4 Powers

4.1 Introduction

Consider the following equations.

$$10 \times 10 = 100$$
$$10 \times 10 \times 10 = 1000$$
$$10 \times 10 \times 10 \times 10 = 10000$$

Each of these equations can be written in a shorter form.

$$10^2 = 100$$
$$10^3 = 1000$$
$$10^4 = 10000$$

In general, a product such as

$$\overbrace{a \times a \times a \times \cdots \times a}^{n\ terms}$$

can always be expressed in the form a^n. In this case a is known as the base, and n as the exponent, power or index. This notation is simple to use, and allows many calculations to be performed more easily.

When performing operations using powers, there are several laws that make calculations easier. These are known as the index or power laws.

Index Laws

$$a^m \times a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{m \times n} \qquad (a \times b)^m = a^m \times b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad a^0 = 1$$

$$a^{-n} = \frac{1}{a^n} \qquad a^{m/n} = (\sqrt[n]{a}\,)^m = \sqrt[n]{a^m}$$

Below we list each rule along with a brief illustration. In practice, it is sufficient to simply apply the rule without including the intermediate working.

1. **Multiplication** $\boxed{a^m \times a^n = a^{m+n}}$

For instance,

$$2^3 \times 2^4 \;=\; (\,2 \times 2 \times 2\,) \;\times\; (\,2 \times 2 \times 2 \times 2\,) \;=\; 2^7 \;=\; 2^{3+4}$$

2. **Division** $\boxed{\dfrac{a^m}{a^n} = a^{m-n}}$

For instance,

$$\frac{3^5}{3^2} \;=\; \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3} \;=\; 3^3 \;=\; 3^{5-2}$$

3. **Powers** $\boxed{(a^m)^n = a^{m \times n}}$

This illustration shows how the power law follows from repeated applications of the multiplication law.

$$(2^4)^3 \;=\; 2^4 \times 2^4 \times 2^4 \;=\; 2^{4+4+4} \;=\; 2^{12} \;=\; 2^{4\times 3}$$

4. **Products** $\boxed{(a \times b)^m = a^m \times b^m}$

For instance,

$$(3 \times 5)^2 \;=\; 3 \times 5 \;\times\; 3 \times 5 \;=\; 3 \times 3 \times 5 \times 5 \;=\; 3^2 \times 5^2$$

5. **Quotients** $\boxed{\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}}$

For instance,

$$\left(\frac{3}{5}\right)^3 \;=\; \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \;=\; \frac{3 \times 3 \times 3}{5 \times 5 \times 5} \;=\; \frac{3^3}{5^3}$$

6. **Zero** $\boxed{a^0 = 1}$

This illustration shows how the zero law follows directly from the quotient law.

$$3^0 \;=\; 3^{1-1} \;=\; \frac{3^1}{3^1} \;=\; \frac{3}{3} \;=\; 1$$

7. **Negatives** $\boxed{a^{-n} = \dfrac{1}{a^n}}$

This illustration shows how the negatives law follows directly from the quotient law and the zero law.

$$2^{-3} \;=\; 2^{0-3} \;=\; \frac{2^0}{2^3} \;=\; \frac{1}{2^3}$$

8. **Fractional Powers** $\boxed{a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}}$

For instance,

$$8^{2/3} \;=\; (8^{1/3})^2 \;=\; (\sqrt[3]{8})^2 \;=\; 2^2 \;=\; 4$$

or

$$8^{2/3} \;=\; (8^2)^{1/3} \;=\; \sqrt[3]{8^2} \;=\; \sqrt[3]{64} \;=\; 4$$

It is important to remember that $x^{1/2}$ is equivalent to $\sqrt{x}$.

A combination of these rules can be used to simplify many equations involving powers. Remember that you can only combine terms that have the same base.

Example *4.1.1* Simplify $\dfrac{x^4 \times x^2}{x^3}$

Solution: $$\frac{x^4 \times x^2}{x^3} = \frac{x^{4+2}}{x^3} = \frac{x^6}{x^3} = x^{6-3} = x^3$$

Example *4.1.2* Simplify $\dfrac{(x^2 \times y^{-3})^2}{x^{-2} \times y^4}$

Solution: $$\frac{(x^2 \times y^{-3})^2}{x^{-2} \times y^4} = \frac{x^{2\times 2} \times y^{-3\times 2}}{x^{-2} \times y^4} = \frac{x^4 \times y^{-6}}{x^{-2} \times y^4} = x^{4-(-2)} \times y^{-6-4} = x^6 \times y^{-10}$$

Example *4.1.3* Evaluate $\dfrac{2^9 \times 2^{-3}}{16}$

Solution: $$\frac{2^9 \times 2^{-3}}{16} = \frac{2^9 \times 2^{-3}}{2^4} = \frac{2^{9-3}}{2^4} = \frac{2^6}{2^4} = 2^{6-4} = 2^2 = 4$$

Example *4.1.4* Evaluate $27^{2/3}$

Solution: $$27^{2/3} = (27^{1/3})^2 = ((3^3)^{1/3})^2 = (3^{3\times 1/3})^2 = (3^1)^2 = 3^2 = 9$$

Example *4.1.5* Simplify $\dfrac{(x^{3/2} \times y^{-2})^2}{y^{-7/2} \times x^3}$

Solution:

$$\frac{(x^{3/2} \times y^{-2})^2}{y^{-7/2} \times x^3} = \frac{x^{3/2 \times 2} \times y^{-2 \times 2}}{y^{-7/2} \times x^3} = \frac{x^3 \times y^{-4}}{y^{-7/2} \times x^3} = x^{3-3} \times y^{-4-(-7/2)}$$

$$= x^0 \times y^{-1/2} = 1 \times y^{-1/2} = \frac{1}{y^{1/2}} \quad \left(= \frac{1}{\sqrt{y}} \right)$$

Practice Problems

Here are some problems for you to practice on, followed by answers. Fully worked solutions to these problems can be found in Section 11.4.

Q4.1 Write 9^5 as a product.

Q4.2 Simplify

(a) $x^2 \times x^3$

(b) $\dfrac{y^5}{y^2}$

(c) $(x^4)^3$

(d) $(2x)^3$

(e) x^{-3}

Q4.3 Evaluate (without a calculator)

(a) $\dfrac{2^{13} \times 2^4}{2^{14}}$

(b) $125^{2/3}$

(c) $(x \times y)^0$

Q4.4 Simplify $\dfrac{(x^7 \times y^{-4})^2}{(y^{12} \times x^3)^{-3}}$

Q4.5 Simplify $\sqrt{\dfrac{(x^{-3} \times y)^{-2}}{x^4 \times y^{-6}}}$ (provided that $x > 0$ and $y \neq 0$)

Q4.6 Simplify $\dfrac{2^{14} \times 9^{-3} \times 5^{-1}}{25 \times 8^2 \times 3^{-11}}$

A4.1 $9 \times 9 \times 9 \times 9 \times 9$

A4.2 **(a)** x^5

(b) y^3

(c) x^{12}

(d) $8x^3$

(e) $\dfrac{1}{x^3}$

A4.3 **(a)** 8

(b) 25

(c) 1

A4.4 $x^{23} \times y^{28}$

A4.5 $x \times y^2$

A4.6 $2^8 \times 3^5 \times 5^{-3}$