5 Trigonometry

5.1 Introduction

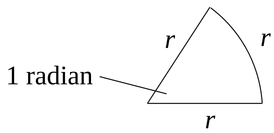
Trigonometry is concerned with various ratios of lengths that are associated with angles. You should be familiar with the three trigonometric functions *sine*, *cosine* and *tangent* (abbreviated to sin, cos, tan respectively). Each of these functions takes an angle as input and returns a number. For example,

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 60^\circ = \frac{1}{2} \quad \text{and} \quad \tan 45^\circ = 1.$$

The numbers returned by these functions are ratios of side lengths of a right-angled triangle.

5.2 Radians

You are probably used to measuring angles in degrees. Another common unit of measurement for angles is the *radian*. By definition, one radian is the angle required to create a sector of a circle with an arc length equal to the radius.



A circle with radius r has a circumference of $2\pi r$ units, so the angle created by a complete revolution is equal to 2π radians. Thus 2π radians equals 360 degrees. From this we get

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \qquad 1 \text{ degree} = \frac{\pi}{180} \text{ radians.}$$

1 radian is approximately 57°.

So a right angle (90°) is $\frac{\pi}{2}$, 60° is $\frac{\pi}{3}$, 45° is $\frac{\pi}{4}$, and 30° is $\frac{\pi}{6}$.

Example *5.2.1* Convert 150° into radians.

Solution: Multiply 150° by $\frac{\pi}{180}$. Thus

$$150 \times \frac{\pi}{180} = \frac{150\pi}{180} = \frac{5\pi}{6}.$$

Hence 150° equals $\frac{5\pi}{6}$ radians (that is, approximately 2.618 radians).

Example *5.2.2* Convert $\frac{5\pi}{4}$ radians into degrees.

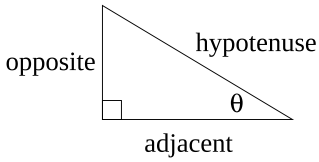
Solution: Multiply $\frac{5\pi}{4}$ by $\frac{180}{\pi}$. Thus

$$\frac{5\pi}{4} \times \frac{180}{\pi} = \frac{900}{4} = 225.$$

Hence $\frac{5\pi}{4}$ radians is 225°.

5.3 Trigonometry of right-angled triangles

In a right-angled triangle with angle θ, we often refer to the sides of the triangle in terms of their position relative to the angle θ.

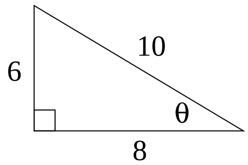


The sides labelled opposite, adjacent and hypotenuse are often abbreviated to *opp*, *adj* and *hyp*, respectively. The three trigonometric functions sine, cosine, and tangent are defined as

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \quad \text{and} \quad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

$$\text{Note that} \quad 0 < \theta < \frac{\pi}{2} \quad \text{and that} \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Example *5.3.1* Determine $\sin\theta$, $\cos\theta$ and $\tan\theta$ in the following triangle.

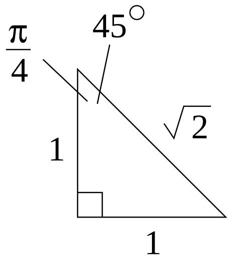


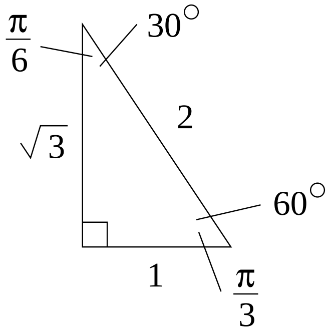
Solution:

$$\begin{aligned}
\sin\theta &= \frac{\text{opp}}{\text{hyp}}, \quad \text{so} \quad \sin\theta = \frac{6}{10} &= \frac{3}{5}\\
\cos\theta &= \frac{\text{adj}}{\text{hyp}}, \quad \text{so} \quad \cos\theta = \frac{8}{10} &= \frac{4}{5}\\
\tan\theta &= \frac{\text{opp}}{\text{adj}}, \quad \text{so} \quad \tan\theta = \frac{6}{8} &= \frac{3}{4}
\end{aligned}$$

You may find the acronym SOHCAHTOA will help you in remembering the trigonometric ratios, where O stands for opposite, A stands for adjacent, H stands for hypotenuse, and S, C and T stand for sine, cosine and tangent respectively.

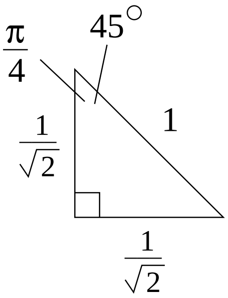
In many cases, the best we can do is find decimal approximations for the trigonometric ratios. However, some of the trigonometric ratios have exact values for certain angles. Here are two triangles that can help you remember the exact values of the trigonometric ratios for $30° = \frac{\pi}{6}$, $45° = \frac{\pi}{4}$ and $60° = \frac{\pi}{3}$. (To help you to remember these triangles, it may help to recall that in any triangle, the shortest side is opposite the smallest angle.)

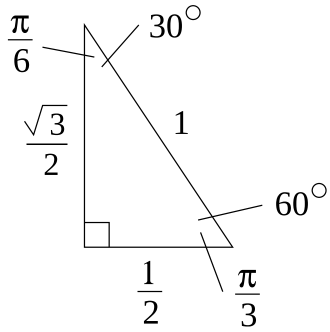




Angle (degrees)	Angle (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

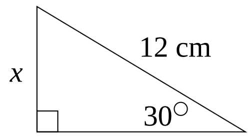
We can re-scale these triangles, to have a hypotenuse of length 1. This will be useful later on.





If you know an angle and one side length of a right-angled triangle, you can use trigonometric ratios to determine the other side lengths.

Example *5.3.2* Find the side length x in the following triangle.

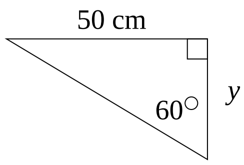


Solution: We are given the length of the hypotenuse and x is the length of the side opposite to the given angle, so the sine ratio is used.

$$\sin 30° = \frac{x}{12}, \quad \text{so } x = 12 \times \sin 30°.$$

From our special triangles $\sin 30° = \frac{1}{2}$, so $x = 12 \times \frac{1}{2} = 6$. The length of the side is 6cm.

Example *5.3.3* Find the side length y in the following triangle.



Solution: We use tangent since the sides involved are opposite and adjacent to the angle marked $60°$.

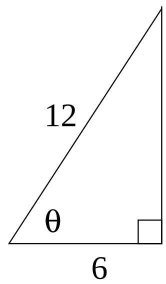
$$\tan 60° = \frac{50}{y}, \quad \text{so } y \times \tan 60° = 50, \quad \text{which means } y = \frac{50}{\tan 60°}.$$

From our special triangles $\tan 60° = \sqrt{3}$, so $y = \dfrac{50}{\sqrt{3}} \approx 28.87$. The length of the side is approximately 28.87cm.

If you are given the value of a trigonometric ratio and asked to find the angle, you may be able to use the special triangles. However if the angle is not $30°$, $45°$ or $60°$ you will need to know how to use the inverse trigonometric functions on your calculator. These often appear above the sin, cos and tan buttons on your calculator. They should look like $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, or asin, acos and atan. You may have heard of these functions as arcsin, arccos, and arctan.

To determine an angle in a right-angled triangle you must know the lengths of at least two sides.

Example *5.3.4* Determine the angle θ in the following triangle.



Solution: The given side lengths are the adjacent and hypotenuse, so we use cosine.

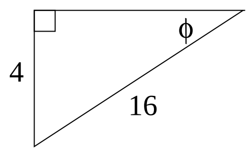
$$\cos\theta = \frac{6}{12} = \frac{1}{2}$$

We can refer to the special triangles to find that the angle with cosine of $\frac{1}{2}$ is 60°. This is written mathematically as

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) \quad \text{so} \quad \theta = 60°.$$

The angle θ is 60°.

Example *5.3.5* Find the angle ϕ in the following triangle.



Solution: We know the opposite and the hypotenuse so we use sine.

$$\sin\phi = \frac{4}{16} = \frac{1}{4}$$

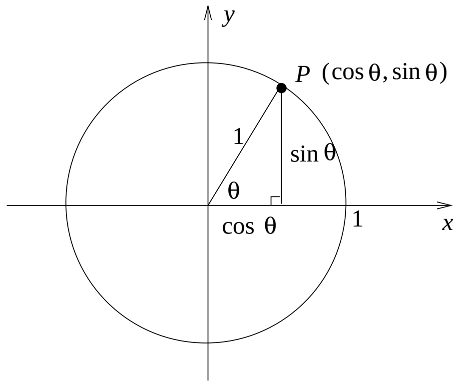
The ratio $\frac{1}{4}$ is not in the special triangles, so we use our calculator to find an angle whose sine is $\frac{1}{4}$.

$$\phi = \sin^{-1}\left(\frac{1}{4}\right) \quad \text{so } \phi \approx 14°$$

The angle ϕ is approximately 14°.

5.4 General definitions of trigonometric functions

The *unit circle* is the circle that has a radius of 1 and is centered at the origin. We can use the unit circle to define the trigonometric ratios for all angles (not just angles between 0° and 90°).



Consider a point P on the circumference of the unit circle, so that the angle measured anticlockwise from the x-axis to P is θ. If P has coordinates (x, y) then the sine and cosine of the angle θ can be defined as follows:

$$\cos\theta = x \quad \text{and} \quad \sin\theta = y.$$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ it follows that $\tan\theta = \frac{y}{x}$. Note that $\tan\theta$ is not defined when $x = 0$, since we cannot divide by 0. Note that these definitions agree with the previous definitions for $0 < \theta < \frac{\pi}{2}$.

Example *5.4.1* Find $\sin\frac{\pi}{2}$, $\cos\frac{\pi}{2}$ and $\tan\frac{\pi}{2}$.

Solution: The point P that makes an angle of $\frac{\pi}{2}$ anticlockwise from the x-axis has coordinates $(x, y) = (0, 1)$.

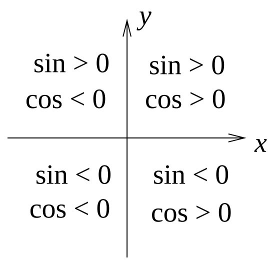
$$\begin{aligned}
\sin\theta &= y \quad \text{so} \quad \sin\tfrac{\pi}{2} = 1.\\
\cos\theta &= x \quad \text{so} \quad \cos\tfrac{\pi}{2} = 0.\\
\tan\theta &= \frac{y}{x} \quad \text{so} \quad \tan\tfrac{\pi}{2} \quad \text{is not defined.}
\end{aligned}$$

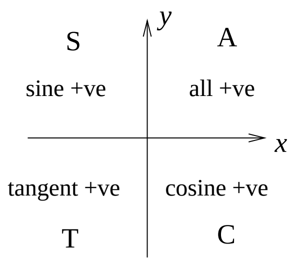
Example *5.4.2* Find $\sin\pi$, $\cos\pi$ and $\tan\pi$.

Solution: The point P that makes an angle of π anticlockwise from the x-axis has coordinates $(x, y) = (-1, 0)$.

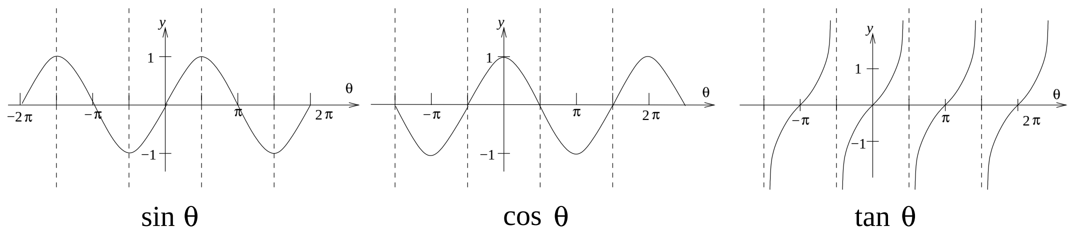
$$\begin{aligned}
\sin\theta &= y \quad \text{so} \quad \sin\pi = 0.\\
\cos\theta &= x \quad \text{so} \quad \cos\pi = -1.\\
\tan\theta &= \frac{y}{x} \quad \text{so} \quad \tan\pi = \frac{0}{-1} = 0.
\end{aligned}$$

From these definitions you can see the trigonometric functions have different signs in different quadrants of the x,y plane. Here are two diagrams which will help you to remember in which quadrants the trigonometric functions are positive and negative. The diagram on the right is often called the CAST rule (in each quadrant starting at the lower right, a letter of the word CAST identifies which of the functions is positive in that quadrant; A means all are positive).

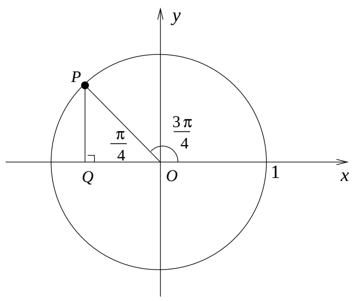




These diagrams are useful when trying to find the value of trigonometric functions for angles larger than $\frac{\pi}{2}$.
Below are the graphs for sine, cosine and tangent. You can see from the graphs that sine and cosine have a period of 2π (so the graphs repeat every 2π) and tangent has a period of π. These graphs are consistent with the CAST diagram: you can clearly see that $\sin\theta$ is positive between 0 and π and negative between π and 2π. From the tangent graph you can see that the function is not defined at $\frac{\pi}{2}$, which we demonstrated in Example 5.4.1.



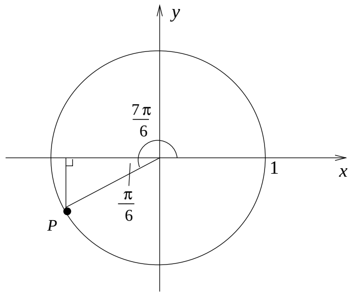
Suppose we want to find the value of $\cos\frac{3\pi}{4}$. First we draw the point P on the unit circle which makes an angle of $\frac{3\pi}{4}$ measured anticlockwise from the x-axis. We must find the x and y coordinates of P. To do this, we draw a line from P to the x-axis to create a right-angled triangle with the right-angle on the x-axis. The angle that this right-angled triangle makes at the origin is $\frac{\pi}{4}$.



Using the special triangle OQP, we see $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Then taking into account that the x-coordinate of a point in the second quadrant is negative (hence cosine is negative) we see that $\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$.

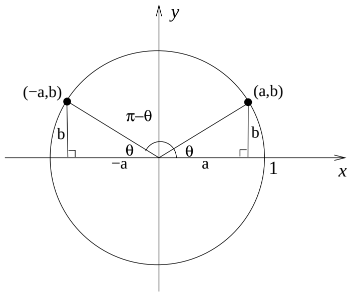
Example *5.4.3* Evaluate $\sin\frac{7\pi}{6}$.

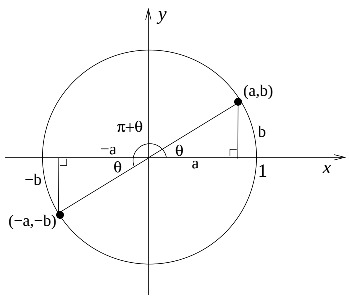
Solution: We draw the point P and create a right angled triangle with an angle of $\frac{\pi}{6}$ at the origin.

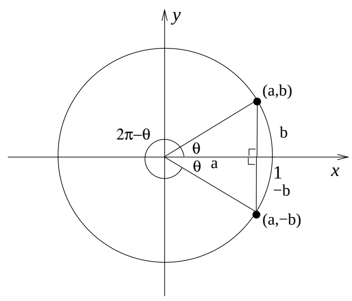


The y-coordinate of P is negative so $\sin \frac{7\pi}{6}$ is negative. Using our special triangles we see that $\sin \frac{7\pi}{6} = -\frac{1}{2}$.

Given an angle θ in the first quadrant, there are related angles in the other quadrants with positive or negative the same x and y coordinates. Thus an angle and its related angles have the same trigonometric values, except possibly for a change in sign.





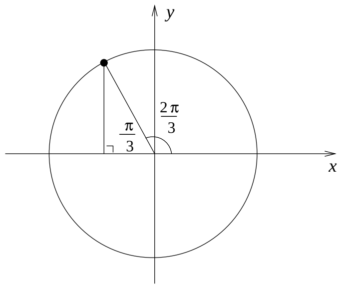


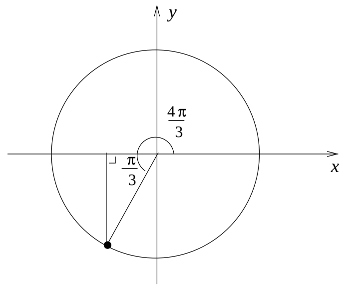
Understanding related angles helps us find all angles with a given trigonometric ratio.

If you are asked to determine an angle that has a particular value of a trigonometric ratio, there is not a unique answer. On the circumference of the unit circle, there are two points with the same x-coordinate (two angles with the same cosine) and two points with the same y-coordinate (two angles with the same sine). Also, if we allow angles greater than 2π radians, then we travel around the unit circle more than once and the values of the ratios repeat themselves.

Suppose we were asked to find two values of θ in the range 0 to 2π such that $\cos\theta = -\frac{1}{2}$. First determine the angle ϕ in the first quadrant that has $\cos\phi = \frac{1}{2}$. This is $\phi = \frac{\pi}{3}$. Find the related angles in the quadrants where cosine (x-coordinate) is negative. These are

$$\pi - \phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \quad \text{and} \quad \pi + \phi = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

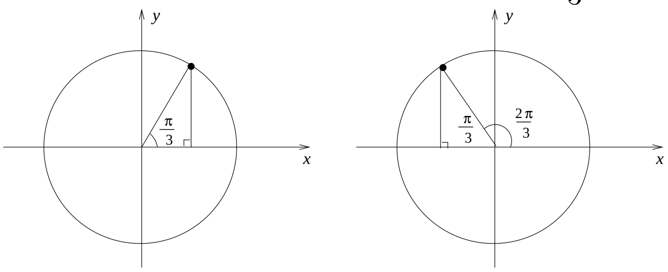




The two angles that satisfy $\cos\theta = -\frac{1}{2}$ are $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

Example *5.4.4* Find two angles in the range 0 to 2π such that $\sin\theta = \frac{\sqrt{3}}{2}$.

Solution: The angle ϕ in the first quadrant that has $\sin\theta = \frac{\sqrt{3}}{2}$ is $\frac{\pi}{3}$. The related angles in the quadrants where sine (y-coordinate) is positive are $\phi = \frac{\pi}{3}$ and $\pi - \phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.



The two angles that satisfy $\sin\theta = \frac{\sqrt{3}}{2}$ are $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$.

Practice Problems

Here are some problems for you to practice on, followed by answers. Fully worked solutions to these problems can be found in Section 11.5.

Q5.1 Convert these angles to radians.

(a) $120°$

(b) $260°$

(c) $315°$

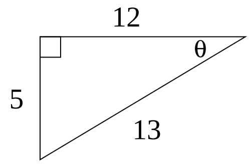
Q5.2 Convert these angles to degrees.

(a) $\frac{4\pi}{9}$

(b) $\frac{3\pi}{5}$

(c) $\frac{\pi}{9}$

Q5.3 From this triangle find the requested values.



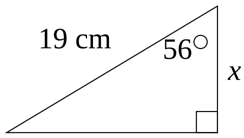
(a) $\sin\theta$

(b) $\cos\theta$

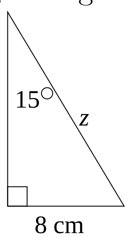
(c) $\tan\theta$

Q5.4 Find the unknown variables.

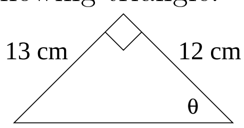
(a) Find the length of side x in the following triangle.



(b) Find the length of side z in the following triangle.



(c) Find the size of the angle θ in the following triangle.



Q5.5 Find exact values (where possible) for the following.

(a) $\sin\dfrac{11\pi}{6}$

(b) $\cos\dfrac{5\pi}{6}$

(c) $\tan\dfrac{4\pi}{3}$

(d) $\sin\dfrac{14\pi}{15}$ (use your calculator for this one).

Q5.6 Find two angles in the range 0 to 2π such that the following are true.

(a) $\sin\theta = -\dfrac{1}{\sqrt{2}}$

(b) $\cos\theta = 1$

(c) $\tan\theta = \dfrac{1}{\sqrt{3}}$

(d) $\cos\theta = \dfrac{1}{4}$

A5.1 **(a)** $\dfrac{2\pi}{3}$

(b) $\dfrac{13\pi}{9}$

(c) $\dfrac{7\pi}{4}$

A5.2 **(a)** $80°$

(b) $108°$

(c) $20°$

A5.3 **(a)** $\sin\theta = \dfrac{5}{13}$

(b) $\cos\theta = \dfrac{12}{13}$

(c) $\tan\theta = \dfrac{5}{12}$

A5.4 **(a)** $x = 10.62$cm

(b) $z = 30.91$cm

(c) $\theta = 47.29°$

A5.5 **(a)** $-\dfrac{1}{2}$

(b) $-\dfrac{\sqrt{3}}{2}$

(c) $\sqrt{3}$

(d) 0.208

A5.6 **(a)** $\dfrac{5\pi}{4}$ and $\dfrac{7\pi}{4}$

(b) 0 and 2π

(c) $\frac{\pi}{6}$ and $\frac{7\pi}{6}$

(d) 1.32 radians and 4.97 radians (approximately)