2 Manipulating fractions

2.1 Introduction

A fraction contains two parts, a numerator and a denominator. In the fraction $\frac{a}{b}$, the numerator is a and the denominator is b. This represents the quantity $a \div b$. It is not possible to divide by zero, so the denominator of a fraction is not allowed to be 0.

2.2 Equivalent fractions

The fractions $\frac{1}{3}$ and $\frac{2}{6}$ are *equivalent*, that is, both fractions represent the same amount. If you take any fraction and multiply or divide both the numerator and the denominator by the same (non-zero) quantity, you obtain an equivalent fraction. For example,

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}.$$

Equivalent fractions help you to write a fraction in its simplest form, also known as lowest terms. To find the simplest form of a fraction you must divide out any common factors from the numerator and denominator. For example, $\frac{14}{21} = \frac{14 \div 7}{21 \div 7} = \frac{2}{3}$. There are no common factors between 2 and 3, so $\frac{2}{3}$ is the simplest form. Unless told otherwise, you should always express fractions in their simplest form.
When dealing with algebraic fractions, be careful that you only divide out common factors and not single terms. For example,

$$\frac{4+2x}{2} = \frac{2(2+x)}{2} = 2 + x.$$

Some students are tempted to divide the 2 into the 4 only and write $\frac{4+2x}{2} = 2 + 2x$. This is WRONG.

Example *2.2.1* Simplify $\dfrac{40xy}{64x}$ (provided that $x \neq 0$).

Solution:

$$\frac{40xy}{64x} = \frac{40xy \div 8x}{64x \div 8x} = \frac{5y}{8}$$

Example *2.2.2* Simplify $\dfrac{9a+27}{6a+18}$ (provided that $a \neq -3$).

Solution:

$$\frac{9a+27}{6a+18} = \frac{9(a+3)}{6(a+3)} = \frac{9(a+3) \div 3(a+3)}{6(a+3) \div 3(a+3)} = \frac{3}{2}$$

One point that must always be remembered is that we cannot divide by 0. Thus in Example 2.2.1 the answer is only valid provided that $x \neq 0$. Again, in Example 2.2.2 the answer is only valid if $a \neq -3$.

2.3 Arithmetic of fractions

Fractions can be added, subtracted, multiplied and divided. Although many calculators can perform calculations with fractions, it is important that you know how to manipulate fractions, because when a fraction involves a variable you cannot simply 'put it in your calculator'.

To **add** fractions they must have the **same denominator**. To determine the sum of two fractions write down equivalent fractions that have the same denominator, then add the numerators together and place that sum over the common denominator.

Example *2.3.1* Simplify $\frac{1}{6} + \frac{3}{8}$.

Solution: What common denominator can we use? We need a number divisible by both 6 and 8 so we use 24.

$$\frac{1}{6} + \frac{3}{8} = \frac{4}{24} + \frac{9}{24} = \frac{13}{24}$$

Example *2.3.2* Simplify $\frac{5a}{6d} + \frac{4a}{3c}$ (provided that $c, d \neq 0$).

Solution: This time the common denominator must be divisible by $3c$ and $6d$ so we use $6cd$.

$$\frac{5a}{6d} + \frac{4a}{3c} = \frac{5ac}{6cd} + \frac{8ad}{6cd} = \frac{5ac + 8ad}{6cd} = \frac{a(5c + 8d)}{6cd}$$

Subtraction is very similar to addition except that the numerators are subtracted from each other, instead of added. Remember that the denominators must be the same.

Example *2.3.3* Simplify $\frac{8}{9} - \frac{2}{3}$.

Solution:

$$\frac{8}{9} - \frac{2}{3} = \frac{8}{9} - \frac{6}{9} = \frac{2}{9}$$

Example *2.3.4* Simplify $\frac{4x}{7y} - \frac{3}{14x}$ (provided that $x, y \neq 0$).

Solution:

$$\frac{4x}{7y} - \frac{3}{14x} = \frac{4x \times 2x}{7y \times 2x} - \frac{3 \times y}{14x \times y} = \frac{8x^2}{14xy} - \frac{3y}{14xy} = \frac{8x^2 - 3y}{14xy}$$

To multiply two fractions, multiply the numerators to obtain the new numerator and multiply the denominators to obtain the new denominator. (There is no need for the denominators to be the same.) You will often be able to simplify your answer.

Example *2.3.5* Simplify $\frac{5}{12} \times \frac{4}{5}$.

Solution:

$$\frac{5}{12} \times \frac{4}{5} = \frac{5 \times 4}{12 \times 5} = \frac{20}{60} = \frac{1}{3}$$

Example *2.3.6* Simplify $\dfrac{ac}{6} \times \dfrac{2}{ba}$ (provided that $a, b \neq 0$).

Solution:

$$\frac{ac}{6} \times \frac{2}{ba} = \frac{2ac}{6ba} = \frac{c}{3b}$$

Dividing by d is the same as multiplying by $\frac{1}{d}$. The reciprocal of a fraction can be obtained by inverting it (swapping its numerator and denominator). To divide fractions, change the division sign to a multiplication sign and invert the second fraction. Then multiply the fractions as shown above.

Example *2.3.7* Simplify $\dfrac{4}{7} \div \dfrac{8}{11}$.

Solution:

$$\frac{4}{7} \div \frac{8}{11} = \frac{4}{7} \times \frac{11}{8} = \frac{44}{56} = \frac{11}{14}$$

Example *2.3.8* Simplify $\dfrac{5ab}{7c} \div \dfrac{15b}{6a}$ (provided that $a, b, c \neq 0$).

Solution:

$$\frac{5ab}{7c} \div \frac{15b}{6a} = \frac{5ab}{7c} \times \frac{6a}{15b} = \frac{30a^2b}{105bc} = \frac{30a^2b \div 15b}{105bc \div 15b} = \frac{2a^2}{7c}$$

Practice Problems

Here are some problems for you to practice on, followed by answers. Fully worked solutions to these problems can be found in Section 11.2.

Q2.1 Simplify the following

- **(a)** $\dfrac{14x}{18x}$ $(x \neq 0)$
- **(b)** $\dfrac{9ab}{27bc}$ $(b, c \neq 0)$
- **(c)** $\dfrac{8+4d}{2e}$ $(e \neq 0)$

Q2.2 Calculate the following

- **(a)** $\dfrac{3}{12} + \dfrac{1}{4}$
- **(b)** $\dfrac{1}{3} + \dfrac{4}{15}$
- **(c)** $\dfrac{6}{7} - \dfrac{2}{3}$
- **(d)** $\dfrac{13}{18} - \dfrac{7}{9}$

Q2.3 Calculate the following

- **(a)** $\dfrac{4}{11} \times \dfrac{7}{12}$
- **(b)** $\dfrac{3}{13} \div \dfrac{8}{9}$
- **(c)** $\dfrac{5}{8} \times \dfrac{2}{7}$
- **(d)** $\dfrac{3}{8} \div \dfrac{5}{6}$

Q2.4 Calculate the following

- **(a)** $\dfrac{ab}{4} + \dfrac{2c}{8d}$ $(d \neq 0)$
- **(b)** $\dfrac{4xy}{9} + \dfrac{3x}{12}$
- **(c)** $\dfrac{30gh}{14} - \dfrac{5h}{7g}$ $(g \neq 0)$
- **(d)** $\dfrac{9a}{24} - \dfrac{5}{8b}$ $(b \neq 0)$

Q2.5 Calculate the following

- **(a)** $\dfrac{12g}{16h} \times \dfrac{h}{2}$ $(h \neq 0)$

(b) $\frac{4e}{45} \times \frac{9d}{f}$ $(f \neq 0)$

(c) $\frac{xy}{7} \div \frac{x}{14}$ $(x \neq 0)$

(d) $\frac{5a}{7} \div \frac{7}{9a}$ $(a \neq 0)$

A2.1 **(a)** $\frac{7}{9}$

(b) $\frac{a}{3c}$

(c) $\frac{4+2d}{e}$

A2.2 **(a)** $\frac{1}{2}$

(b) $\frac{3}{5}$

(c) $\frac{4}{21}$

(d) $-\frac{1}{18}$

A2.3 **(a)** $\frac{7}{33}$

(b) $\frac{27}{104}$

(c) $\frac{5}{28}$

(d) $\frac{9}{20}$

A2.4 **(a)** $\frac{abd+c}{4d}$

(b) $\frac{x(16y+9)}{36}$

(c) $\frac{5h(3g^2-1)}{7g}$

(d) $\frac{3ab-5}{8b}$

A2.5 **(a)** $\frac{3g}{8}$

(b) $\frac{4ed}{5f}$

(c) $2y$

(d) $\frac{45a^2}{49}$