6 Solving quadratic equations

6.1 Introduction

Here are six quadratic equations in the variable x. Each equation includes a term that involves x^2, but no higher powers of x appear.

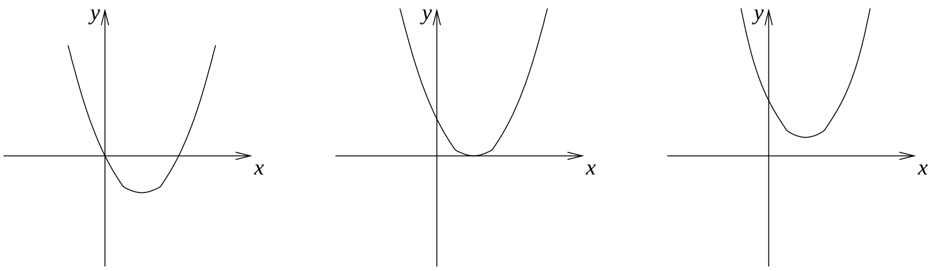
$7x^2 + 4x + 6 = 0$	$6x^2 = 3x - 17$	$7x^2 + x - 16 = 0$
$5x^2 = 0$	$x^2 = 8$	$\frac{1}{2}x^2 + 8x = 40$

A *quadratic equation* is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b and c are constants (with $a \neq 0$) and x is the unknown variable. We say a is the coefficient of x^2, b is the coefficient of x and c is the constant term. In a quadratic equation, b and/or c can be zero, however a cannot be zero. Each of the examples listed above can be rearranged and written in the form $ax^2 + bx + c = 0$.

An important feature of quadratic equations is that they can have two real solutions, or one real solution, or no real solutions. The graph of a quadratic function $y = ax^2 + bx + c$ is a parabola. Refer to Section 9 for information on how to graph a parabola. The solutions of $ax^2 + bx + c = 0$ are the values of x where the parabola touches the x-axis. If you imagine the shape of a parabola (opening upwards or downwards), it should be easy to see that it can touch the x-axis twice, once, or not at all.



It is important for you to be able to solve quadratic equations. We will discuss two methods for doing so.

- Factorisation (Section 6.2)
- The quadratic formula (Section 6.3)

Which method you choose to use will depend on the quadratic equation you need to solve. If the quadratic equation can be factorised, then that is usually the quickest method to obtain a solution. However, sometimes it is hard, or even impossible, to factorise a quadratic equation. In that case, the quadratic formula can be used.

6.2 Factorisation

If you can factorise a quadratic expression $ax^2 + bx + c$, then it is easy to obtain the solutions from the factored form of the equation. The reason for this is that if the product of two things equals zero, then one (or both) of those things must be zero. If you do not know how to factorise a quadratic expression, refer to Section 1.

Example *6.2.1* Solve $x^2 + x - 20 = 0$.
Solution: We start by factorising $x^2 + x - 20 = (x + 5)(x - 4)$.

$$\begin{aligned} x^2 + x - 20 &= 0 \\ (x + 5)(x - 4) &= 0 \\ x + 5 = 0 \quad &\text{or} \quad x - 4 = 0 \\ x = -5 \quad &\text{or} \quad x = 4 \end{aligned}$$

The solutions to $x^2 + x - 20 = 0$ are $x = -5$ and $x = 4$.

Note that in the above example, since the product of $(x + 5)$ and $(x - 4)$ equals zero, one or both of $(x + 5)$ and $(x - 4)$ must be equal to zero.

Example *6.2.2* Solve $2x^2 + 9x + 4 = 0$.
Solution: Note that in this case the coefficient of x^2 is not 1, however we can factorise $2x^2 + 9x + 4$ as $(2x + 1)(x + 4)$.

$$\begin{aligned} 2x^2 + 9x + 4 &= 0 \\ (2x + 1)(x + 4) &= 0 \\ 2x + 1 = 0 \quad &\text{or} \quad x + 4 = 0 \\ 2x = -1 \quad &\text{or} \quad x = -4 \\ x = -1/2 \quad &\text{or} \quad x = -4 \end{aligned}$$

The solutions to $2x^2 + 9x + 4 = 0$ are $x = -1/2$ and $x = -4$.

6.3 The quadratic formula

Given an equation of the form $ax^2 + bx + c = 0$, any solutions can be found by substituting the values of a, b and c into the following formula.

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The number of solutions to the quadratic equation $ax^2 + bx + c = 0$ depends on the value of $b^2 - 4ac$. If $b^2 - 4ac$ is negative, then the equation will have no real solutions since there is no real number that is the square root of a negative number. If $b^2 - 4ac$ is equal to zero, then the equation will have one solution. If $b^2 - 4ac$ is greater than zero, then the $\pm$ in the numerator ensures that we obtain two solutions to the equation.

Example *6.3.1* Solve $3x^2 + 5x - 2 = 0$.
Solution: We apply the quadratic formula with $a = 3$, $b = 5$ and $c = -2$.

$$
\begin{aligned}
x &= \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times (-2)}}{6} \\
x &= \frac{-5 \pm \sqrt{49}}{6} \\
x &= \frac{-5 \pm 7}{6}
\end{aligned}
$$

$$x = \frac{-5 + 7}{6} \quad \text{or} \quad x = \frac{-5 - 7}{6}$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -2$$

The solutions to $3x^2 + 5x - 2 = 0$ are $x = 1/3$ and $x = -2$.

Example *6.3.2* Solve $2x^2 + 4x + 2 = 0$.
Solution: We apply the quadratic formula with $a = 2$, $b = 4$ and $c = 2$.

$$
\begin{aligned}
x &= \frac{-4 \pm \sqrt{16 - 16}}{4} \\
x &= \frac{-4 \pm \sqrt{0}}{4} \\
x &= \frac{-4}{4} \\
x &= -1
\end{aligned}
$$

The only solution to $2x^2 + 4x + 2 = 0$ is $x = -1$.

Example *6.3.3* Solve $4x^2 + 8x + 5 = 0$.
Solution: We apply the quadratic formula with $a = 4$, $b = 8$ and $c = 5$.

$$
\begin{aligned}
x &= \frac{-8 \pm \sqrt{64 - 80}}{8} \\
x &= \frac{-8 \pm \sqrt{-16}}{8}
\end{aligned}
$$

The equation $4x^2 + 8x + 5 = 0$ has no real solutions.

Practice Problems

Here are some problems for you to practice on, followed by answers. Fully worked solutions to these problems can be found in Section 11.6.

Q6.1 Which of the following are quadratic equations?

- **(a)** $x^2 = \frac{1}{2}x - 4$
- **(b)** $x^3 + x + 4 = 0$
- **(c)** $(x + 3)(x - 2) = 0$
- **(d)** $x + 1 = \frac{2x}{x - 2}$

Q6.2 Solve using factorisation.

- **(a)** $2x^2 + 10x + 12 = 0$
- **(b)** $4x^2 - 25 = 0$
- **(c)** $x^2 + 4x = 0$
- **(d)** $y^2 + 6y + 9 = 0$

Q6.3 Solve using the quadratic formula.

- **(a)** $x^2 + 6x + 9 = 0$
- **(b)** $7p^2 + 6p + 1 = 0$
- **(c)** $2x^2 - 10x + 13 = 0$

A6.1
- **(a)** Yes
- **(b)** No
- **(c)** Yes
- **(d)** Yes

A6.2
- **(a)** $x = -2$ or $x = -3$
- **(b)** $x = \frac{5}{2}$ or $x = -\frac{5}{2}$
- **(c)** $x = 0$ or $x = -4$
- **(d)** $y = -3$

A6.3
- **(a)** $x = -3$
- **(b)** $p = \frac{-3 + \sqrt{2}}{7}$ or $p = \frac{-3 - \sqrt{2}}{7}$
- **(c)** No real solutions