8 Sigma Notation

8.1 Introduction

It is easy to write out expressions involving the sum of a few terms. For example, $3x+4x+5x$ and $4+12x+x^2$ can easily be written by hand. There are, however, many cases where we want to write down the sum of many terms and it becomes far too long to write in this fashion. One way to deal with this is to use a series of dots (an ellipsis) to represent terms that are left out, as in $1+2+\cdots+64$. The problem with this notation is that it is often unclear what should be included in the space left by the dots. Was that the sum, for instance, of all the numbers between 1 and 64, or all the powers of 2 from 2^0 to 2^6? Both of these interpretations could make sense on this occasion. To overcome these problems, *sigma* notation is used. The letter Σ is a Greek capital sigma standing for sum. The notation

$$\sum_{i=1}^{64} i$$

represents the sum of all the numbers between 1 and 64, while

$$\sum_{n=0}^{6} 2^n$$

represents the sum of all powers of 2 from 0 to 6.

In general, sigma notation is of the form

$$\sum_{lower}^{upper} expression$$

In this notation, *lower* usually has the form of *variable* = *integer* and *upper* is usually a larger integer. Together *lower* and *upper* define an interval of integers that the variable will take on. The *expression* is usually some formula involving the variable. The expression is evaluated for each of the possible integer values of the variable, advancing by 1 each time, and the resulting terms are added together. For example,

$$\sum_{j=2}^{5} 2j+3 = (2\times 2+3)+(2\times 3+3)+(2\times 4+3)+(2\times 5+3) = 7+9+11+13 = 40.$$

The variable used in the notation is usually referred to as a dummy variable, as changing it has no effect on the sum the notation represents.

Example *8.1.1* Write out the sums described by $\sum_{m=1}^{4}(m/5)$ and $\sum_{n=1}^{4}(n/5)$.

Solution: Both expressions represent the same sum, $\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}$.

The next example is used to demonstrate that there is always more than one way in which a sum can be written in sigma notation. However, a sum in sigma notation will only correspond to one expanded sum.

Example *8.1.2* Write $1+3+5+7+9$ using sigma notation.

Solution: In Sigma notation, the value of the variable always increases by 1 at each step. Here we need the values of the terms in the sum to increase by 2 each time. If the variable takes on the values 1,2,3 and so on in turn, then we can make the terms in the sum increase by 2 each time by multiplying the variable by 2.
So we can think of the terms as $2\times 1-1$, $2\times 2-1$, $2\times 3-1 \cdots 2\times 5-1$. So the sum is

$$\sum_{n=1}^{5} 2n-1.$$

Or we can think of the terms as $2\times 0+1$, $2\times 1+1$, $2\times 2+1 \cdots 2\times 4+1$. So the sum is

$$\sum_{n=0}^{4} 2n+1.$$

In the next example care must be taken to determine which variable is taking the values from 1 to 5.

Example *8.1.3* What is the expanded form of the sum $\sum_{i=1}^{5} ij$?

Solution: This corresponds to $j+2j+3j+4j+5j$.

The variable can also be used in subscripts.

Example *8.1.4* What is the expanded form of the sum $\sum_{j=1}^{5} x_j$?

Solution: This expands to $x_1+x_2+x_3+x_4+x_5$.

Example *8.1.5* Evaluate $\sum_{n=1}^{4}(-1)^n\, n^2$.

Solution:

$$\begin{aligned}\sum_{n=1}^{4}(-1)^n\, n^2 &= (-1)^1\,1^2+(-1)^2\,2^2+(-1)^3\,3^2+(-1)^4\,4^2\\ &= -1+4-9+16\\ &= 10\end{aligned}$$

Note that $(-1)^n$ is -1 if n is odd and $+1$ if n is even.

Practice Problems

Here are some problems for you to practice on, followed by answers. Fully worked solutions to these problems can be found in Section 11.8.

Q8.1 Write the following sums out in full.

(a) $\sum_{n=1}^{3} 2n$

(b) $\sum_{n=-1}^{2} 3^n$

(c) $\sum_{i=0}^{5} i(i+2)$

Q8.2 Write the following sums in sigma notation.

(a) $3+4+5+6+7+8$

(b) $-2+1+4+7$

(c) $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{100}$

Q8.3 Write out in full $\sum_{j=3}^{6}(-1)^j\,(2j+n^{j-2})$.

Q8.4 Write in sigma notation $3x_1 - 5x_2 + 7x_3 - 9x_4$.

A8.1 **(a)** $2+4+6$.

(b) $\frac{1}{3}+1+3+9$.

(c) $0+3+8+15+24+35$.

A8.2 There are many possible answers to these problems. Those presented below are just examples.

(a) $\sum_{i=3}^{8} i$.

(b) $\sum_{i=0}^{3} 3i-2$.

(c) $\sum_{i=1}^{100} \frac{1}{i}$.

A8.3 $-(6+n)+(8+n^2)-(10+n^3)+(12+n^4)$.

A8.4 $\sum_{i=1}^{4}(-1)^{i-1}\,(2i+1)x_i$.