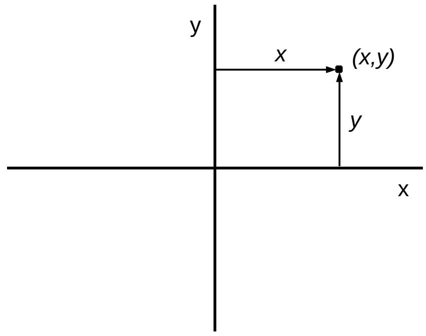
9 Graphing lines and parabolas

9.1 Introduction

The graph of an equation is a picture representation of the set of points that are solutions of the equation. It shows the relationship between the variables, in a coordinate system. In this section we will use the Cartesian Plane as our coordinate system and deal with equations that have no more than two variables. Each point in the Cartesian Plane corresponds to an ordered pair (x, y), where x and y are the x-coordinate and y-coordinate respectively.



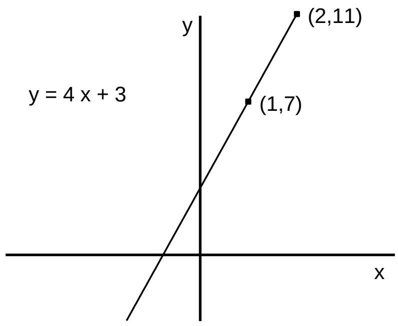
9.2 Lines

The graph of a linear equation is a line. Linear equations have a general form of

$$y = mx + c,$$

where m is the gradient or slope of the line, c is the y-intercept, x is the independent variable and y is the dependent variable.

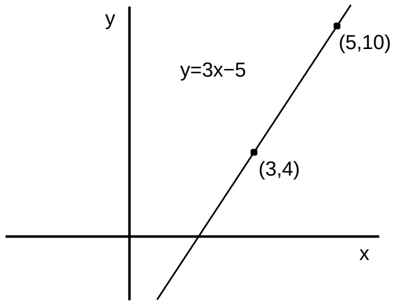
There are several ways to sketch a line: all you require is two points, or one point and the gradient. An easy way to sketch a line is to substitute two different values of x into the equation and calculate the two corresponding values of y. This will give you two points on the line. Plot those points and then draw a straight line through them.
For example consider the equation $y = 4x + 3$. If we substitute the x values of 1 and 2 into the equation, the corresponding y values are 7 and 11 respectively. This gives us two points, $(1, 7)$ and $(2, 11)$, which can be plotted. Draw a straight line through the points giving a sketch of $y = 4x + 3$.



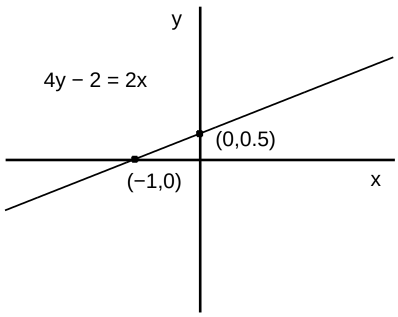
Example *9.2.1* Sketch $y = 3x - 5$ using x values of 3 and 5.

Solution: Substitute the x values into the equation.

For $x = 3$, $y = 9 - 5 = 4$
For $x = 5$, $y = 15 - 5 = 10$
The points are $(3, 4)$ and $(5, 10)$.



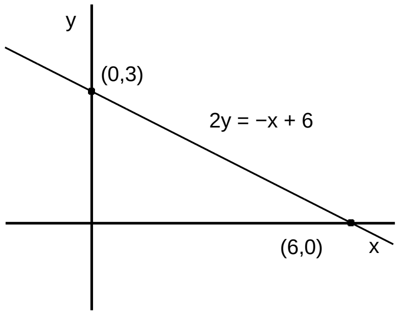
It is often useful to choose the x and y intercepts as your two points. The x intercept is where the line crosses the x-axis. At this point $y = 0$. So to find the x-intercept, substitute zero for y (so we are solving the equation for $y = 0$). The y-intercept is where the line crosses the y-axis. At this point $x = 0$. So to find the y-intercept substitute zero for x (so we are solving the equation for $x = 0$).
For example, consider the equation $4y - 2 = 2x$. To find the x-intercept we solve $0 - 2 = 2x$ which gives $x = -1$. To find the y-intercept we solve $4y - 2 = 0$ which gives $y = 1/2$. Thus these points are $(-1, 0)$ and $(0, 1/2)$, and can be plotted on the graph. Draw a straight line through the points and the result is a sketch of $4y - 2 = 2x$. The method of finding intercepts will also be used to sketch other types of functions.



If the equation is in the form $y = mx + c$ the y-intercept is c (using $x = 0$) and can be immediately read from the equation.

Example *9.2.2* Sketch $2y = -x + 6$ by finding the x and y intercepts.

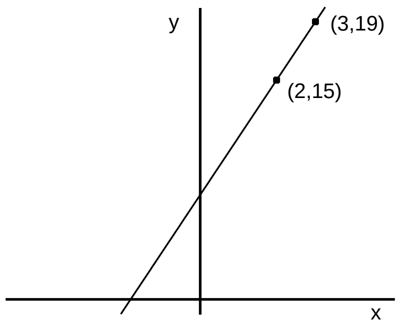
Solution:
To find the x-intercept, let $y = 0$. So $0 = -x + 6$, that is $x = 6$.
To find the y-intercept, let $x = 0$. So $2y = 6$, that is $y = 3$.
The points are $(6, 0)$ and $(0, 3)$.



It is useful to know how to find the equation of a line from the graph of the line. As was the case when drawing the graph, all that is needed to find the equation is two points on the line. Using these two points we can find the gradient (slope) of the line and then solve the equation for the y-intercept. These two values can be used to write the equation of the line in the form $y = mx + c$, where m is the gradient and c is the y-intercept.
The gradient is defined as the change in y divided by the change in x, often written as $\frac{\Delta y}{\Delta x}$. It can be calculated by substituting two points on the line (x_1, y_1) and (x_2, y_2) into the following formula.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that it doesn't matter which point you choose as (x_1, y_1) and which you choose as (x_2, y_2). You will get the same answer both ways. Consider the graph below.



The two points given are $(2, 15)$ and $(3, 9)$. We can now calculate the gradient of the line.

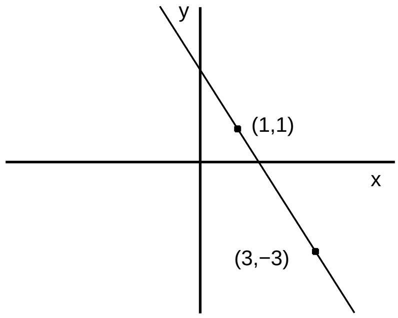
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{19 - 15}{3 - 2} = 4$$

To find c we solve $y = mx + c$ using the value for m that we just calculated, along with one of the points on the line. (Again, it doesn't matter which point you choose.) Using the first point $(2, 15)$, this gives

$$\begin{aligned} y &= mx + c \\ 15 &= 4 \times 2 + c \\ 15 &= 8 + c \\ 7 &= c \end{aligned}$$

Substituting m and c into the general equation we now have the equation for the line given in the graph above, $y = 4x + 7$.

Example *9.2.3* Determine the equation of line in the graph given below.



Solution: The two points given are $(1, 1)$ and $(3, -3)$.
First we find the gradient m:

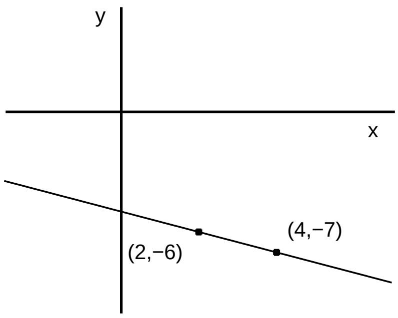
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-1}{3-1} = \frac{-4}{2} = -2$$

Next we find the y-intercept, c, by using the point (1,1):

$$\begin{aligned} y &= mx + c \\ 1 &= -2 \times 1 + c \\ 1 &= -2 + c \\ 3 &= c \end{aligned}$$

Substituting m and c into the general equation we now have the equation for the line given in the graph above, $y = -2x + 3$.

Example *9.2.4* Determine the equation of line in the graph given below.



Solution: The two points given are $(2, -6)$ and $(4, -7)$.
First we find the gradient m:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7-(-6)}{4-2} = -\frac{1}{2}$$

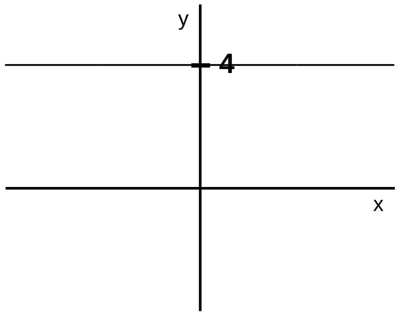
Next we find the y-intercept, c, by using the point $(2, -6)$:

$$\begin{aligned} y &= mx + c \\ -6 &= -\frac{1}{2} \times 2 + c \\ -6 &= -1 + c \\ -5 &= c \end{aligned}$$

Substituting m and c into the general equation we now have the equation for the line given in the graph above, $y = -\frac{1}{2}x - 5$.

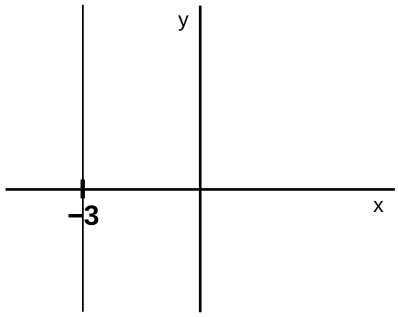
There are two types of lines with special equations: horizontal lines and vertical lines. A horizontal line has 0 slope, so its equation has $m = 0$, so it is of the form $y = c$, where c is a constant. A vertical line has an infinite slope, so its equation must have a different form. In fact, every vertical line has the equation $x = b$, where b is a constant.

Example *9.2.5* Determine the equation of line in the graph given below.



Solution: This is a horizontal line, so it has 0 slope. For any x value, y is always 4. Hence the equation for this line is $y = 4$.

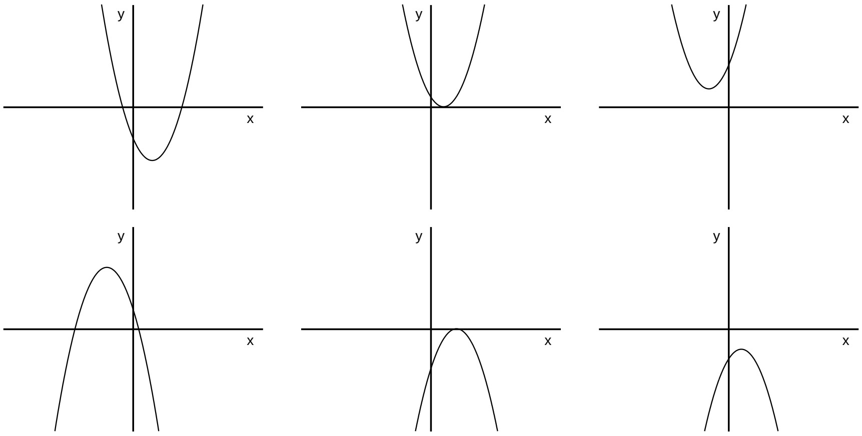
Example *9.2.6* Determine the equation of line in the graph given below.



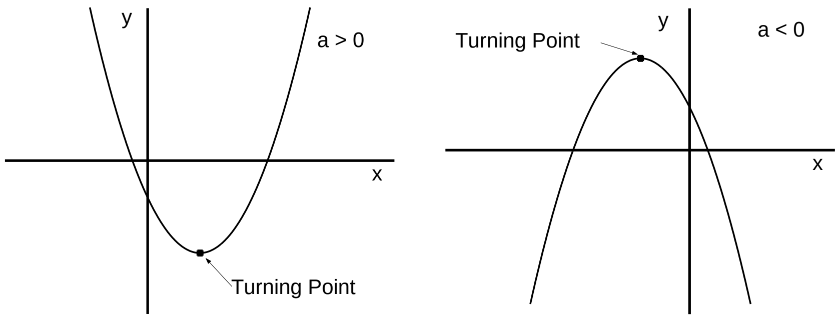
Solution: This is a vertical line, so it has infinite slope. For any y value, x is always -3. Hence the equation for this line is $x = -3$.

9.3 Parabolas

The graph of a quadratic equation is a parabola. Quadratic equations have the general form of $y = ax^2 + bx + c$, where a,b and c are constants (a is not zero), and x, y are the independent and dependent variables respectively. The general shape of a parabola is shown in the following figure. Parabolas can have two, one or no x-intercepts. The x-intercepts are called *roots*, and are solutions to the quadratic equation. The method for solving quadratic equations is given in Section 6.



In addition to any roots, parabolas have two other important points: the *turning point* and the y-intercept. The turning point is the point at which the parabola changes direction, creating a maximum (hill) or minimum (valley). Whether the parabola has a maximum or a minimum is determined by the sign of the constant a in the quadratic equation. If a is positive, the parabola will have a minimum. If a is negative, the parabola will have a maximum. All parabolas are symmetric about the vertical line through the turning point. This fact will help when sketching the curve.



To sketch a parabola we must find all the important points. The x-intercepts are the roots, and can be found by substituting 0 for y and solving the quadratic equation. The y-intercept can be found by substituting zero for x; the y-intercept will be equal to the constant term from the equation. The turning point can be found by using a formula. The x-coordinate of the turning point has the value $\frac{-b}{2a}$, where a and b are the constants in the quadratic equation. To find the y-coordinate, substitute the x-value you have just calculated into the equation. Once all the important points have been found, we can plot them and draw a smooth curve passing through them, remembering that the graph will be a parabola that is symmetric about the vertical line through the turning point.

Example *9.3.1* Sketch $2x^2 + 9x + 4$.

Solution:
The roots:
Substituting 0 for y gives $0 = 2x^2 + 9x + 4$. After factorisation we have $0 = (2x + 1)(x + 4)$

$$\text{which gives } x = -\frac{1}{2} \text{ and } x = -4.$$

The y-intercept:

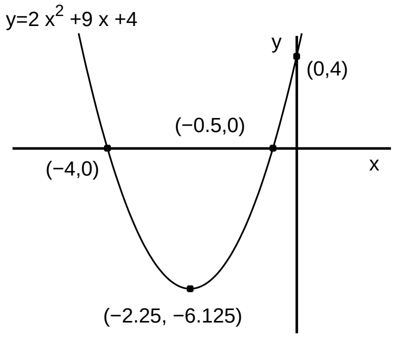
$$\text{Substituting 0 for } x \text{ gives } y = 4.$$

The turning point:

$$\text{The } x\text{-coordinate is } \frac{-b}{2a} = \frac{-9}{4}.$$

$$\text{The } y\text{-coordinate is } y = 2 \times \left(\frac{-9}{4}\right)^2 + 9 \times \frac{-9}{4} + 4 = -\frac{49}{8}.$$

The points to plot are: $(-0.5, 0)$, $(-4, 0)$, $(0, 4)$ and $(-2.25, -6.125)$.



Example *9.3.2* Sketch $-x^2 + 6x - 9$.

Solution:
The roots:

Substituting 0 for y gives $0 = -x^2 + 6x - 9$. After factorisation we have $0 = (-x + 3)(x - 3)$ which gives $x = 3$.

The y-intercept:

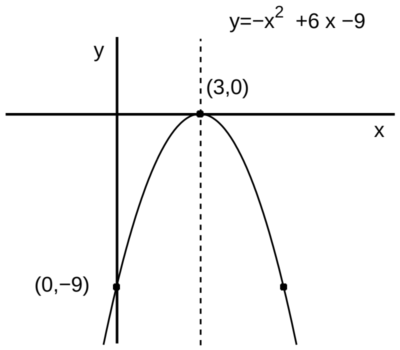
Substituting 0 for x gives $y = -9$.

The turning point:

$$\text{The } x\text{-coordinate is } \frac{-b}{2a} = \frac{-6}{-2} = 3.$$

$$\text{The } y\text{-coordinate is } y = -(3)^2 + 6 \times 3 - 9 = 0.$$

The points to plot are: $(3, 0)$ and $(0, -9)$. Use the fact the graph is symmetric about the vertical line through the turning point to help you draw the curve.



Example *9.3.3* Sketch $4x^2 - 10x + 7$.

Solution:
The roots:

$$\text{Substituting 0 for } y \text{ gives } 0 = 4x^2 - 10x + 7.$$
Using the quadratic formula we find there are no real roots.

The y-intercept:

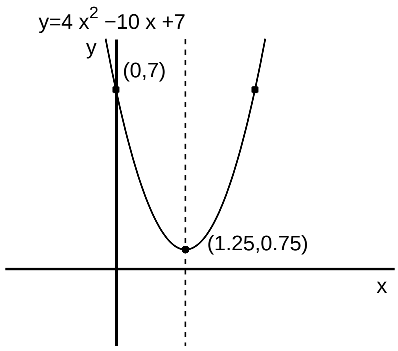
$$\text{Substituting 0 for } x \text{ gives } y = 7.$$

The turning point:

$$\text{The } x\text{-coordinate is } \frac{-b}{2a} = \frac{10}{8} = \frac{5}{4}.$$

$$\text{The } y\text{-coordinate is } y = 4 \times \left(\frac{5}{4}\right)^2 - 10 \times \frac{5}{4} + 7 = \frac{3}{4}.$$

The points to plot are: $(0, 7)$ and $(1.25, 0.75)$. Use the fact the graph is symmetric about the vertical line through the turning point to help you draw the curve.



Practice Problems

Here are some problems for you to practice on, followed by answers. Fully worked solutions to these problems can be found in Section 11.9.

Q9.1 Sketch the following lines.

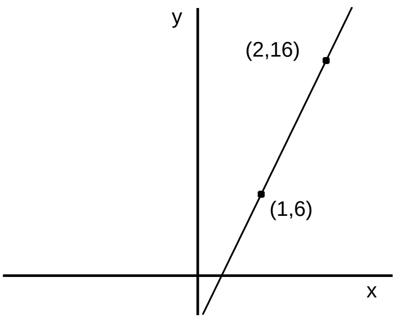
(a) $y = 4x - 7$

(b) $3y = -2x + 4$

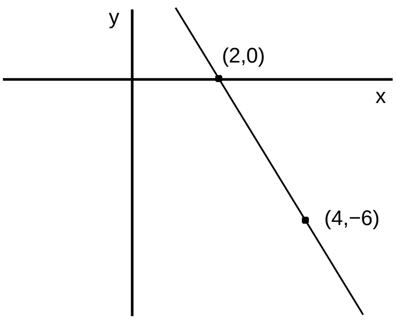
(c) $-2y = \frac{x}{2} - 3$

Q9.2 Determine the equations for the following lines.

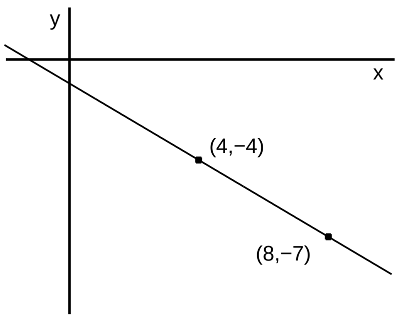
(a)



(b)



(c)



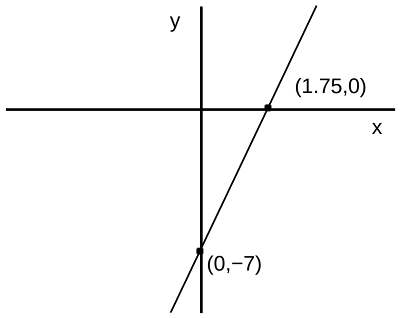
Q9.3 Sketch the following parabolas.

(a) $y = 4x^2 - 25$

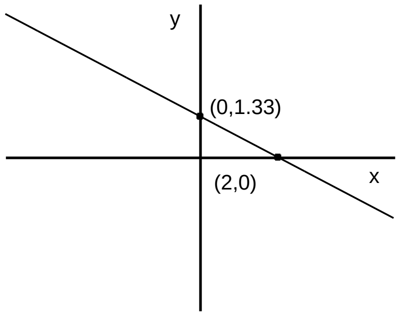
(b) $y = x^2 + 4x + 5$

(c) $y = -x^2 + 4x - 4$

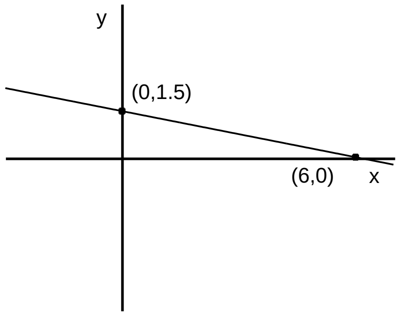
A9.1 **(a)** $y = 4x - 7$



(b) $3y = -2x + 4$



(c) $-2y = \dfrac{x}{2} - 3$

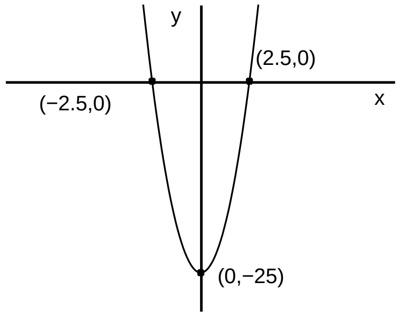


A9.2 **(a)** $y = 10x - 4$

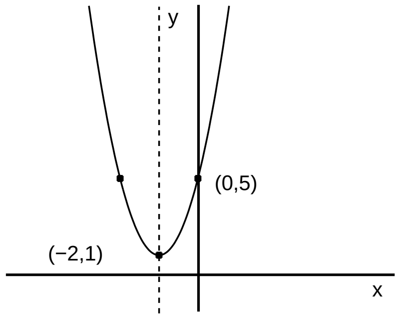
(b) $y = -3x + 6$

(c) $y = -\frac{3}{4}x - 1$

A9.3 **(a)** $y = 4x^2 - 25$



(b) $y = x^2 + 4x + 5$



(c) $y = -x^2 + 4x - 4$