, $f(10) = 10 + 2 = 12$ and $f(a) = a + 2$. There are a few ways to write functions. The most common are $f(x) = x + 2$ and $y = x + 2$. In the second, $y$ is equal to the output variable. The symbols $y$ and $f(x)$ are interchangeable.

## 10.1 What is a derivative?

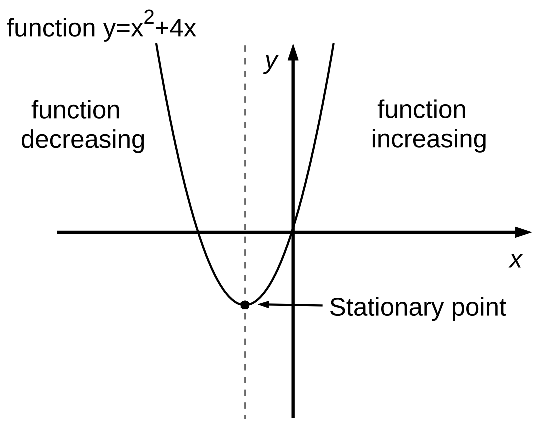
Given a function $f$, the *derivative* of $f$ at point $a$ is the rate of change of $f$ at $a$, which is the rate at which the value of $f$ changes near $a$ compared to the change in the value of $x$. This is the same as saying that the derivative of a function at point $a$ is the slope of the function at that point $a$. The derivative of $f$ at $x$ is written as $f'(x)$ or $\dfrac{df}{dx}$.
Consider the function $f(x) = 4x + 3$. We know that this represents a straight line, because it is in the linear form $y = mx + c$, where $m$ is the gradient of the function. Thus the rate of change of this function is the gradient of the function, which is 4. For each unit increase in $x$, $y$ increases by 4. If we take a closer look at the graph of $y = 4x + 3$ we can see that as $x$ increases, the function (that is, the $y$ value) also increases. The function has a positive slope, which means that the derivative must be positive. This can be seen by graphing the derivative $y' = 4$ (here, the derivative is the gradient, which is a constant). The derivative is positive because it is above the $x$-axis.



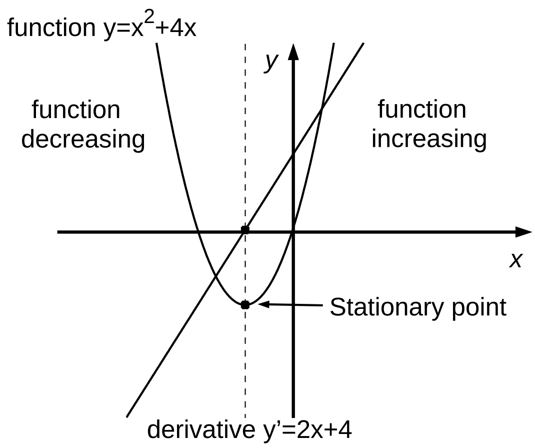

Now consider the function $y = -4x + 3$. The slope of this function is $-4$. Looking at the graph of the function we can see that as $x$ increases, the function decreases. This means the derivative must be negative, which we can see because $y' = -4$ is below the $x$-axis.




These same principles apply when we move onto more complex functions like quadratics. Consider the function $y = x^2 + 4x$, shown in the following figure.

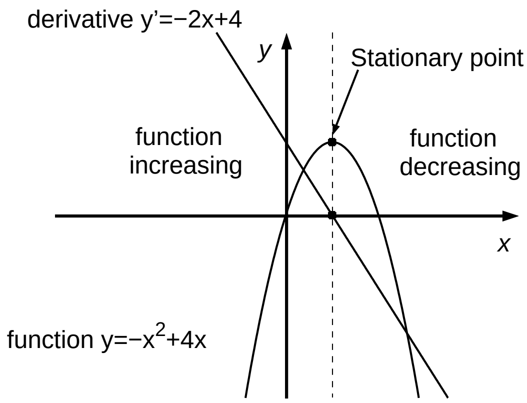



Now we have to be more careful when talking about the slope of the function. (With straight lines, the slope was the same at every point on the line.) Clearly, this function does not have a constant slope: it changes depending on at which point of the function you are looking. (Because such functions do not have a constant slope or gradient, we tend to use the phrase "rate of change" rather than slope. Sometimes we might say "instantaneous slope".) However we do know that as $x$ increases, the function first decreases and then increases (these sections of the function are separated by the dashed line on the diagram). This means the derivative must first be negative, then equal to zero (where the function is flat) and then positive. In fact, the derivative of a quadratic is linear (a straight line). This is shown on the following figure; in this case, the derivative is $y' = 2x + 4$.




If you look at the derivative on that diagram, you can see that is negative (below the $x$ axis) where the parabola is decreasing. You can also see that the derivative is positive (above the $x$-axis) where the parabola is increasing. There is one more important point, when the function is at a *stationary point*. A stationary point is when the rate of change of the function is zero, so the function is neither decreasing nor increasing. When there is a stationary point in the function, the derivative must be zero, so the graph of the derivative of that function will touch the $x$-axis at the same $x$ value that corresponds with the stationary point. You can see that this happened on the graph.

Now consider $y = -x^2 + 4x$. This function increases and then decreases. You can see how the derivative matches the function: it is positive, then zero, then negative. Once again the derivative crosses the $x$ axis at the same $x$ value as the stationary point in the function.




## 10.2 Basic rules for finding derivatives

There are three simple rules that allow the derivative of any polynomial to be calculated.

**Rule 1.** $f(x) = ax^n$ gives $f'(x) = nax^{n-1}$ if $a$ is a constant ($n \neq 0$).

**Rule 2.** $f(x) = c$ gives $f'(x) = 0$ if $c$ is a constant.

**Rule 3.** $f(x) = g(x) + h(x)$ gives $f'(x) = g'(x) + h'(x)$.

**Example** *10.2.1* Let $f(x) = 4$. Find $f'(x)$.

*Solution:* By Rule 2, $f'(x) = 0$. (This makes sense because $f(x) = 4$ is a horizontal line, so its slope is 0.)

**Example** *10.2.2* Let $f(x) = x^2$. Find $f'(x)$.

*Solution:* Using Rule 1 with $a = 1$ and $n = 2$ gives $f'(x) = 2 \times 1x^{2-1} = 2x$.

**Example** *10.2.3* Let $f(x) = 3x^2$. Find $f'(x)$.

*Solution:* By Rule 1, $f'(x) = 2 \times 3x^{2-1} = 6x$.

**Example** *10.2.4* Let $f(x) = 3x^2 + 4$. Find $f'(x)$.

*Solution:* We know the derivatives of $3x^2$ and 4, so by Rule 3, $f'(x) = 6x + 0 = 6x$.

**Example** *10.2.5* Let $f(x) = 3x^2 + 5x - 1$. Find $f'(x)$.

*Solution:* Let $f(x) = g(x) + h(x) + k(x)$, with $g(x) = 3x^2$, $h(x) = 5x$ and $k(x) = -1$.
Then by Rule 3, $f'(x) = g'(x) + h'(x) + k'(x)$.
By Rule 1, as $g(x) = 3x^2$, we know $g'(x) = 2 \times 3x^{2-1} = 6x$.
Similarly, as $h(x) = 5x$, $h'(x) = 1 \times 5x^0 = 5$.
Finally, by Rule 2, $k(x) = -1$ gives $k'(x) = 0$.
So $f'(x) = g'(x) + h'(x) + k'(x) = 6x + 5$.

We can use the derivative of a function to find the gradient (or slope) of the function at any point on the function. For example, consider the function $f(x) = 3x^2 + 4$. We know from Example 10.2.4 that the derivative of this function is $f'(x) = 6x$. Say that we wanted to find the gradient at the point $x = 4$. To do this, we substitute the value $x = 4$ into the derivative, giving $f'(4) = 6 \times 4 = 24$. We say that the derivative evaluated at $x = 4$ is equal to 24. This means that the gradient of the function $f(x) = 3x^2 + 4$ at the point $x = 4$ is 24.

**Example** *10.2.6* What is the gradient of the function $f(x) = -2x^2 + 4x$ at $x = 1$?

*Solution:* The gradient of the function $f(x)$ at any point is the value of the derivative $f'(x)$ at that point. From the rules above we have $f'(x) = -4x + 4$. So at $x = 1$, the gradient is $f'(1) = -4 + 4 = 0$. So at that point the gradient is zero.

Negative powers are handled in exactly the same way as positive powers.

**Example** *10.2.7* Let $f(x) = x^{-4} + 11x$. Find $f'(x)$.

*Solution:* Let $f(x) = g(x) + h(x)$, with $g(x) = x^{-4}$ and $h(x) = 11x$.
We know $f'(x) = g'(x) + h'(x)$.
Using Rule 1, $g'(x) = -4x^{-4-1} = -4x^{-5}$ and $h'(x) = 11$.
So $f'(x) = g'(x) + h'(x) = -4x^{-5} + 11$.

Indeed, it is also possible to find derivatives in this way when the powers involved are any rational numbers, not just integers.

**Example** *10.2.8* Let $f(x) = \sqrt{x}$. Find $f'(x)$.

*Solution:* We know $\sqrt{x} = x^{1/2}$. So using Rule 1, $f'(x) = \dfrac{1}{2} \times x^{-1/2} = \dfrac{1}{2x^{1/2}} = \dfrac{1}{2\sqrt{x}}$.

If you have trouble with manipulating powers in the example above, refer to Section 4.

You may notice that any constant in front of an $x^n$ term is essentially unchanged when the derivative is taken. This follows from a general rule, which governs the derivative of a term consisting of a constant multiplied by a function of $x$.

**Rule 4.** Let $f(x) = a \times g(x)$, with $a$ a constant. Then $f'(x) = a \times g'(x)$.

## 10.3 Derivatives of $\sin(x)$ and $\cos(x)$

Often it is necessary to find derivatives of the trigonometric functions. The rules governing this are surprisingly simple.

**Rule 5.** If $f(x) = \sin x$ then $f'(x) = \cos x$.

**Rule 6.** If $f(x) = \cos x$ then $f'(x) = -\sin x$.

Combining this with the previous rules allows a variety of derivatives to be found.

**Example** *10.3.1* Let $f(x) = 3x^2 + \cos x + 4$. Find $f'(x)$.

*Solution:* Using the rules above, we have $f'(x) = 6x - \sin x$.

**Example** *10.3.2* Let $f(x) = 4\sin x$. Find $f'(x)$.

*Solution:* Since 4 is a constant, $f'(x) = 4\cos x$.

## 10.4 Derivatives of $e^x$ and $\ln x$

The rules for the derivatives of $e^x$ and $\ln x$ are also quite simple.

**Rule 7.** If $f(x) = e^x$ then $f'(x) = e^x$.

**Rule 8.** If $f(x) = \ln(x)$ then $f'(x) = \dfrac{1}{x}$.

It is here that the significance of the number $e$ becomes more apparent. The function $f(x) = e^x$ has the unique property that $f'(x) = f(x)$. So the derivative of the function at any point along the function is equal to the value of the function at that point.

**Example** *10.4.1* Let $f(x) = -3e^x$. Find $f'(x)$.

*Solution:* Since $-3$ is a constant, $f'(x) = -3e^x$.

One of the most significant applications of derivatives comes from the fact that they allow functions to be maximised or minimised. A local maximum or minimum in a function will correspond to a position where the gradient of the function, and therefore the derivative of the function at that point, will be zero. Since functions can be used to model all manner of problems in science, business, economics and information technology, this ability to find local maxima and minima is extremely valuable.

## Practice Problems

Here are some problems for you to practice on, followed by answers. Fully worked solutions to these problems can be found in Section 11.10.

**Q10.1** Find the derivative of the following.

- **(a)** $f(x) = 4x^2 + 2x + 6$
- **(b)** $f(x) = 9x^3 - 5x^{-4} + 5$
- **(c)** $f(x) = \sqrt{x} + 2x^2$
- **(d)** $f(a) = a^4 - a^2 + a$

**Q10.2** Find the derivative of the following.

- **(a)** $f(x) = 4\cos x$
- **(b)** $f(x) = 2\sin x + 8x$
- **(c)** $f(a) = a^2 + \sin a - 4$
- **(d)** $f(x) = x^3 - 3\cos x + 2x$

**Q10.3** Find the derivative of the following.

- **(a)** $f(x) = 5e^x$
- **(b)** $f(x) = -7e^x + x^2$
- **(c)** $f(x) = 2\ln(x)$
- **(d)** $f(a) = 4a^2 + \ln(a)$

**A10.1** **(a)** $f'(x) = 8x + 2$

- **(b)** $f'(x) = 27x^2 + 20x^{-5}$
- **(c)** $f'(x) = \dfrac{1}{2\sqrt{x}} + 4x$
- **(d)** $f'(a) = 4a^3 - 2a + 1$

**A10.2** **(a)** $f'(x) = -4\sin x$

- **(b)** $f'(x) = 2\cos x + 8$
- **(c)** $f'(a) = 2a + \cos a$
- **(d)** $f'(x) = 3x^2 + 3\sin x + 2$

**A10.3** **(a)** $f'(x) = 5e^x$

- **(b)** $f'(x) = -7e^x + 2x$
- **(c)** $f'(x) = \dfrac{2}{x}$
- **(d)** $f'(a) = 8a + \dfrac{1}{a}$