1 Factorisation

1.1 Introduction

Suppose you were asked to evaluate 2(3+4), using the correct order of operations (sometimes called BEDMAS). You would first add the numbers in the brackets (giving 7) and then multiply 2 by 7. Your steps would be

$$2(3+4) = 2(7) = 14$$

However we can evaluate this expression in a different way. We first multiply the number outside the brackets by each term inside the brackets, and then add the results. This is called *expanding*, and the steps are

$$2(3+4) = 2 \times 3 + 2 \times 4 = 6 + 8 = 14$$

So given a product of terms involving brackets, like a(b+c), it is easy to multiply them together to expand the product, giving ab+ac.

Example *1.1.1* Expand $3(7-2)$.

Solution:

$$3(7-2) = 3 \times 7 - 3 \times 2 = 21 - 6 = 15$$

Example *1.1.2* Expand $2x(4+y)$.

Solution:

$$2x(4+y) = 2x \times 4 + 2x \times y = 8x + 2xy$$

The opposite process to expanding is called *factorising*. Given a sum of terms, we try to find a way to rewrite the sum as a product involving brackets. To do this we look for a *common factor* in the terms, which is an expression that evenly divides into all or some of the terms. Then the common factor goes outside the brackets. For example, consider the expression $32 + 24$. These two numbers are both divisible by 8, which is their highest common factor. We can now write $32 + 24$ as products of 8, so $32 + 24 = 8 \times 4 + 8 \times 3 = 8(4+3)$. The factored form of 32+24 is 8(4+3). Note that you can always check your answer by expanding.

Example *1.1.3* Factorise $3x^2 + 6x$.

Solution:

$$3x^2 + 6x = 3x \times x + 3x \times 2 = 3x(x+2)$$

Example *1.1.4* Factorise $2ab + 8b + 14bc$.

Solution:

$$2ab + 8b - 14bc = 2b \times a + 2b \times 4 - 2b \times 7c = 2b(a + 4 - 7c)$$

Example *1.1.5* Factorise $13n + 9d$.

Solution:

There are no common factors in these two terms, so the expression cannot be factorised.

You can also factorise an expression that includes terms that have already been arranged in factored form. For example, $2(x+1)+y(x+1)$ has two terms which have the common factor of $x+1$. This expression can be factorised further giving the expression $(x+1)(2+y)$. This process will become useful in factoring quadratics.

Example *1.1.6* Factorise $3xy + 6y + 2x + 4$.

Solution: $3xy + 6y + 2x + 4 = 3y(x+2) + 2(x+2) = (x+2)(3y+2)$

To expand expressions like $(x+2)(3y+2)$ you must multiply each term in the first brackets by each term in the second brackets. This method is sometimes called FOIL, because you multiply the first terms (F), the outer terms (O), the inner terms (I) and the last terms (L). For example,

$$\begin{aligned}(x+2)(3y+2) &= x \times 3y + x \times 2 + 2 \times 3y + 2 \times 2 \\ &= 3xy + 2x + 6y + 4\end{aligned}$$

Example *1.1.7* Expand $(x-3)(y+4)$.

Solution:

$$(x-3)(y+4) = x \times y + x \times 4 + -3 \times y + -3 \times 4 = xy + 4x - 3y - 12$$

1.2 Special cases

There are some special factorisations that are useful to remember. They are *perfect squares* and *the difference of two squares*.

Perfect squares

Here are two perfect squares in expanded and factorised form.

- $a^2 + 2ab + b^2 = (a+b)(a+b) = (a+b)^2$
  For example, $x^2 + 4x + 4 = (x+2)^2$ by putting $x = a$, $b = 2$.
- $a^2 - 2ab + b^2 = (a-b)(a-b) = (a-b)^2$
  For example, $x^2 - 6x + 9 = (x-3)^2$ by putting $x = a$, $b = 3$.

Example *1.2.1* Factorise $a^2 + 8a + 16$.

Solution:

$$a^2 + 8a + 16 = (a+4)(a+4) = (a+4)^2$$

Example *1.2.2* Factorise $x^2 - 4xy + 4y^2$.

Solution:

$$x^2 - 4xy + 4y^2 = (x - 2y)(x - 2y) = (x - 2y)^2$$

The difference of two squares

- $a^2 - b^2 = (a + b)(a - b)$
  For example, $x^2 - 49 = (x + 7)(x - 7)$ by putting $x = a$, $b = 7$.

Example *1.2.3* Factorise $x^2 - 1$.

Solution:

$$x^2 - 1 = (x)^2 - 1^2 = (x - 1)(x + 1)$$

Example *1.2.4* Factorise $4x^2 - 25$.

Solution:

$$4x^2 + 25 = (2x)^2 + 5^2 = (2x - 5)(2x + 5)$$

1.3 Factorising quadratics

A *quadratic* is an expression that can be written in the form

$$ax^2 + bx + c$$

where a, b and c are constants (with $a \neq 0$) and x is the unknown variable. We say a is the coefficient of x^2, b is the coefficient of x and c is the constant term. In a quadratic, b and/or c can be zero, however a can never be zero.

If a quadratic factorises, it must break up into two linear factors. For example,

$$x^2 + 8x + 7 = (x + 7)(x + 1).$$

Let's see how to factorise this quadratic. We are trying to rewrite $x^2 + 8x + 7$ in the form $(x + d)(x + e)$, where d and e are unknown numbers. Now $(x + d)(x + e)$ expands to $x^2 + (d + e)x + de$. Thus we have $x^2 + 8x + 7 = x^2 + (d + e)x + de$. Equating coefficients, we need to find d and e such that $d + e = 8$ and $de = 7$. Since the only factors of 7 are 1 and 7, we should take $d = 7$ and $e = 1$ (or vice versa). Now we can write the quadratic in factored form, $x^2 + 8x + 7 = (x + 7)(x + 1)$. Note that $(d + e)$ is the x coefficient b and de is the constant term c in the general quadratic expression.

Example *1.3.1* Factorise $x^2 + 8x + 12$.

Solution: First we need to find two numbers that multiply to 12 and add to 8. There are many factors of 12: (12,1), (6,2) and (4,3), but only one of these combinations adds to

8. That combination is 2 and 6. Therefore,

$$\begin{aligned} & x^2 + 8x + 12 \\ = \; & x^2 + 2x + 6x + 12 \\ = \; & x(x+2) + 6(x+2) \\ = \; & (x+2)(x+6) \end{aligned}$$

Example *1.3.2* Factorise $x^2 + x - 20$.

Solution: First we need to find two numbers that multiply to -20 and add to 1. These are -4 and 5. Therefore,

$$\begin{aligned} & x^2 + x - 20 \\ = \; & x^2 + 5x - 4x - 20 \\ = \; & x(x+5) - 4(x+5) \\ = \; & (x+5)(x-4) \end{aligned}$$

When the coefficient of x^2 is not 1, the method only changes slightly. To factorise ax^2+bx+c, we still need to find two numbers that add to give b, but now they must multiply to give $a \times c$. For example, to factorise $2x^2+9x+4$, two numbers that multiply to give 8 and add to give 9 are 8 and 1. Written in expanded form the expression $2x^2+9x+4$ becomes $2x^2+8x+x+4$, which this is easily factorised into $x(2x+1)+4(2x+1)$ and then $(2x+1)(x+4)$.

Example *1.3.3* Factorise $10x^2 + 11x + 3$.

Solution: First we need to find two numbers that multiply to give $10 \times 3 = 30$ and add to give 11. These two numbers are 5 and 6. Therefore,

$$\begin{aligned} & 10x^2 + 11x + 3 \\ = \; & 10x^2 + 5x + 6x + 3 \\ = \; & 5x(2x+1) + 3(2x+1) \\ = \; & (2x+1)(5x+3) \end{aligned}$$

Practice Problems

Here are some problems for you to practice on, followed by answers. Fully worked solutions to these problems can be found in Section 11.1.

Q1.1 Factorise the following.

- **(a)** $4y^2 - 16y$
- **(b)** $32ab + 16b + 8abc$
- **(c)** $6xyz + 3yz + 18wyz$

Q1.2 Factorise the following.

- **(a)** $4 + 4b + b^2$
- **(b)** $25a^2 - 10ad + d^2$
- **(c)** $x^2 + 12xy + 36y^2$
- **(d)** $144 - 48a + 4a^2$

Q1.3 Factorise the following.

- **(a)** $9y^2 - 36$
- **(b)** $64 - 121g^4$
- **(c)** $16x^6 - 49y^8$

Q1.4 Factorise the following.

- **(a)** $x^2 - 5x + 4$
- **(b)** $x^2 + 2x - 15$
- **(c)** $y^2 - 3y + 2$
- **(d)** $x^2 + 13x + 42$
- **(e)** $x^2 - 11x + 24$
- **(f)** $a^2 + 7a + 12$

Q1.5 Factorise the following.

- **(a)** $3x^2 + 17x + 10$
- **(b)** $2y^2 + 12y + 16$
- **(c)** $4x^2 - 8x - 12$
- **(d)** $2z^2 - z - 15$

A1.1 **(a)** $4y(y - 4)$
- **(b)** $8b(4a + 2 + ac)$
- **(c)** $3yz(2x + 1 + 6w)$

A1.2 **(a)** $(2 + b)^2$

(b) $(5a - d)^2$

(c) $(x + 6y)^2$

(d) $(12 - 2a)^2$

A1.3 **(a)** $(3y - 6)(3y + 6)$

(b) $(8 - 11g^2)(8 + 11g^2)$

(c) $(4x^3 - 7y^4)(4x^3 + 7y^4)$

A1.4 **(a)** $(x - 4)(x - 1)$

(b) $(x - 3)(x + 5)$

(c) $(y - 2)(y - 1)$

(d) $(x + 7)(x + 6)$

(e) $(x - 8)(x - 3)$

(f) $(a + 3)(a + 4)$

A1.5 **(a)** $(x + 5)(3x + 2)$

(b) $(y + 4)(2y + 4)$

(c) $(2x - 6)(2x + 2)$ or $(x - 3)(4x + 4)$ or $(x + 1)(4x - 12)$

(d) $(z - 3)(2z + 5)$