MATH1050

Mathematical Foundations

Course notes

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About these notes

These are the course notes for MATH1050. We will use these notes very heavily, so it is important that you get your own copy. You can buy these notes fairly cheaply at the University Copy shop, or you can download them from the web and print them yourself. Do not try to re-use a copy from last year: the notes have changed, and in addition it is important for you to write things in your own words.

In lectures, we will use overheard projectors and slides. These notes contain copies of all the slides used in lectures, so you have time to listen and think in class, rather than spending your whole time writing. However, there are many spaces in your notes for examples and solutions. We'll work through the examples in lectures, and you should write down all the solutions to the examples, as well as any annotations that you feel will help you to understand the course material when you look through your notes at a later date.

The notes are divided into sections. The table of contents at the start of the notes and the index at the end of the notes should help you to find your way around. At the back of the notes there are additional practice problems (with solutions) for each section of the notes.

Each year, some people accidentally lose their notes, which causes big problems for them. You might like to write your name and some contact details on the front cover just in case you misplace them.

These notes have been prepared very carefully, but there will inevitably be some errors in them. If you find any errors, or have any suggestions on how to improve the notes, please tell your lecturer (in person, by email, or using the MATH1050 forum on the web).

Table of contents.

1 Matrices

- Matrices represent a structured way of storing and using groups of data in mathematically valid ways.

- They were initially developed for solving systems of simultaneous equations, such as

$$
\begin{aligned}
2x - 3y + z &= -13 \\
x + 4y &= 0 \\
x - 2y + 3z &= 2.
\end{aligned}
$$

- Matrices are used very heavily in computer software, for solving complex problems from science, business and engineering. Most supercomputers spend a lot of their time solving large matrix problems.

- Most programming languages allow the use of *arrays*, which are closely related to matrices.

- Topics in this section are:

  - Introduction to matrices
  - Adding and subtracting matrices
  - Scalar multiplication of matrices
  - Multiplying matrices
  - The transpose of a matrix
  - Identity and inverse matrices
  - The determinant of a square matrix
  - Solving systems of linear equations

1.1 Introduction to matrices

- We will start with an example to illustrate what matrices are and how they can be used in a familiar situation.

Example *1.1.1* Suppose that this week you buy 1 kg of apples, 0.6 kg of oranges and 0.76 kg of bananas, and next week you buy 0.75 kg of apples, 1 kg of oranges and 0.8 kg of bananas. If the cost of apples is \$2.99 per kg, oranges is \$3.99 per kg, and bananas is \$1.89 per kg, how much did you spend on fruit each week?

This information can be presented as follows.

	A	O	B
Week 1	1	0.6	0.76
Week 2	0.75	1	0.8

	cost per kg
A	2.99
O	3.99
B	1.89

To determine how much money was spent on fruit each week we combine the infomation in a row of the first table with a column of the second table.

$$
\begin{pmatrix} 1 & 0.6 & 0.76 \\ 0.75 & 1 & 0.8 \end{pmatrix} \begin{pmatrix} 2.99 \\ 3.99 \\ 1.89 \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 \times 2.99 + 0.6 \times 3.99 + 0.76 \times 1.89 \\ 0.75 \times 2.99 + 1 \times 3.99 + 0.8 \times 1.89 \end{pmatrix} = \begin{pmatrix} 6.82 \\ 7.74 \end{pmatrix}
$$

You spent \$6.82 in the first week and \$7.74 in the second week.

- The rows and columns of numbers enclosed in brackets are examples of matrices, and this combining operation is an example of matrix multiplication. We will now define matrices and operations on matrices formally.

- A *matrix* is a rectangular array of numbers, enclosed in brackets.

- An $m \times n$ matrix has m rows and n columns. The *size* or *order* of a matrix is its number of rows and number of columns. An $m \times n$ matrix has size "m by n".

- The plural of matrix is *matrices*.

Example *1.1.2* $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is a 2×3 matrix.

- Common notation for a general $m \times n$ matrix A is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}).$$

- The notation a_{ij} is used to represent the *element* or *entry* in the ith row and jth column of the matrix A.

- We commonly use an upper case letter to refer to a matrix and the corresponding lower-case letter (with subscripts) to refer to the elements of that matrix.

Example *1.1.3*   The 2×3 matrix $A = (a_{ij})$ with entries $a_{ij} = i - j$ is

---

- A matrix with exactly one row may be called a *row vector*. A matrix with exactly one column may be called a *column vector*. Row and column vectors are often denoted by lower-case letters in bold type.

- A matrix with one row **and** one column is just a number (as we're all familiar with). Sometimes this is called a *scalar*, to distinguish it from matrices with multiple rows or columns, and we write it without any brackets.

- A matrix in which every entry is 0 is called a *zero matrix*. The $m \times n$ zero matrix will sometimes be written as $\mathbf{0}_{m \times n}$.

- A matrix with the same number of rows and columns is called a *square matrix*.

Example *1.1.4*

$(1 \ 2 \ 3)$ is a 1×3 matrix, also called a row vector.

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a 2×1 matrix, also called a column vector.

$2, 3, -4$ and π are all examples of scalars.

$\mathbf{0}_{2 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the 2×2 zero matrix, and it is also a square matrix.

- Matrices $A = (a_{ij})$ and $B = (b_{ij})$ are *equal* if and only if
  - A and B have the same size, and
  - $a_{ij} = b_{ij}$ for all values of i and j.

1.2 Adding and subtracting matrices

- Two matrices can be added or subtracted provided that they have the same size. Two matrices of different sizes cannot be added or subtracted!

- *Matrix addition*: Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then $A + B = C = (c_{ij})$ is the $m \times n$ matrix with $c_{ij} = a_{ij} + b_{ij}$.

- *Matrix subtraction*: Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then $A - B = C = (c_{ij})$ is the $m \times n$ matrix with $c_{ij} = a_{ij} - b_{ij}$.

Example *1.2.1*

Let $A = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 & \frac{1}{2} \\ 2 & -5 & 3 \end{pmatrix}$. Then

$A + B =$

$A - B =$

Example *1.2.2*   Three teams (X, Y and Z) compete in a hockey league. Each team has played a number of home matches and a number of away matches. Their results are summarised in the following table (P = played, W = won, L = lost, D = drawn):

Team	Home matches				Away matches			
	P	W	L	D	P	W	L	D
X	9	6	2	1	6	4	2	0
Y	7	4	1	2	9	5	3	1
Z	8	4	3	1	6	3	1	2

a) A matrix representing the home games of the three teams is

$$H = \begin{pmatrix} 9 & 6 & 2 & 1 \\ 7 & 4 & 1 & 2 \\ 8 & 4 & 3 & 1 \end{pmatrix}$$

b) A matrix representing the away games of the three teams is

$$A = \begin{pmatrix} 6 & 4 & 2 & 0 \\ 9 & 5 & 3 & 1 \\ 6 & 3 & 1 & 2 \end{pmatrix}$$

c) A matrix representing all the games of the three teams is

$$H + A =$$

Example *1.2.3*   Evaluate each of the following where possible:

a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 1 \\ 3 & -2 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ -3 & 4 \end{pmatrix} - \begin{pmatrix} x & -2y \\ 1 & -1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$

Properties of matrix addition

For $m \times n$ matrices A, B and C:

(1) $A + B = B + A$ (commutative);

(2) $A + (B + C) = (A + B) + C$ (associative);

(3) $A + \mathbf{0}_{m \times n} = A$.

1.3 Scalar multiplication of matrices

- It is possible to multiply any matrix by any scalar (number); this is called *scalar multiplication.*

- Let $A = (a_{ij})$ be an $m \times n$ matrix, and let k be any scalar (number). Then $kA = C = (c_{ij})$ is the $m \times n$ matrix with $c_{ij} = k\,a_{ij}$.

Example 1.3.1 Let $A = \begin{pmatrix} 0 & 2 & 4 \\ 6 & 8 & 2 \end{pmatrix}$. Then

$$\tfrac{1}{2}A = \begin{pmatrix} \frac{1}{2} \cdot 0 & \frac{1}{2} \cdot 2 & \frac{1}{2} \cdot 4 \\ \frac{1}{2} \cdot 6 & \frac{1}{2} \cdot 8 & \frac{1}{2} \cdot 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 1 \end{pmatrix}.$$

- Scalar multiplication of a matrix A by a positive integer k is the same as adding k copies of A together.

Example 1.3.2 Let $B = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$.

(a) Calculate $B + B + B$.

(b) Calculate $3B$.

- We usually write $-A$ for $(-1)A$. Note that matrix subtraction is just a combination of matrix addition and scalar multiplication since $A - B = A + (-B)$.

Example *1.3.3*

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ -2 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Evaluate each of

$2A + 3C$

$-3(A - B)$

1.4 Multiplying matrices

- The product AB of matrices A and B exists if and only if the number of columns in A equals the number of rows in B.

- *Matrix multiplication*: Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n \times p$ matrix. Then $AB = C = (c_{ij})$ is the $m \times p$ matrix with $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

***Example** 1.4.1*   The product of a 3×4 matrix and a 4×2 matrix is a 3×2 matrix. (Note that $\cdot$ represents multiplication.)

$$
\begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 4 & 3 & 2 & 5 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 4 & 5 \\ 3 & 4 \\ 2 & 4 \end{pmatrix}
$$

$$
= \begin{pmatrix} 2\cdot 5 + 3\cdot 4 + 1\cdot 3 + 0\cdot 2 & 2\cdot 6 + 3\cdot 5 + 1\cdot 4 + 0\cdot 4 \\ 0\cdot 5 + 4\cdot 4 + 3\cdot 3 + 2\cdot 2 & 0\cdot 6 + 4\cdot 5 + 3\cdot 4 + 2\cdot 4 \\ 4\cdot 5 + 3\cdot 4 + 2\cdot 3 + 5\cdot 2 & 4\cdot 6 + 3\cdot 5 + 2\cdot 4 + 5\cdot 4 \end{pmatrix}
$$

$$
= \begin{pmatrix} 25 & 31 \\ 29 & 40 \\ 48 & 67 \end{pmatrix}
$$

***Example** 1.4.2*

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Which of the following products are defined?

a) AA **b)** CA **c)** AC **d)** BB

- Some people get confused by how to work out exactly which row should be multiplied by which column. The following method may help you.

To multiply matrix A by matrix B (assuming possible):

(a) Write matrix A on the page.

(b) Write matrix B **above** and **to the right** of matrix A.

(c) Draw an empty answer matrix, to the right of A and below B. This matrix will have the same number of rows as A, and the same number of columns as B.

(d) Imagine a horizontal line through the first row of A.

(e) Imagine a vertical line through the first column of B.

(f) The entry in the answer matrix where the two lines intersect is obtained by multiplying the first row of A by the first column of B.

(g) Repeat (d)-(f) for each row of A and column of B.

Example *1.4.3*   Calculate the product AB where
$$A = \begin{pmatrix} -1 & 3 & 1 \\ 0 & 4 & -2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -5 \\ 1 & 3 \\ -2 & 4 \end{pmatrix}.$$

First, check that it **is** possible to find the product AB. A is 2×3 and B is 3×2, so AB will be 2×2. We apply the above method on the next page.

Example *1.4.3 (continued)*

Answer matrix is represented as a 2×2 rectangular array of dots. Draw (or imagine) horizontal and vertical lines, and look at where they intersect in the answer matrix.

$$
B \quad \begin{pmatrix} 2 & -5 \\ 1 & 3 \\ -2 & 4 \end{pmatrix}
$$

$$
A \quad \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & -2 \end{pmatrix} \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}
$$

The cell at position $(1,1)$ in the answer matrix is found by multiplying entries in the first row of A by entries in the first column of B (and adding the products to get the answer value). Continue until the answer matrix is full.

$$
B \quad \begin{pmatrix} 2 & -5 \\ 1 & 3 \\ -2 & 4 \end{pmatrix}
$$

$$
A \quad \begin{pmatrix} -1 & 3 & 1 \\ 0 & 4 & -2 \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix}
$$

Example *1.4.4*   Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$.
Calculate AB and BA.

Properties of matrix multiplication
For matrices of appropriate sizes:

- $A(BC) = (AB)C$ (associative);

- $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$
  (distributive);

- The order of multipication is important. Matrix
  multiplication is not commutative, so in general $AB \neq BA$.

Example _1.4.5_ Find all possible x such that

$$\begin{pmatrix} x & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -7 \\ x \end{pmatrix} = 0$$

- For a square matrix A, we can define matrix powers by repeated matrix multiplication.

$$A^1 = A, \qquad A^2 = A\,A, \qquad A^3 = A\,A\,A = A^2\,A,$$
$$\ldots, \qquad A^{k+1} = A^k\,A.$$

- The usual exponent laws hold for matrix powers, so

$$A^k \cdot A^r = A^{k+r} \quad \text{and} \quad (A^k)^r = A^{kr}.$$

Example *1.4.6* Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$. Calculate A^4.

1.5   The transpose of a matrix

- Informally, the transpose of an $m \times n$ matrix A is the $n \times m$ matrix obtained by exchanging the rows and columns of A.

- Formally, given an $m \times n$ matrix $A = (a_{ij})$, the *transpose* of A is the $n \times m$ matrix $C = (c_{ij})$ where

$$c_{ij} = a_{ji}, \text{ for all } i, j.$$

- The transpose of A is usually denoted by A^T.

Example 1.5.1

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 3 & 5 \end{pmatrix}^T =$$

$$\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}^T =$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^T =$$

Properties of transposes

For matrices of appropriate sizes:

- $(A + B)^T = A^T + B^T$;

- $(A^T)^T = A$;

- $(AB)^T = B^T \ A^T$ (Be careful!).

1.6   Identity and inverse matrices

- In the multiplication of real numbers, the number 1 plays a special role because $1 \cdot k = k \cdot 1 = k$ for all real numbers k. In matrix multiplication, this role is played by the identity matrix.

- The *identity matrix* of order n, denoted I or I_n, is the $n \times n$ matrix $A = (a_{ij})$ where

$$a_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{array} \right.$$

Example *1.6.1*   The identity matrices of orders 2 and 3 are:

$$I_2 = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \qquad I_3 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

- If A is any $m \times m$ matrix, then

$$I_m \, A = A \, I_m = A.$$

- More generally, if A is any $m \times n$ matrix, then

$$I_m \, A = A \, I_n = A.$$

- When dealing with matrix powers, if A is an $m \times m$ matrix, we define $A^0 = I_m$.

Example 1.6.2   Let $B = \begin{pmatrix} -1 & 2 & 0 \\ -2 & 4 & 3 \end{pmatrix}$. Verify that

$I_2\, B = B$ and $B\, I_3 = B$.

- The identity matrix is an example of a *diagonal matrix.*

- A square matrix $A = (a_{ij})$ is a *diagonal matrix* if and only if $a_{ij} = 0$ whenever $i \neq j$.

- Examples of diagonal matrices are:

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -3 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

- Matrix powers of diagonal matrices are easy to calculate. For example

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^3 = \begin{pmatrix} 2^3 & 0 \\ 0 & (-1)^3 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}.$$

The inverse of a square matrix

- A square matrix A is *non-singular* if there exists a matrix B such that $AB = BA = I$. The matrix B is called the *inverse* of A, and is denoted A^{-1}.

- A non-singular matrix is said to be *invertible.*

Example *1.6.3*

Let $A = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$. Show that $A^{-1} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$.

- Not every square matrix is invertible.

- A matrix that does not have an inverse is said to be *singular* or *non-invertible.*

Example *1.6.4* Show that the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ is singular.

Properties of matrix inverses

If A and B are non-singular $n \times n$ matrices, then

- $(A^{-1})^{-1} = A$;

- $(AB)^{-1} = B^{-1}A^{-1}$ (Be careful!);

- $(A^T)^{-1} = (A^{-1})^T$.

The inverse of a 2×2 matrix

Consider the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- A is invertible if and only if $ad - bc \neq 0$.

- If $ad - bc \neq 0$, then the inverse of A is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example *1.6.5* Suppose that $ad - bc \neq 0$ and verify that $AA^{-1} = A^{-1}A = I$.

Example _1.6.6_ Let $A = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -\frac{2}{3} \\ 6 & 4 \end{pmatrix}$.

Determine, if possible, A^{-1} and B^{-1}.

- Note that after you have calculated the inverse of a matrix A, it is a good idea (and easy) to check your work by multiplication. If $AA^{-1} \neq I$ then you have made an error.

Example *1.6.7*   Prove that a matrix A has at most one inverse.

- Suppose that A and B are square matrices of the same size. If A is invertible and $AB = I$, then B is the inverse of A and thus BA must equal I.

$$
\begin{aligned}
AB &= I \\
\text{if and only if} \quad A^{-1}AB &= A^{-1}I \\
\text{iff} \quad IB &= A^{-1}I \\
\text{iff} \quad B &= A^{-1}
\end{aligned}
$$

1.7   The determinant of a square matrix

- With each square matrix A we associate a number called the *determinant* of A, denoted $\det(A)$ or $|A|$.

- A square matrix A is invertible if and only if $\det(A) \neq 0$.

- The determinant of a matrix A is often indicated by writing the elements of A inside two vertical bars.

For the 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

For the 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Example 1.7.1

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \begin{array}{l} = \end{array} 1 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 1(1 - 0) - 2(0 - 0) + 3(0 - 3)$$
$$= -8.$$

- Memorise the formula for the determinant of a 2×2 matrix.

- For a 3×3 (or larger) matrix, understand the following method for calculating determinants.

Calculating the determinant of a square matrix

For the element a_{ij} in a matrix A, define the *cofactor* of a_{ij} to be $(-1)^{i+j}$ times the determinant of the matrix obtained from A by deleting row i and column j.

To calculate the determinant of an $n \times n$ matrix A, choose one row (or column) of A along which you will expand the determinant. Multiply each element of that row (or column) by its cofactor, and sum the results.

Observe that $(-1)^{i+j}$ gives the pattern

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & & & & \end{pmatrix},$$

so if you remember this pattern, you don't need to work out $(-1)^{i+j}$ each time you calculate a cofactor.

On the previous slide we calculated the determinant of a 3×3 matrix by expanding along the first row of the matrix. We could have calculated the determinant in many ways.

Expanding along the second row we have:

$$
\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = -0 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}
$$

$$
= -0(2 - 6) + 1(1 - 9) - 0(2 - 6)
$$

$$
= -8.
$$

Expanding along the first column we have:

$$
\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} =
$$

$$
=
$$

$$
=
$$

Expanding along the second column we have:

$$
\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} =
$$

$$
=
$$

$$
=
$$

1.8   Solving systems of linear equations

Consider the following pair of simultaneous linear equations in two unknowns.

$$
\begin{aligned}
2x + y &= 7 \\
3x + 2y &= 12
\end{aligned}
$$

We can write these using matrices as follows,

$$
\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}
\begin{pmatrix} x \\ y \end{pmatrix}
=
\begin{pmatrix} 7 \\ 12 \end{pmatrix}.
$$

Now the inverse of $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

We can solve for x and y by multiplying both sides of the above matrix equation by A^{-1} (on the left).

$$
\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}
\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}
\begin{pmatrix} x \\ y \end{pmatrix}
=
\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}
\begin{pmatrix} 7 \\ 12 \end{pmatrix}
$$

$$
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\begin{pmatrix} x \\ y \end{pmatrix}
=
\begin{pmatrix} 2 \\ 3 \end{pmatrix}
$$

$$
\begin{pmatrix} x \\ y \end{pmatrix}
=
\begin{pmatrix} 2 \\ 3 \end{pmatrix}
$$

Thus, the solution is $x = 2$, $y = 3$.

How to use matrices to solve a system of simultaneous linear equations

Consider a system of n simultaneous linear equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$
\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n & = & b_2 \\
\vdots & = & \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n & = & b_n
\end{array}
$$

These equations can be expressed in matrix form as $A\mathbf{x} = \mathbf{b}$ where

$$
A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.
$$

If A is invertible, then we can multiply both sides of $A\mathbf{x} = \mathbf{b}$ by A^{-1} (on the left!), to give

$$
A^{-1}\, A\, \mathbf{x} = A^{-1}\, \mathbf{b},
$$

which gives the solution

$$
\mathbf{x} = A^{-1}\, \mathbf{b}.
$$

- If the matrix A is singular, then the system of equations has either no solution, or an infinite number of solutions, and we cannot use the above method.

Example *1.8.1*   Use matrices to solve (if possible) the following pairs of simultaneous equations.

a) $x - 2y = 0$ and $x + 3y = 5$

b) $2x = y - 3$ and $2y - 4x = 6$

Example *1.8.2*

Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ so $A^{-1} = \dfrac{1}{2} \begin{pmatrix} -1 & 1 & 2 \\ 3 & -1 & -6 \\ 3 & -1 & -4 \end{pmatrix}$.

Use these matrices to solve the following system of equations

$$
\begin{aligned}
x - y + 2z &= 3 \\
3x + y &= -5 \\
-y + z &= 2
\end{aligned}
$$

Example *1.8.3*   Ali Baba's rug store is having a sale on persian rugs and mats. Peter buys 3 mats and 2 rugs for $1300, and Dave buys 2 mats and 3 rugs for $1700. How much will it cost to purchase 4 mats and 4 rugs, at these prices?

2 Vectors

- In modelling the real world, some quantities, such as length, area, time and temperature, can be described by real numbers. We call these *scalar* quantities. However, the description of other quantities, such as displacement, velocity and force, require more than just one real number. They are described by both a magnitude and a direction. We call these *vector* quantities.

- You have already seen the word vector used in the context of matrices, and it may seem strange that we use the word vector also to describe something having both magnitude and direction, but you will see how the two uses of the word are related.

- Vectors can be represented in many ways and have many applications.

- In this section we look at three representations of vectors: geometric form, matrix form and component form.

- Topics in this section are:

  - Introduction to vectors (geometric and matrix form)
  - Addition of vectors
  - Scalar multiplication of vectors
  - Position vectors
  - The norm of a vector
  - Trigonometry review
  - Component form of a vector
  - More trigonometry

2.1   Introduction to vectors

- A *vector* quantity is something whose specification requires both a magnitude and a direction.

- We normally use bold lower-case letters to represent vectors, or, when handwriting, a lower-case letter with the $\sim$ symbol below it, for example $\underset{\sim}{v}$.

- Throughout this section we will refer to the (x, y)-plane as 2-space, denoted $\mathbb{R}^2$, and (x, y, z)-space as 3-space, denoted $\mathbb{R}^3$. All of our vectors can be depicted in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Geometric representation of a vector

- A vector can be represented geometrically in $\mathbb{R}^2$ or $\mathbb{R}^3$ by an arrow.

- The length of the arrow represents the *magnitude* of the vector, and the *direction* of the vector is indicated by the direction the arrow is pointing.

- The actual location of the arrow in the diagram is irrelevant, only its magnitude and direction matter.

Example 2.1.1   The vector **v** represents a velocity of 10 km/h in the north-east direction, while the vector **w** represents a velocity of 5 km/h in the north-west direction.
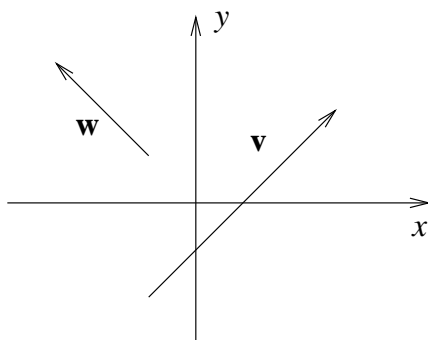
Example 2.1.2 The three arrows shown below each represent the same vector.
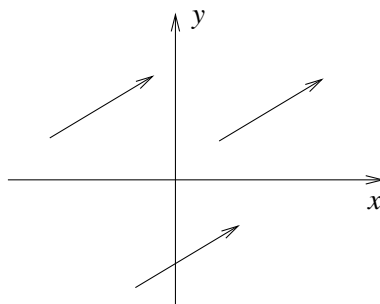
- If P and Q are points in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\overrightarrow{PQ}$ denotes the vector from P to Q. The point P is the tail of the vector and the point Q is the head of the vector.

- Suppose that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are two representations of the same vector in the (x, y)-plane. Let the coordinates of the four points be $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$, $R = (x_R, y_R)$ and $S = (x_S, y_S)$. Since $\overrightarrow{PQ} = \overrightarrow{RS}$, the triangles PQA and RSB are congruent.
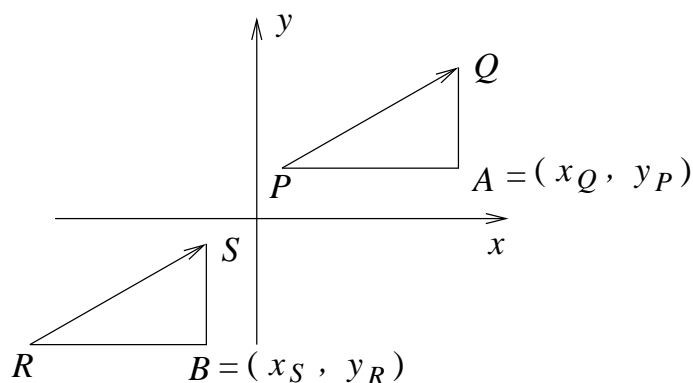
  Thus

$$PA = RB, \quad \text{so} \quad x_Q - x_P = x_S - x_R,$$
$$AQ = BS, \quad \text{so} \quad y_Q - y_P = y_S - y_R.$$

- For every geometric representation of a particular vector $\mathbf{v}$ in $\mathbb{R}^2$, the change in x-coordinate is a fixed quantity, and the change in y-coordinate is a fixed quantity.

Matrix representation of a vector

- For a (geometric) vector $\mathbf{v} \in \mathbb{R}^2$ with tail at the point (x_1, y_1) and head at the point (x_2, y_2), the matrix form of the vector is the 2×1 matrix

$$\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}.$$

- The matrix form of a vector is the same for all geometric representations of the vector.

- The usual notation for writing a general vector $\mathbf{v}$ in matrix form is $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ or $\mathbf{v} = (v_1, v_2)$.

- Although we usually write the vector $\mathbf{v}$ as a column vector, we may occasionally write it as a row vector.

- Given a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in matrix form, you can find a geometric representation of $\mathbf{v}$ by picking any point in the plane as the tail of the vector, moving v_1 units in the x-direction and then v_2 units in the y-direction to find the point that is the head of the vector.

- For a (geometric) vector $\mathbf{v} = \overrightarrow{PQ}$ in $\mathbb{R}^3$, where $P = (x_P, y_P, z_P)$ and $Q = (x_Q, y_Q, z_Q)$, the matrix form is

$$\mathbf{v} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \\ z_Q - z_P \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ or } \mathbf{v} = (v_1, v_2, v_3).$$

- The entries v_1 and v_2 (or v_1, v_2 and v_3) are called the components of the vector.

Converting from matrix to geometric form

***Example 2.1.3*** On the (x, y)-axes below, draw geometric representations of each of the following vectors.

a) $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
b) $\mathbf{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$
c) $\mathbf{w} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$
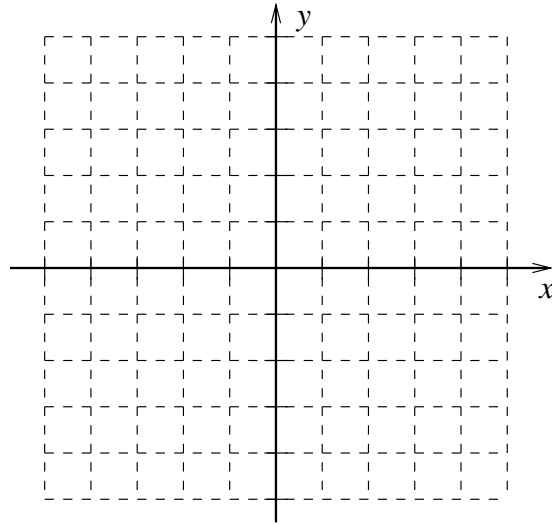
Converting from geometric to matrix form

Example 2.1.4 Determine the matrix form of each of the vectors drawn below.
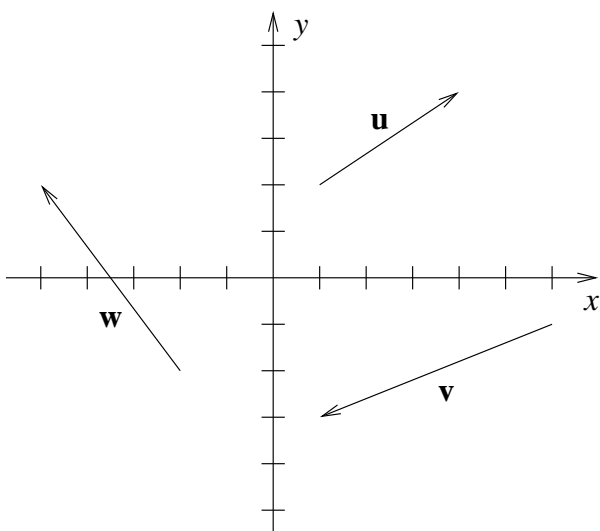
2.2   Addition of vectors

Geometric addition of vectors

- Given vectors $\mathbf{v}$ and $\mathbf{w}$, we define the sum $\mathbf{v} + \mathbf{w}$ by the triangle rule for vector addition.

- Let P, Q and R be points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) such that $\mathbf{v} = \overrightarrow{PQ}$ and $\mathbf{w} = \overrightarrow{QR}$. Then $\mathbf{v} + \mathbf{w} = \overrightarrow{PR}$.
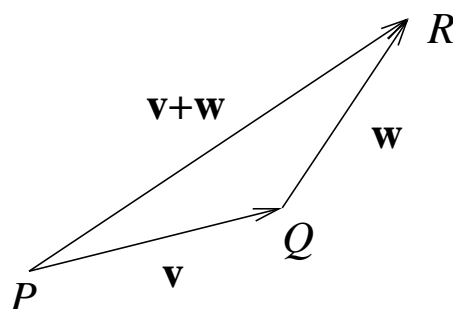
- Note that when adding vectors $\mathbf{v}$ and $\mathbf{w}$ geometrically you put the tail of $\mathbf{w}$ at the head of $\mathbf{v}$ and then draw the sum $\mathbf{v} + \mathbf{w}$ from the tail of $\mathbf{v}$ to the head of $\mathbf{w}$.

Matrix addition of vectors

- If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, then

$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix},$$

  in agreement with regular matrix addition.

- Addition of vectors in $\mathbb{R}^3$ is similar.

To see that geometric and matrix addition of vectors are equivalent, consider the following diagram showing the addition of vectors $\mathbf{v}$ and $\mathbf{w}$.
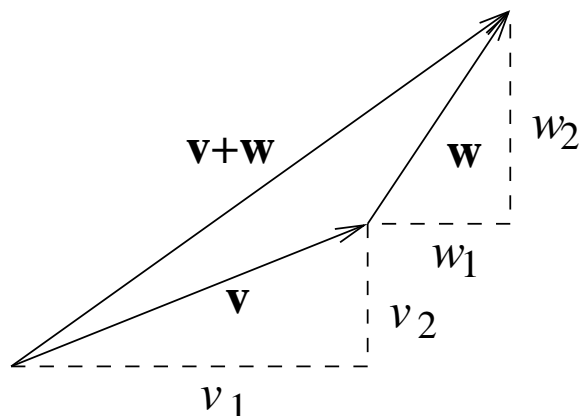
Example *2.2.1*

Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$.

Determine the following vector sums, using matrix addition of vectors and using geometric addition of vectors.

a) $\mathbf{u} + \mathbf{v}$
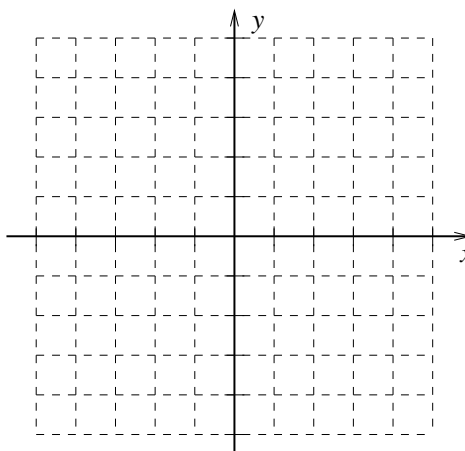
b) $\mathbf{u} + \mathbf{w}$
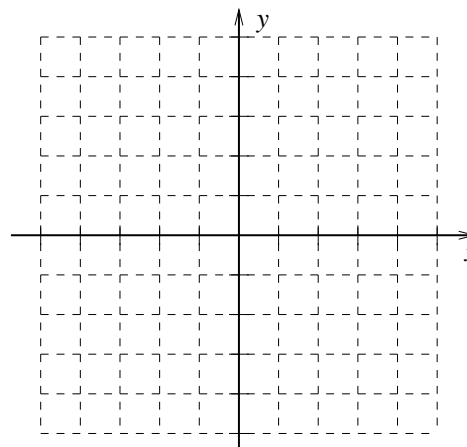
- We define the *zero vector* to be the vector (of an appropriate size) with each component equal to zero, and denote it by **0**.

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- The zero vector has zero magnitude and unspecified direction, so can be represented geometrically as a point.

Properties of vector addition

(1) Vector addition is *commutative*, that is, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

- We have already seen that matrix addition is commutative.

- To illustrate the commutativity of addition geometrically, consider four points P, Q, R and S, arranged in 2-space so that

$$\mathbf{v} = \overrightarrow{PQ} = \overrightarrow{SR} \quad \text{and} \quad \mathbf{w} = \overrightarrow{PS} = \overrightarrow{QR}.$$
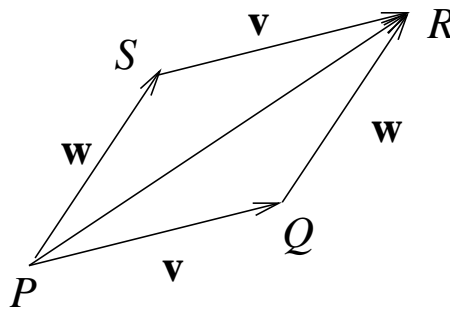
Then $\mathbf{v} + \mathbf{w} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SR} = \mathbf{w} + \mathbf{v}$.

(2) Vector addition is *associative*, that is
$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

(3) $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$.

2.3   Scalar multiplication of vectors

Geometric scalar multiplication of vectors

- Given a vector $\mathbf{v}$ and a real number t, we define the scalar multiple $t\mathbf{v}$ to be the vector whose magnitude is $|t|$ times the magnitude of $\mathbf{v}$, and whose direction is the same as $\mathbf{v}$ if $t > 0$ and opposite to $\mathbf{v}$ if $t < 0$.

- Note that if $t = 0$, then the scalar multiple $t\mathbf{v}$ is the zero vector.
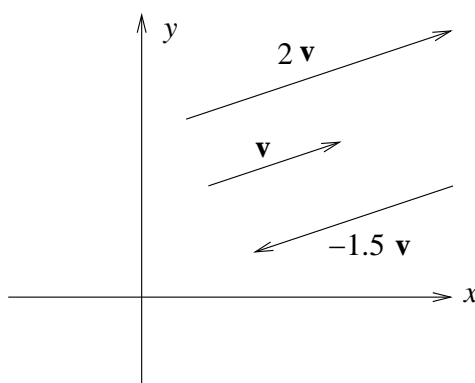
Matrix scalar multiplication of vectors

- If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and t is a real number, then

$$t\mathbf{v} = \begin{pmatrix} t \cdot v_1 \\ t \cdot v_2 \end{pmatrix},$$

  in agreement with regular matrix scalar multiplication.

- Scalar multiplication of vectors in $\mathbb{R}^3$ is similar.

To see that geometric and matrix scalar multiplication of vectors are equivalent, consider the following diagram showing the multiplication of vector **v** by the scalar t.
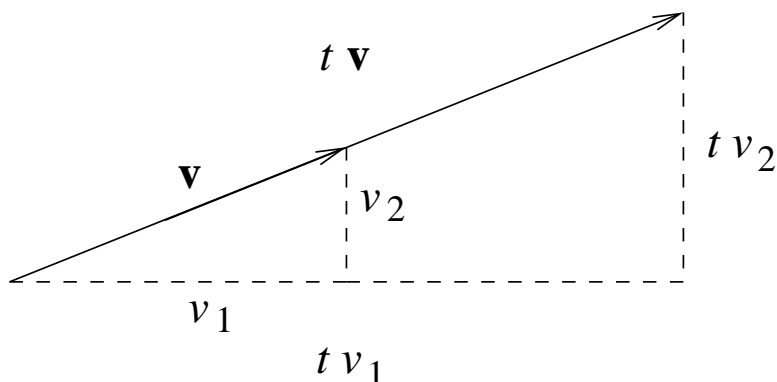
Example 2.3.1 Let $\mathbf{u} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$. Determine the following vector scalar multiples, using matrix scalar multiplication of vectors and using geometric scalar multiplication of vectors.
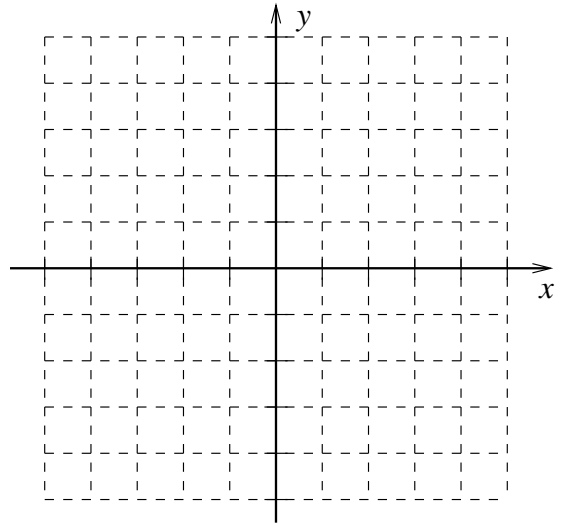
a) $2\mathbf{u}$
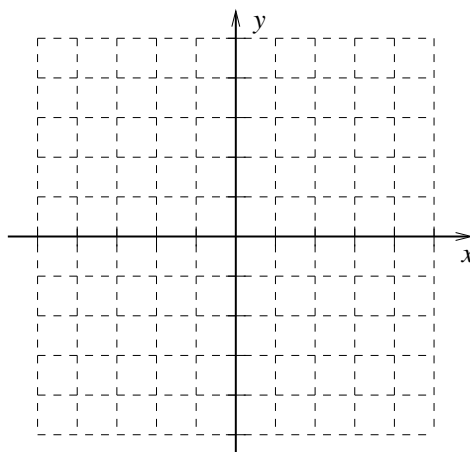
b) $-1\mathbf{u}$
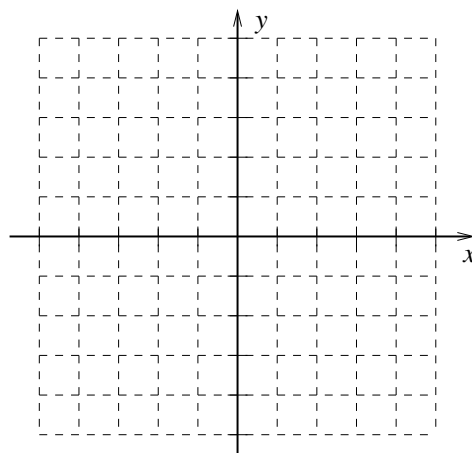
- Note that we usually write $-1\mathbf{v}$ as $-\mathbf{v}$.

- We can define *vector subtraction* as a combination of vector addition and scalar multiplication. If $\mathbf{v}$ and $\mathbf{w}$ are two vectors, then

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}).$$

***Example* 2.3.2** Let $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$.

Determine the vectors resulting from the following operations, in both matrix form and geometric form.

a) $\mathbf{u} - \mathbf{v}$

b) $\mathbf{v} - 2\mathbf{u}$
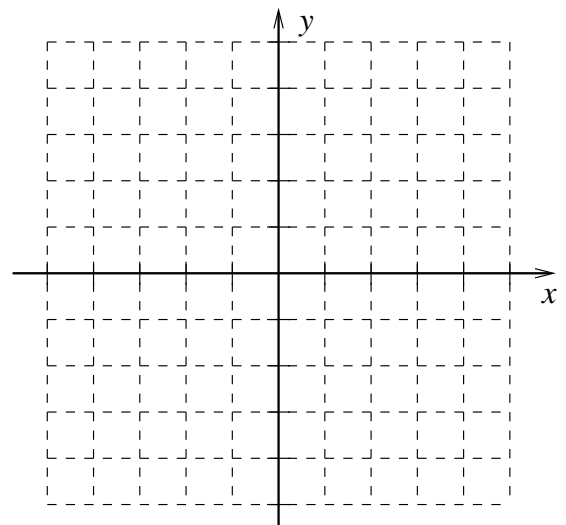
2.4   Position vectors

- Of all the geometric representations of a vector $\mathbf{v}$, the one with its tail at the origin is special.

- In $\mathbb{R}^2$, let P be the point with coordinates (x_P, y_P). The vector $\overrightarrow{OP}$ with its tail at the origin O and its head at P is called the *position vector* of P.

- The matrix form of $\overrightarrow{OP}$ is $\begin{pmatrix} x_P - 0 \\ y_P - 0 \end{pmatrix} = \begin{pmatrix} x_P \\ y_P \end{pmatrix}$, which can also be written as $\overrightarrow{OP} = (x_P, y_P)$.

- Similarly, in $\mathbb{R}^3$, let P be the point with coordinates (x_P, y_P, z_P). The vector $\overrightarrow{OP}$ with its tail at the origin O and its head at P is called the *position vector* of P, and the matrix form of $\overrightarrow{OP}$ is $\begin{pmatrix} x_P - 0 \\ y_P - 0 \\ z_P - 0 \end{pmatrix} = \begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix}$, which can also be written as $\overrightarrow{OP} = (x_P, y_P, z_P)$.

- The coordinates of the point P are the components of the position vector of P.
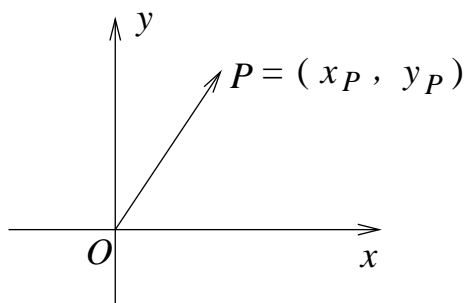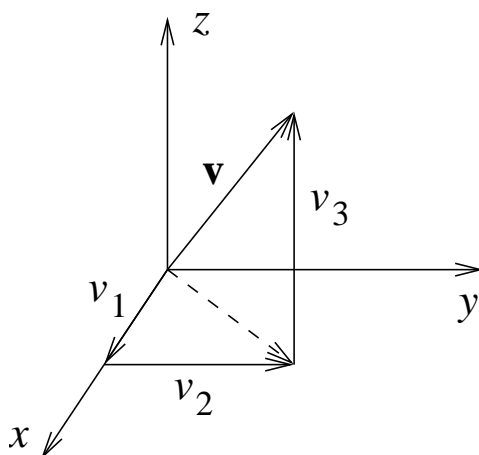
2.5 The norm of a vector

- The *norm* (or *length* or *magnitude*) of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the (shortest) distance between points P and Q.

- The norm of the vector $\mathbf{v}$ is denoted $||\mathbf{v}||$.

- In $\mathbb{R}^2$, if $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$, then
$$\mathbf{v} = \overrightarrow{PQ} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \text{ and } ||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}.$$
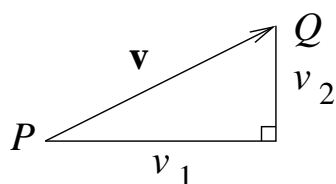
In $\mathbb{R}^3$, if $P = (x_P, y_P, z_P)$ and $Q = (x_Q, y_Q, z_Q)$, then
$$\mathbf{v} = \overrightarrow{PQ} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \\ z_Q - z_P \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \text{ and }$$

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

- Note that for most vectors $\mathbf{v}$ and $\mathbf{w}$, $||\mathbf{v} + \mathbf{w}|| \neq ||\mathbf{v}|| + ||\mathbf{w}||$.

- For any vector $\mathbf{v}$ and any real number t, $||t\mathbf{v}|| = |t| \, ||\mathbf{v}||$.

- A vector with norm 1 is called a *unit vector.*

- The notation $\hat{\mathbf{v}}$ will be used to denote a unit vector having the same direction as the vector $\mathbf{v}$.

- For a given vector $\mathbf{v}$, with norm $||\mathbf{v}||$, the vector

$$\hat{\mathbf{v}} = \frac{1}{||\mathbf{v}||}\mathbf{v}$$

is a unit vector in the direction of $\mathbf{v}$.

***Example* 2.5.1** Determine $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ in matrix form where

$$\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}.$$

2.6   Trigonometry review

- So far, the geometric representations of our vectors have fallen nicely at grid points of the (x, y)-axes so it has been easy to convert them to matrix form, and to perform addition, scalar multiplication, etc. In general, this will not be the case. In this section we quickly review some trigonometry that will be required in our work with vectors.

- We define the trigonometric ratios of sine and cosine as follows. If P is a point on the unit circle, and the line segment OP makes an angle θ measured anti-clockwise from the positive x-axis, then the point P has coordinates $(\cos\theta, \sin\theta)$.

- If we scale the unit circle, so that it becomes a circle of radius r, and let A be the point where that circle meets the extension of the line OP, then the coordinates of A are $(r\cos\theta, r\sin\theta)$.
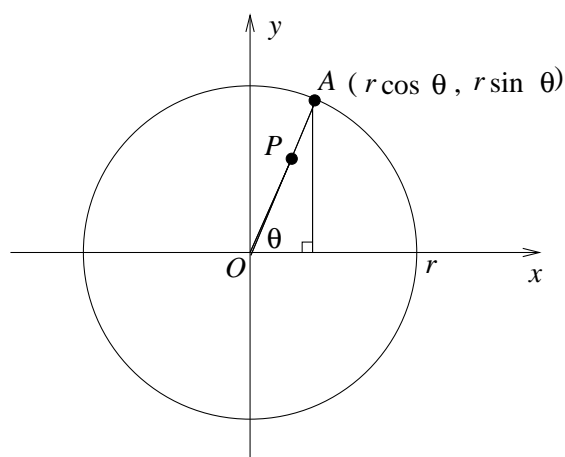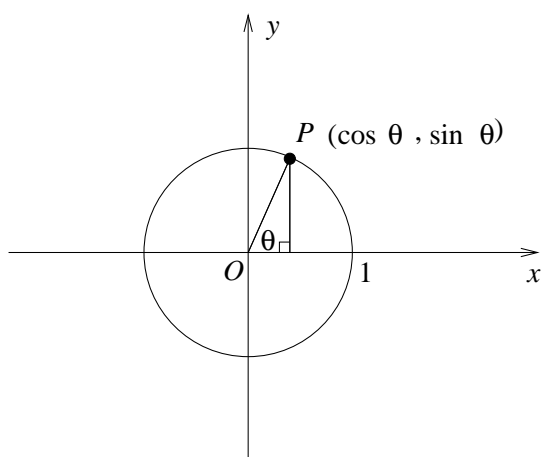

- Thus, the vector $\overrightarrow{OA}$ has magnitude r and direction θ (measured anti-clockwise from the positive x-axis). Its matrix form is $\overrightarrow{OA} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$.

Example *2.6.1*   Let $\mathbf{v}$ be the vector in $\mathbb{R}^2$ with magnitude 5 such that $\mathbf{v}$ makes an angle of $\frac{5\pi}{7}$ radians with the positive x-axis. Find the matrix form of $\mathbf{v}$ (2 decimal place accuracy).

- The trigonometric function "tangent" is defined as $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

- In this course we will usually measure angles in radians. To convert between radians and degrees, use the fact that π radians is equal to 180 degrees.

- Angles will be measured anti-clockwise from the positive x-axis, unless otherwise indicated.

- For a right-angled triangle with angle θ as shown, the values of the trigonometric ratios are
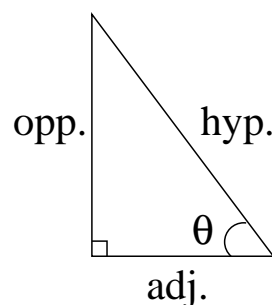
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

- You will need to remember how to calculate the sine and cosine of the special angles, $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their related angles in the (x, y)-plane.

- The two special triangles and the CAST rule (illustrated below) can be used in calculations of trigonometic ratios.
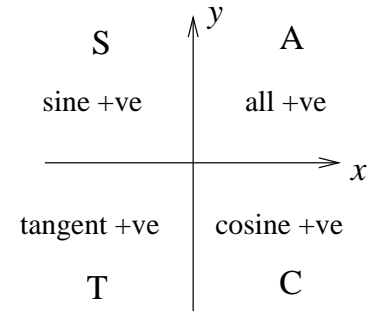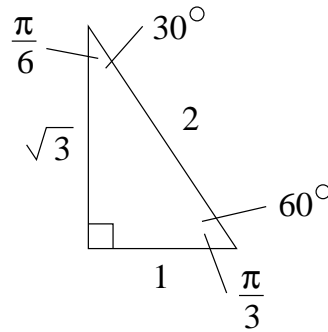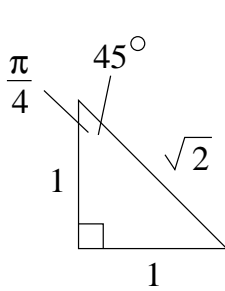
***Example* 2.6.2**   Calculate the following trigonometric ratios.

a) $\cos(\frac{5\pi}{3})$

b) $\tan(\pi)$

c) $\sin(\frac{5\pi}{4})$

Example *2.6.3* Calculate the values of the angle θ that satisfy the following equations, where $0 \leq \theta < 2\pi$.

a) $\sin \theta = \frac{1}{2}$ (Equivalently $\theta = \arcsin(\frac{1}{2})$ or $\theta = \sin^{-1}(\frac{1}{2})$.)

b) $\tan \theta = -1$ (Equivalently $\theta = \arctan(-1)$ or $\theta = \tan^{-1}(-1)$.)

c) $\cos \theta = 0.53$ (Use your calculator for this one.)

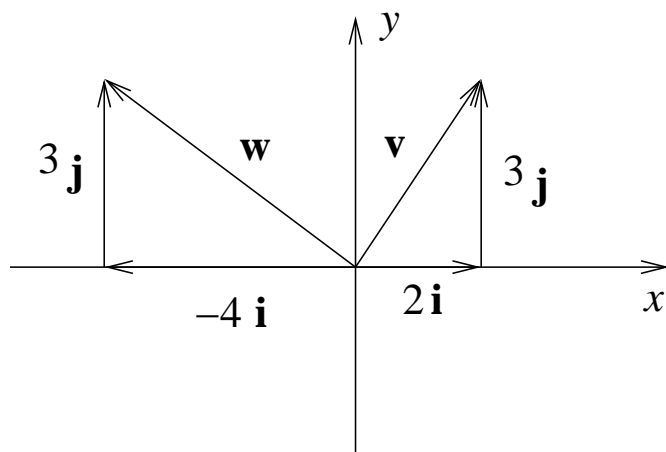# 2.7   Component form of a vector

Component form in 2-space

- In the (x, y)-plane, there are two important unit vectors. The unit vector in the direction of the x-axis is denoted $\mathbf{i}$, and the unit vector in the direction of the y-axis is denoted $\mathbf{j}$, so

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- Note that $\mathbf{i}$ and $\mathbf{j}$ can also be written as row vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

- Any vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{R}^2$ can be written as the sum of scalar multiples of $\mathbf{i}$ and $\mathbf{j}$, since

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\mathbf{i} + b\mathbf{j}.$$

- The component form of the vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.

$$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{w} = -4\mathbf{i} + 3\mathbf{j}$$

Component form in 3-space

- In 3-space, there are three important unit vectors. The unit vector in the direction of the x-axis is denoted $\mathbf{i}$, the unit vector in the direction of the y-axis is denoted $\mathbf{j}$, and the unit vector in the direction of the z-axis is denoted $\mathbf{k}$, so

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$
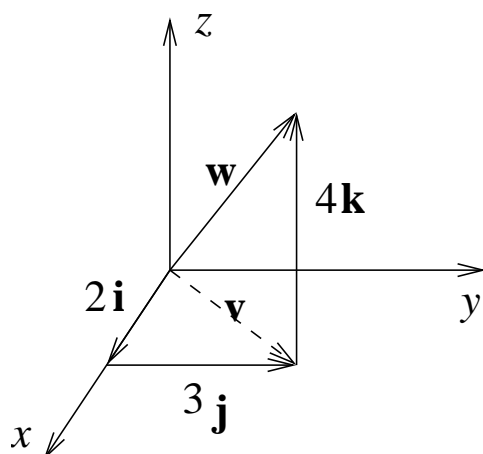
- Note that $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ can also be written as row vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

- Any vector $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbb{R}^3$ can be written as the sum of scalar multiples of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, since

$$\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

- The component form of the vector $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is

$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$

$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$

$\mathbf{w} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

Converting vectors from geometric to component form

A vector $\mathbf{v}$ in $\mathbb{R}^2$, with magnitude $||\mathbf{v}||$ and direction θ measured anti-clockwise from the positive x-axis, has component form

$$\mathbf{v} = ||\mathbf{v}|| \cos\theta\,\mathbf{i} + ||\mathbf{v}|| \sin\theta\,\mathbf{j}.$$

Example *2.7.1*   **a)** The vector $\mathbf{v}$ in $\mathbb{R}^2$ has magnitude 4 and direction $\frac{2\pi}{3}$. Write $\mathbf{v}$ in component form and in matrix form.

b) The vector $\mathbf{w}$ in $\mathbb{R}^2$ has magnitude 3 and direction $\frac{3\pi}{2}$. Write $\mathbf{w}$ in component form and in matrix form.

Converting vectors from component to geometric form

A vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ has magnitude $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}$. If $\mathbf{v}$ is non-zero, then its direction θ, measured anti-clockwise from the positive x-axis, is obtained as follows.

- Sketch the position vector $\mathbf{v}$ on a set of (x, y)-axes.

- If $\mathbf{v} = v_1\mathbf{i} + 0\mathbf{j}$ or $\mathbf{v} = 0\mathbf{i} + v_2\mathbf{j}$, then θ will be one of 0, $\frac{\pi}{2}$, π or $\frac{3\pi}{2}$, and you can determine which it is from the sketch.

- If neither of v_1 nor v_2 is zero, then calculate $\phi = \arctan\left(\left|\frac{v_2}{v_1}\right|\right)$. The value of ϕ will be between 0 and $\frac{\pi}{2}$.

- The value of θ will be one of $\phi, \pi - \phi, \pi + \phi$, or $2\pi - \phi$. You can identify which it is from your sketch.
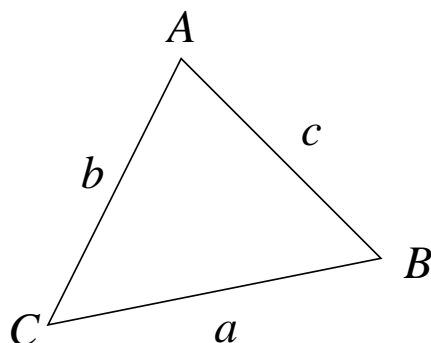

***Example* 2.7.2**
a) Find the magnitude and direction of the vector $\mathbf{v} = \mathbf{i} - \sqrt{3}\mathbf{j}$.

b) Find the magnitude and direction of the vector $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$.

2.8　More trigonometry

- When you work with triangles that are not right-angled triangles, the following two rules are useful.

- Let ABC be a triangle with side lengths a (opposite the angle at A), b (opposite the angle at B), and c (opposite the angle at C).


Cosine Rule　　　$a^2 = b^2 + c^2 - 2bc \cos A$

Sine Rule　　　$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

- Although the Sine Rule is usually quicker to apply than the Cosine Rule, you have to be careful with the Sine Rule if your triangle contains an obtuse angle (greater than a right angle).

- Every angle θ within a triangle must satisfy $0 < \theta < \pi$, but $\sin \theta = \sin(\pi - \theta)$ so there are two angles between 0 and π that have the same sine value.

Example *2.8.1* Determine the angles (in degrees) in the following triangle.
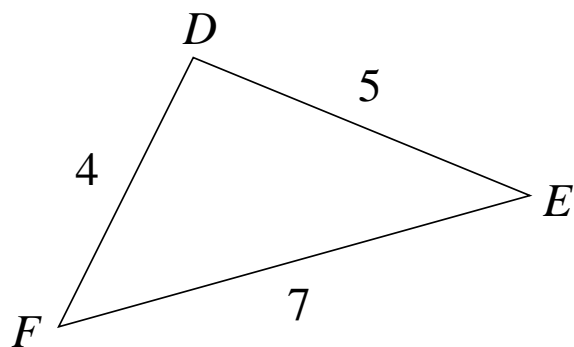
- When manipulating algebraic expressions involving the trigonometric functions, it is useful to be aware of their relationships to one another.

- An equation that involves trigonometric functions is called a trigonometric identity. We will now prove several of these. For future reference, we will label them TI1 through TI19 for trigonometric identities 1 through 19.

- Recall that angles are measured anticlockwise from the x-axis, so a negative angle can be interpreted in terms of moving clockwise from the x-axis.

- From the definitions of sine and cosine in terms of the unit circle, we see that

$$\sin(-\theta) = -\sin\theta \; (\textbf{TI1}) \; \text{ and } \; \cos(-\theta) = \cos\theta \; (\textbf{TI2}).$$
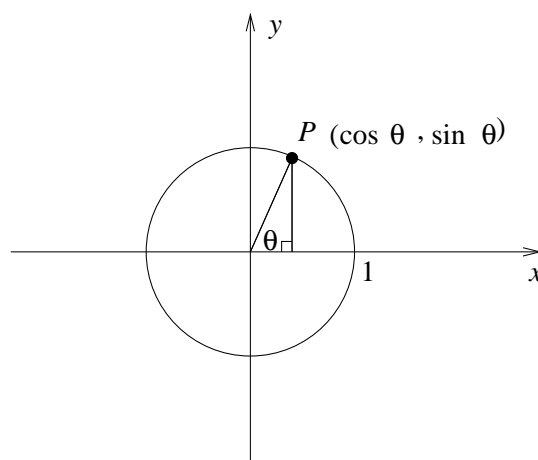
- The reciprocal trigonometic functions are cosecant, secant and cotangent. There are defined as:

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \text{and} \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

- By considering a point P on the unit circle, with coordinates $(x, y) = (\cos\theta, \sin\theta)$, and using the equation for the unit circle $x^2 + y^2 = 1$, we have

$$\sin^2\theta + \cos^2\theta = 1 \textbf{ (TI3)}.$$

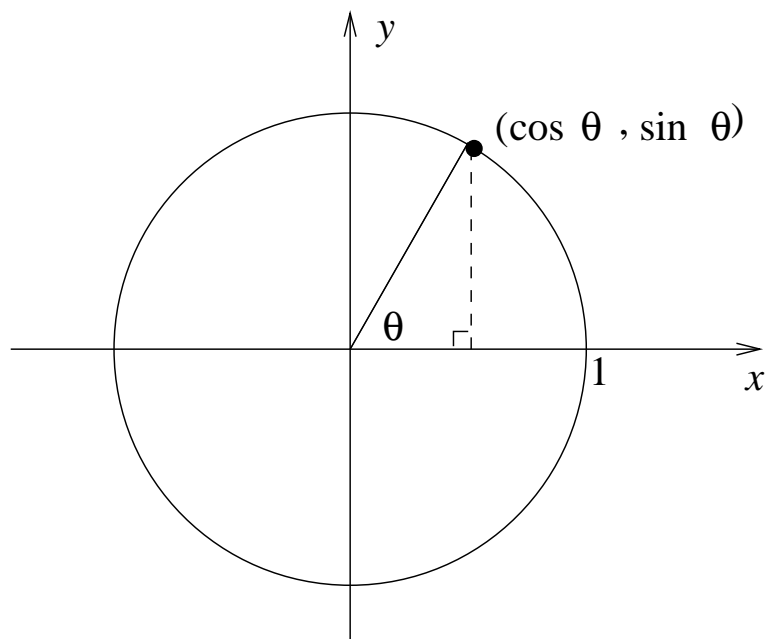***Example*** *2.8.2* Prove the following trigonometric identities.

(a) $1 + \tan^2\theta = \sec^2\theta$ **(TI4)**

(b) $\cot^2\theta + 1 = \csc^2\theta$ **(TI5)**

***Example* 2.8.3** Show that $\cos(\frac{\pi}{2} - \theta) = \sin\theta$ and that $\sin(\frac{\pi}{2} - \theta) = \cos\theta$, for $0 < \theta < \frac{\pi}{2}$.

- In fact, the identities

  $\cos(\frac{\pi}{2} - \theta) = \sin\theta$ **(TI6)**   and $\sin(\frac{\pi}{2} - \theta) = \cos\theta$ **(TI7)**

  are true for all θ.

- Similarly, it can be shown that

  $\cos(\frac{\pi}{2} + \theta) = -\sin\theta$ **(TI8)**   and $\sin(\frac{\pi}{2} + \theta) = \cos\theta$ **(TI9)**.

- The sum and difference formulas are trigonometric identities that involve the sine or cosine of the sum or difference of two angles.

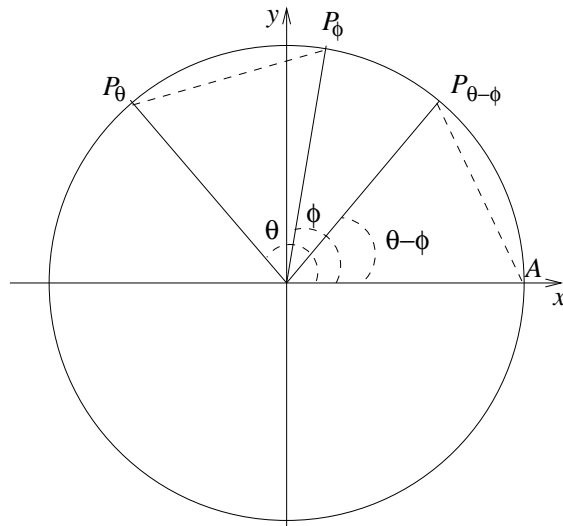$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \textbf{ (TI10)}$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \textbf{ (TI11)}$$
$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \textbf{ (TI12)}$$
$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi \textbf{ (TI13)}$$

Example *2.8.4*   Prove that $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$.

Consider the points $P_\theta = (\cos\theta, \sin\theta)$, $P_\phi = (\cos\phi, \sin\phi)$, $P_{\theta-\phi} = (\cos(\theta - \phi), \sin(\theta - \phi))$ and $A = (1, 0)$.

- The sum and difference formulas can be used to calculate the exact value of the trigonometic ratios for certain angles by relating them to the special angles $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

Example 2.8.5 Determine the exact value of $\sin(\frac{7\pi}{12})$.

- If we let $\phi = \theta$ in the sum formulas, we get identities that are often called the double-angle formulas.

$$
\begin{array}{rcl}
\sin(2\theta) & = & 2\sin\theta\cos\theta \ \textbf{(TI14)} \\
\cos(2\theta) & = & \cos^2\theta - \sin^2\theta \ \textbf{(TI15)}
\end{array}
$$

- Applying $\sin^2\theta + \cos^2\theta = 1$ to TI15, we see that

$$
\begin{array}{rcl}
\cos(2\theta) & = & 2\cos^2\theta - 1 \ \textbf{(TI16)} \\
\cos(2\theta) & = & 1 - 2\sin^2\theta \ \textbf{(TI17)}
\end{array}
$$

- Finally from TI16 and TI17, we get the half-angle formulas.

$$
\sin^2\theta = \frac{1 - \cos(2\theta)}{2} \ \textbf{(TI18)} \quad \cos^2\theta = \frac{1 + \cos(2\theta)}{2} \ \textbf{(TI19)}
$$

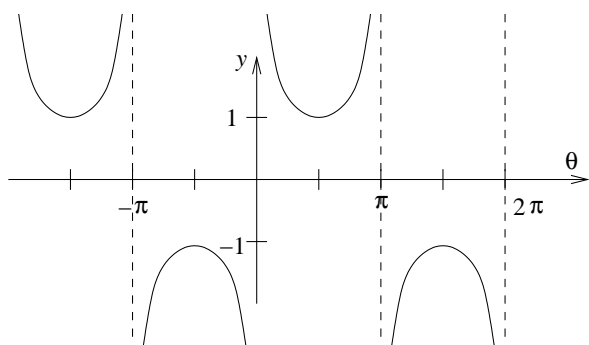- The double-angle formulas and the half-angle formulas are useful in integration.

Graphs of the trigonometric functions

We can draw the graphs of $y = \sin\theta$ and $y = \cos\theta$ by considering what happens to the x and y coordinates of the points as we move around the circumference of the unit circle.
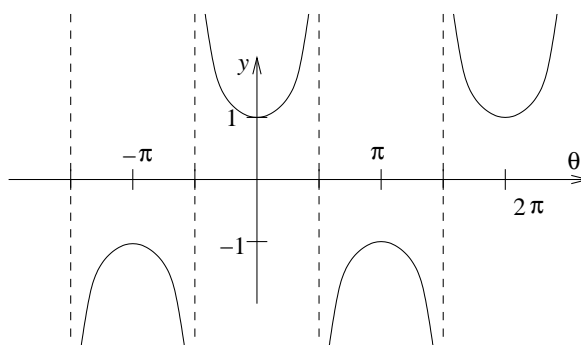
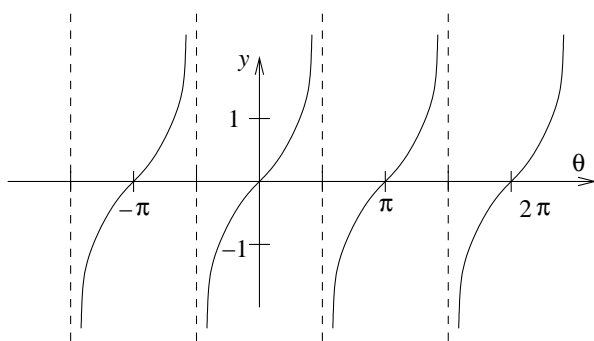***Example*** *2.8.6*   On the axes below, draw the graphs of $y = \sin\theta$ and $y = \cos\theta$.
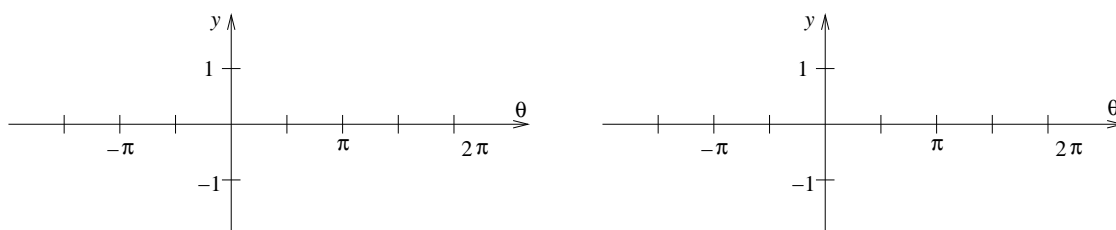
The graphs of $y = \csc\theta$, $y = \sec\theta$, $y = \tan\theta$ and $y = \cot\theta$ are shown. You should think about how these relate to the graphs of $y = \sin\theta$ and $y = \cos\theta$ above.
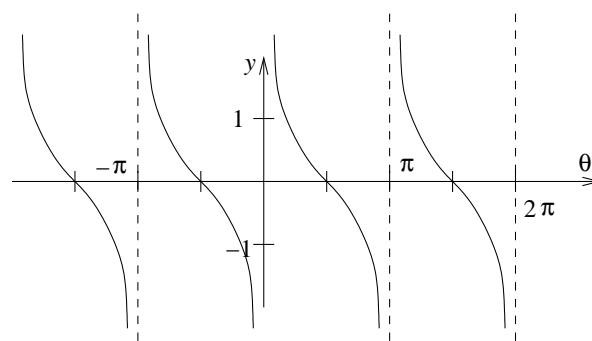
$$y = \csc\theta = \frac{1}{\sin\theta}$$

$$y = \sec\theta = \frac{1}{\cos\theta}$$
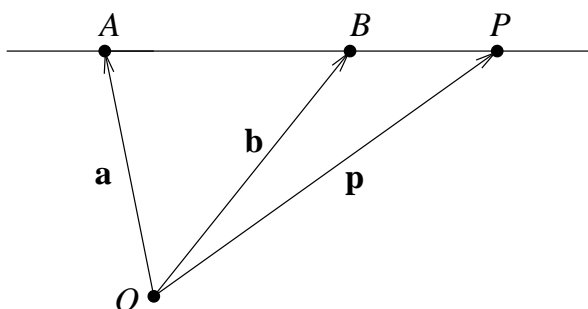
$$y = \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$y = \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

3 Applications of vectors

- Now that we are familiar with the different representations of vectors, we can investigate some of the ways vectors are used to solve mathematical problems.

- In pure mathematics, vectors can be used to solve problems in geometry.

- In applied mathematics, vectors can be used to model quantities such as displacement, velocity, or forces acting on an object, and thus answer questions about such vector quantities.

- There are two operations on vectors, the scalar product and the vector product, that also have useful applications.

- In this section we look at several applications of vectors and investigate the scalar and vector products.

- Topics in this section are:

  - Vectors in geometry
  - Forces
  - Displacement, velocity and momentum
  - The scalar product
  - The vector product

3.1 Vectors in geometry

- Let P be a point on the line through points A and B. We can use vectors to describe the position vector of P in terms of the position vectors of A and B.

- Let $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$.
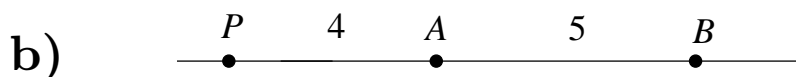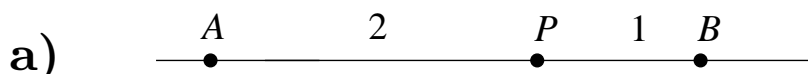

Now $\mathbf{a} + \overrightarrow{AB} = \mathbf{b}$ so $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$. Thus,

$$\mathbf{p} = \mathbf{a} + \overrightarrow{AP} = \mathbf{a} + t\overrightarrow{AB} = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = (1-t)\mathbf{a} + t\mathbf{b},$$

where t is the real number such that $\overrightarrow{AP} = t\overrightarrow{AB}$.

Example *3.1.1* Express $\mathbf{p}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ in each of the following cases.

a)

b)

- A *median* of a triangle is the line segment from a vertex of the triangle to the midpoint of the opposite side.

- The *centroid* of a triangle is the point where the three medians meet.

- The centroid of a triangle is its center of mass.

Example 3.1.2 Use vectors to show that the centroid of a triangle lies $\frac{2}{3}$ of the way along each median.

$AD = DB$
$BE = EC$
$CF = FA$

Example *3.1.2 (continued)*   Extra space for your working.
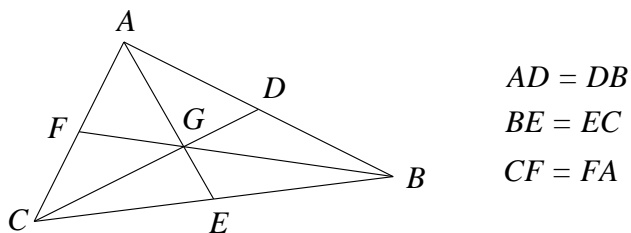
***Example*** *3.1.2 (continued)*   Extra space for your working.

3.2 Forces

- A force (a push or a pull on an object) can be modelled as a vector.

- The standard unit of measurement for forces is the newton, denoted N.

- **Weight:** This is the force due to gravity, and we will denote it by $\mathbf{W}$. The weight vector associated with an object of mass m kilograms has magnitude $9.8m$ newtons and has direction vertically downwards. (The acceleration due to gravity $g = 9.8$ m/s^2 is the magnitude of the acceleration with which an object close to the surface of the Earth falls back to Earth.)

- **Tension:** This is a force along a string or similar object, that counteracts a pulling force (such as weight) at the end of the string. We will denote it by $\mathbf{T}$. The tension adjusts its magnitude to exactly counteract the pulling force.

- **Normal reaction:** This is a force that pushes at a right-angle to a surface, and we will denote it by $\mathbf{N}$. For example, the normal reaction of a horizontal table-top pushes upwards to counteract the weight of a book placed on the table. Like tension, the normal reaction adjusts its magnitude to exactly counteract the downwards force.

- **Friction:** This is a force in the direction parallel to a surface. It is the force that counteracts an object sliding along a surface. We will denote it by $\mathbf{F}$.

- In our modelling of forces, we will assume that our objects can be represented by a particle, so we don't have to worry about which part of an object the forces act on. We will also make other simplifying assumptions in our modelling.

- For an object to be at rest, the forces acting on that object must be in balance. This is called the equilibrium condition for forces, and it means that the sum of all the forces acting on that object must be the zero vector.


Example *3.2.1*   An object with a mass of 2 kg is suspended from the ceiling by a string. Determine the tension acting on the object.

- If exactly three forces are acting on an object, and that object is at rest, then the three vectors representing the forces must sum to zero. Geometrically, this means that the three vectors must form a triangle, with the arrows pointing in a consistent direction around the triangle.
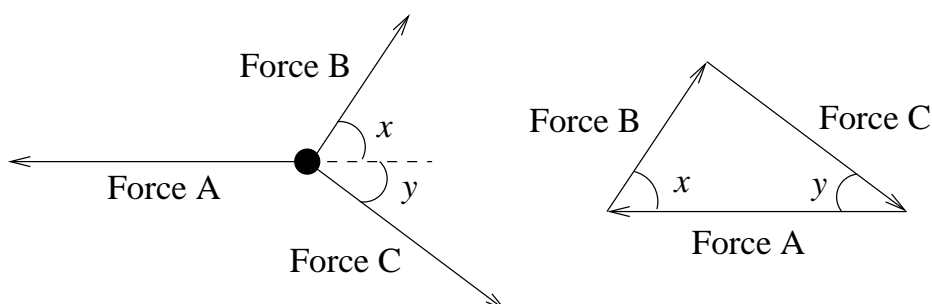
Example *3.2.2* A 3 kg brick is sitting on a ramp, inclined at $\pi/6$ radians to the horizontal. The only forces acting on it are weight, normal reaction from the ramp, and friction, as depicted in the following diagram. Determine the magnitude of each of the three forces.

Example *3.2.2 (continued)* Extra space for your working.
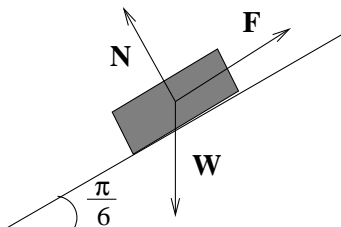
Example *3.2.2 (continued)* Extra space for your working.

3.3 Displacement, velocity and momentum

Example 3.3.1 A surveyor walks 200 metres due North. He then turns clockwise through an angle of $\frac{2\pi}{3}$ radians and walks 100 metres. Finally he turns and walks 300 metres due West. Find his resulting displacement, relative to his starting point.

Example _3.3.1 (continued)_   Extra space for your working.

Example _3.3.1 (continued)_   Extra space for your working.

Example *3.3.2*   A pilot wishes to fly in the direction N69°W to reach her destination. The light plane has a speed of 85 knots and there is a steady wind blowing from the north-east at 15 knots. What heading should the pilot set and what will be the speed of the plane relative to the ground?

Example *3.3.2 (continued)*   Extra space for your working.

Example *3.3.2 (continued)*   Extra space for your working.

Example *3.3.3* A river flows due East at a speed of 0.3 metres per second. A boy in a rowing boat, who can row at 0.5 m/s in still water, starts from a point on the South bank and steers due North. The boat is also blown by a wind with speed 0.4 m/s from a direction of N 20° E.

a) Find the resultant velocity of the boat.

b) If the river has a constant width of 10 metres, how long does it take the boy to cross the river, and how far upstream or downstream has he then travelled?

Example *3.3.3 (continued)* Extra space for your working.

Example *3.3.3 (continued)* Extra space for your working.

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- In order to understand the movement of two objects after they collide, we need the concept of momentum.

- Momentum is a vector property of a moving object. It is a scalar multiple of the velocity of the object, and is given by momentum $=$ mass times velocity.

- The standard unit for the magnitude of momentum is newton seconds, denoted Ns. To obtain momentum in Ns, your mass must be in kilograms and your velocity in metres per second.

- The important property of momentum is that it is conserved in collisions. That is, when objects collide, the total momentum before collision is equal to the total momentum after collision.

Example *3.3.4* A car weighing 1900 kg is travelling east at 22 m/s. A second car travelling north and weighing 2400 kg passes through a stop sign travelling at 14 m/s. It collides with the first car, and after the collision the cars move off together. Calculte their speed and direction after the collision.

(Note that we make some simplifying assumptions, namely that the cars do not brake, and that there is no friction from the road or air resistance to slow them down.)
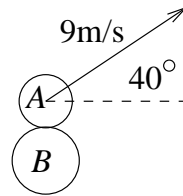
continued...

Example *3.3.4 (continued)*   Extra space for your working.

Example *3.3.4 (continued)*   Extra space for your working.

Example *3.3.5*   In outer space, object A weighing 2 kg, collides with object B weighing 3 kg, moving as shown in diagram (a). After the collision, A moves off as shown in diagram (b). Calculate the speed and direction of B after the collision.
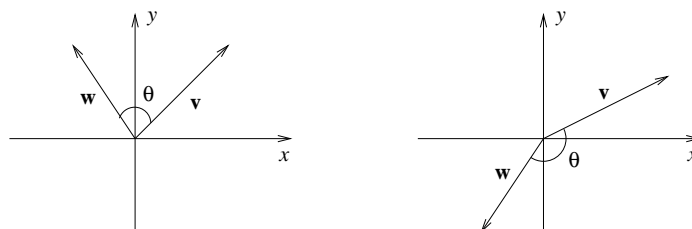

(a)   (b)

Example *3.3.5 (continued)* Extra space for your working.
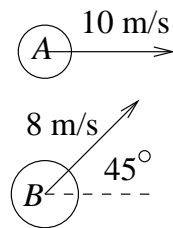
Example *3.3.5 (continued)* Extra space for your working.

3.4   The scalar product

- Let $\mathbf{v} = \overrightarrow{OP} \neq \mathbf{0}$ and $\mathbf{w} = \overrightarrow{OQ} \neq \mathbf{0}$ be two vectors. Then the angle between $\mathbf{v}$ and $\mathbf{w}$ is the angle θ between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ at the origin, with $0 \leq \theta \leq \pi$.

- The *scalar product* of two vectors $\mathbf{v}$ and $\mathbf{w}$ is

$$\mathbf{v} \cdot \mathbf{w} = \begin{cases} 0 & \text{if } \mathbf{v} = \mathbf{0} \text{ or } \mathbf{w} = \mathbf{0}, \\ \|\mathbf{v}\| \, \|\mathbf{w}\| \, \cos \theta & \text{otherwise,} \end{cases}$$

  where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

- The scalar product is also called the *dot product* or *inner product*.

- If $\mathbf{v}$ and $\mathbf{w}$ are in matrix form, then the scalar product is easy to calculate. If $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$, then

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2.$$

  If $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$, then

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

- The calculation $\mathbf{v} \cdot \mathbf{w}$ in matrix form is simply matrix multiplication of $\mathbf{v}$ as a row vector and $\mathbf{w}$ as a column vector.

$$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v_1 w_1 + v_2 w_2$$

Example *3.4.1*

a) The vector **u** has magnitude 5 and direction $\frac{5\pi}{4}$ radians. The vector **v** has magnitude 3 and direction $\frac{7\pi}{12}$ radians. Calculate **u** · **v**.

b) The vector **v** $= (-3, 2)$ lies in the 2nd quadrant. Find a vector in the 3rd quadrant that is perpendicular to **v**.

- If we are given two vectors **v** and **w** in matrix or component form, then we can use the scalar product to calculate the angle between **v** and **w**.

Example *3.4.2* Calculate the angle (in both radians and degrees) between vectors $\mathbf{u} = (4, 3, 1)$ and $\mathbf{v} = (2, -3, -2)$.

Properties of the scalar product

(1) The scalar product of two vectors is a *scalar*, not a vector.

(2) $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$ since the angle between $\mathbf{v}$ and itself is 0 and $\cos 0 = 1$.

(3) $\mathbf{v} \cdot \mathbf{w} = 0$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are perpendicular. (Perpendicular vectors are also called orthogonal vectors.)
   Proof If $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$, then

$$\mathbf{v} \cdot \mathbf{w} = 0 \quad \text{iff} \quad ||\mathbf{v}|| \, ||\mathbf{w}|| \, \cos\theta = 0,$$
$$\text{iff} \quad \cos\theta = 0,$$
$$\text{iff} \quad \theta = \pi/2.$$

(4) For vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.

(5) For vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

(6) For vectors $\mathbf{v}$ and $\mathbf{w}$ and any real number t, $(t\mathbf{v}) \cdot \mathbf{w} = t(\mathbf{v} \cdot \mathbf{w})$.

Resolving vectors

- In vector applications we often want to know what the component of a vector $\mathbf{v}$ is in a particular direction. If that direction is given by a vector $\mathbf{w}$, then we call this the *projection* of $\mathbf{v}$ onto $\mathbf{w}$, or the component of $\mathbf{v}$ in direction $\mathbf{w}$.

- For a vector $\mathbf{v} = (v_1, v_2)$ the projection of $\mathbf{v}$ onto $\mathbf{i}$ is $v_1\mathbf{i} = (v_1, 0)$. The projection of $\mathbf{v}$ onto $\mathbf{j}$ is $v_2\mathbf{j} = (0, v_2)$.
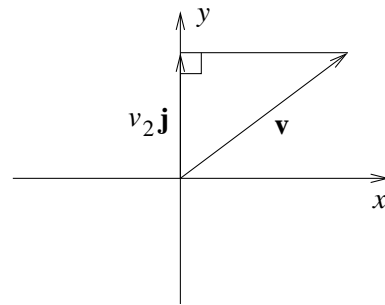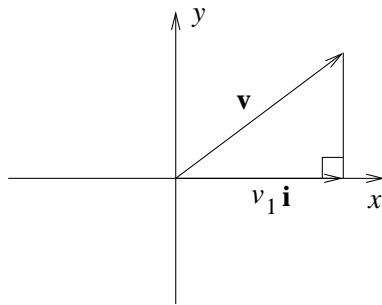
- Sometimes we are interested in the projection itself (a vector) and sometimes we are only interested in the length of the projection (a scalar). The length (magnitude) of the projection of $\mathbf{v}$ onto $\mathbf{w}$ is effectively 'how much' of $\mathbf{v}$ is in the direction of $\mathbf{w}$.

- To find the projection of a vector $\mathbf{v}$ onto a vector $\mathbf{w}$, we are effectively finding two vectors $\mathbf{w}_1$ and $\mathbf{w}_1^\perp$ such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_1^\perp$, where $\mathbf{w}_1$ is in the direction of $\mathbf{w}$ and $\mathbf{w}_1^\perp$ is perpendicular to $\mathbf{w}_1$. We call this a resolution of the vector $\mathbf{v}$. The vector $\mathbf{w}_1$ is the projection of $\mathbf{v}$ onto $\mathbf{w}$.
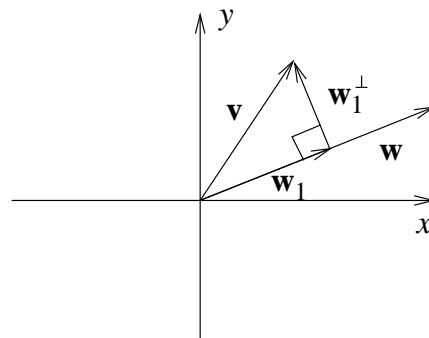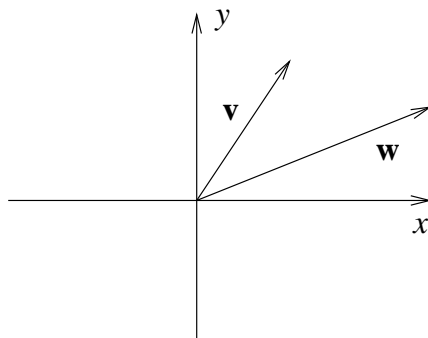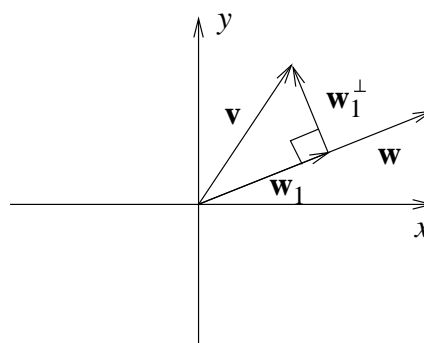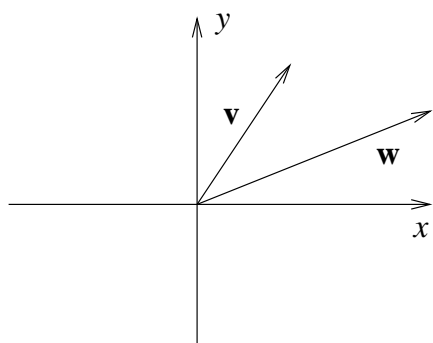
The projection, $\mathbf{w}_1$, of a vector $\mathbf{v}$ onto a vector $\mathbf{w}$ is a scalar multiple of the vector $\mathbf{w}$. It is given by

$$\mathbf{w}_1 = \frac{\mathbf{w} \cdot \mathbf{v}}{||\mathbf{w}||^2}\mathbf{w}.$$

The length of the projection of $\mathbf{v}$ onto $\mathbf{w}$ is

$$||\mathbf{w}_1|| = \frac{|\mathbf{w} \cdot \mathbf{v}|}{||\mathbf{w}||}.$$

Proof Now $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_1^{\perp}$ where $\mathbf{w}_1 = t\mathbf{w}$ for some real number t, and $\mathbf{w}_1^{\perp} \cdot \mathbf{w} = 0$ since $\mathbf{w}_1^{\perp}$ and $\mathbf{w}$ are perpendicular. Thus

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= (t\mathbf{w} + \mathbf{w}_1^{\perp}) \cdot \mathbf{w} \\
&= t\mathbf{w} \cdot \mathbf{w} + \mathbf{w}_1^{\perp} \cdot \mathbf{w} \\
&= t\mathbf{w} \cdot \mathbf{w} \\
&= t||\mathbf{w}||^2.
\end{aligned}$$

Thus $t = \dfrac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2}$ and so

$$\mathbf{w}_1 = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2}\mathbf{w} \quad \text{and} \quad ||\mathbf{w}_1|| = \frac{|\mathbf{v} \cdot \mathbf{w}|}{||\mathbf{w}||}.$$

Example 3.4.3

a) The vector **v** has magnitude 5 and direction $\pi/4$. The vector **w** has magnitude 8 and direction $\pi/6$. Find the projection of **v** onto **w**.

**b)** The vector $\mathbf{u} = (-2, 4)$ and the vector $\mathbf{v} = (1, 5)$. Find the projection, $\mathbf{v}_1$, of **u** onto **v**. Then determine $\mathbf{v}_1^{\perp}$ such that $\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_1^{\perp}$.

Work

- The scalar quantity *work* is a measure of the energy transfered to an object when a force $\mathbf{f}$ moves the object through a displacement $\mathbf{d}$.

- If the force and the displacement have the same direction, then the enery gained is $||\mathbf{f}|| \, ||\mathbf{d}||$. If the force and the displacement do not have the same direction, then only the component of $\mathbf{f}$ in the direction of $\mathbf{d}$ contributes to the energy gained by the object. The component of $\mathbf{f}$ in the direction of $\mathbf{d}$ is $||\mathbf{f}|| \cos\theta$, where θ is the angle between $\mathbf{f}$ and $\mathbf{d}$. Thus Work $= \mathbf{f} \cdot \mathbf{d}$.

- If $||\mathbf{f}||$ is in newtons, and $||\mathbf{d}||$ is in metres, then Work is measured in Joules.


Example 3.4.4 An ice skater exerts a force of 90 N on his skating partner as shown. Calculate the work done by the skater in moving his partner a distance of 6 metres at the angle shown.
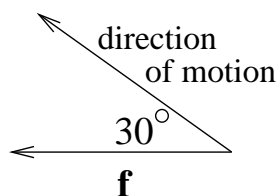
3.5   The vector product

- Given a pair of vectors, it is often useful to find a vector which is perperdicular to both of them. For example, in 3-space, any two non-collinear vectors define a plane, and we often need to find a vector which is perpendicular to such a plane.

- Consider the vectors $\mathbf{v}$ and $\mathbf{w}$ shown below along with the plane they define. There are two possible directions for a vector that is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ (and hence perpendicular to the plane), the vector can either go upwards from the plane $\mathbf{u}_1$, or downwards, $\mathbf{u}_2$.
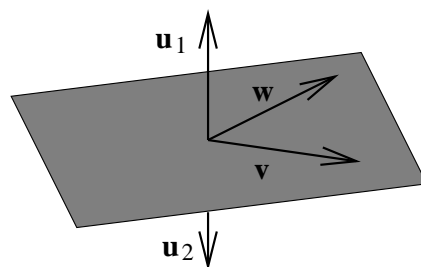

- The vector $\mathbf{u}_1$ is said to be in a *right-hand direction* from $\mathbf{v}$ to $\mathbf{w}$. If you consider your right hand with your thumb pointing outwards then if you curl your fingers in the direction from $\mathbf{v}$ to $\mathbf{w}$, then your thumb points in the direction of $\mathbf{u}_1$.

- The *vector product* of two vectors $\mathbf{v}$ and $\mathbf{w}$ is the vector $\mathbf{v} \times \mathbf{w}$ satisfying

  (1) $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}|| \, ||\mathbf{w}|| \, \sin \theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, and

  (2) $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ and in a right-hand direction from $\mathbf{v}$ to $\mathbf{w}$.

- The vector product is also called the *cross product*.

***Example* 3.5.1** Consider the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in 3-space.

Compute the nine vector products to fill in the following table.

First vector

	Second vector		
$\times$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$			
$\mathbf{j}$			
$\mathbf{k}$			
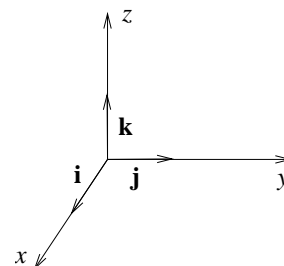
- Note that in any diagram of 3-space, the arrangement of the axes must satisfy the right-hand rule so that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

Properties of the vector product

- The vector product is a vector, *not* a scalar.

- For two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, $||\mathbf{u} \times \mathbf{v}|| = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel or antiparallel.

$$||\mathbf{u} \times \mathbf{v}|| = 0 \quad \text{iff} \quad \sin\theta = 0,$$
$$\text{iff} \quad \theta = 0 \text{ or } \theta = \pi.$$

- For any two vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

- The vector product is not associative, so for most vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.
  For example, $(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$ but $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times -\mathbf{j} = -\mathbf{k}$.

- For vectors $\mathbf{u}$ and $\mathbf{v}$ and any real number t

$$t(\mathbf{u} \times \mathbf{v}) = (t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v}).$$

- For vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$,

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}, \text{ and } (\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}.$$

We can use the properties of the vector product and the table of vector products of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ to calculate the vector product of any pair of vectors expressed in component form.

Example *3.5.2*   If $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, calculate $\mathbf{v} \times \mathbf{w}$.

Although this method works, it can be a bit tedious. We can take a shortcut by making use of a determinant.

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ then

$$\mathbf{u} \times \mathbf{v} = \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

So $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ where the determinant must be expanded along its top row.

Warning! This is not a real determinant, we are just using it as a calculation aid.

Example *3.5.3*   For vectors $\mathbf{v} = (2, -1, 4)$ and $\mathbf{w} = (1, 2, 3)$ from the previous example, show that the determinant method gives the same answer for $\mathbf{v} \times \mathbf{w}$.

Area of a triangle

- The area of a triangle with base length b and height h is given by $\frac{1}{2}bh$.

- The triangle formed by the vectors $\mathbf{v}$ and $\mathbf{w}$ has base of length $||\mathbf{v}||$, and height given by $||\mathbf{w}||\sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.
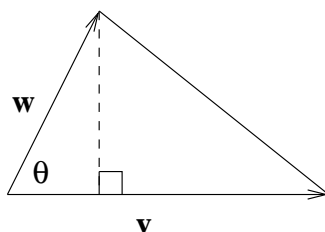
- Thus, the area of the triangle formed by vectors $\mathbf{v}$ and $\mathbf{w}$ is

$$\text{Area } = \tfrac{1}{2}||\mathbf{v}|| \, ||\mathbf{w}|| \, \sin\theta = \tfrac{1}{2}||\mathbf{v} \times \mathbf{w}||.$$

Example *3.5.4*   Calculate the area of the triangle OAB where A is the point with coordinates $(3, -1, 1)$ and B is the point with coordinates $(-2, 2, 1)$.
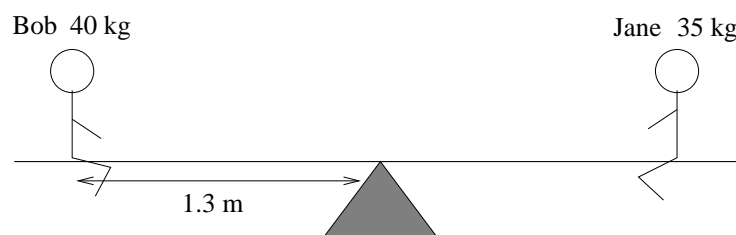
Torque

- When you use a spanner to turn a nut, or use different gears while riding a bicycle, your choice of size of spanner or particular gear on the bike is based on a turning force called *torque.*

- The magnitude of the torque is given by $||\mathbf{r} \times \mathbf{f}||$ where $\mathbf{f}$ is the force and $\mathbf{r}$ is the vector from the point of application of the force to the point (or axis) about which the object is turning.

- The standard unit for torque is Newton metres, denoted Nm. To obtain torque in Nm, $\mathbf{f}$ should be in newtons and $\mathbf{r}$ should be in metres.

- Since $\sin \theta$ reaches its maximum at $\pi/2$ radians ($90°$), the torque is a maximum when the angle between $\mathbf{f}$ and $\mathbf{r}$ is $\pi/2$ radians.

Example *3.5.5* Bob and Jane are sitting on a horizontal seesaw. Bob has a mass of 40 kg and Jane has a mass of 35 kg.
a) If Bob is sitting 1.3 metres from the center, calculate the magnitude of the torque that Bob's weight exerts about the centre point.
b) Where should Jane sit to keep the seesaw balanced in a horizontal position (with no feet on the ground).

Example *3.5.5 (continued)*

4   Sequences and Series

- A *sequence* is an ordered set of numbers, and we call the numbers in the sequence the *terms* of the sequence.

- Some sequences have an identifiable pattern, allowing us to develop a formula to calculate any particular term of the sequence directly, or a formula to calculate the value of a term from the values of preceding terms.

- As well as illustrating the beauty of mathematics, sequences have many applications. For instance, the calculation of mortgage repayments and estimating population growth.

- A *series* is the sum of the terms of a sequence. We will investigate mathematical induction, a proof technique that is useful in proving results about series.

- Topics in this section are:

  - Introduction to sequences
  - Review of logarithms
  - Arithmetic and geometric sequences
  - The Fibonacci sequence
  - Arithmetic and geometric series
  - Finite differences and telescoping sums
  - Applications of sequences
  - Mathematical induction

4.1   Introduction to sequences

- A *sequence* is an ordered set of numbers.

- We call each number in a sequence a *term* of the sequence.

- A sequence with a fixed number of terms is a *finite* sequence, and a sequence with an infinite number of terms is an *infinite* sequence.

- To describe a sequence, we usually choose a particular letter and combine that letter with subscripts to represent the terms of the sequence. For example, in the sequence

$$a_1, a_2, a_3, \ldots, a_{10}$$

  a_1 represents the first term, a_2 the second term, and so on, and a_{10} is the 10th and final term of the sequence. The notation a_n represents a general term of the sequence, and we use the notation $\{a_n\}$ or $\{a_n\}_{n=1}^{10}$ to represent the entire sequence.

- The symbol ∞ is used for infinity, and so an infinite sequence might be written as $\{a_n\}_{n=1}^{\infty}$.

Example *4.1.1*   Let $\{a_n\}_{n=0}^{\infty}$ be a sequence with $a_n = 2n - 1$. List the first four terms of the sequence.

(We could also write this sequence as $\{2n - 1\}_{n=0}^{\infty}$.)

- A general rule that allows you to calculate the terms of a sequence can be specified in two ways.

- A formula that allows direct calculation of the value of any particular term of a sequence is called a *closed form* for the sequence. For example, the sequence $3, 5, 7, 9, 11$ has closed form $3 + 2n$ ($n = 0, 1, 2, 3, 4$).

- A formula that specifies how to calculate the value of a term using the value(s) of one (or more) preceding terms is called a *recurrence relation*. A *recursive* definition of a sequence is a recurrence relation combined with initial conditions giving the value(s) of one (or more) initial terms of the sequence. For example, the sequence $3, 5, 7, 9, 11$ can be defined recursively as $a_n = a_{n-1} + 2$ ($n = 1, 2, 3, 4$) with initial condition $a_0 = 3$.

Example *4.1.2*   Find both a closed form and a recursive definition of a sequence whose first four terms are

$$2, 4, 8, 16.$$

- It is important to be aware that specifying the first few terms of a sequence does not adequately describe the sequence.

- You probably thought that the next term of the sequence in the previous example was 32. The sequence with closed form

$$a_n = 2^n \quad (n = 1, 2, \dots)$$

certainly has $2, 4, 8, 16, 32$ as its first five terms.

- However, in the previous example, if we had chosen the closed form

$$a_n = (n-1)(n-2)(n-3)(n-4) + 2^n \quad (n = 1, 2, \dots),$$

then the first five terms of the sequence would be

$$2, \quad 4, \quad 8, \quad 16, \quad 56.$$

- Simply writing down the first few terms of a sequence is never enough to specify the remainder of the sequence!

- In order to describe a sequence, if you cannot write down all the terms of the sequence, then you must give a general description of the sequence such as a closed form or a recurrence relation with initial conditions.

4.2 Review of logarithms

- Recall the sequence $\{a_n\}_{n=1}^{\infty}$ with closed form $a_n = 2^n$ from the previous section. Which term would be the first term to exceed 7000? To answer this, we need to find the smallest value of n such that $2^n > 7000$. To find n, we could solve $2^x = 7000$ and then choose n to be the smallest integer that is greater than or equal to x.

- Solving equations such as $2^x = 7000$ is best done using logarithms, which we review briefly here.

- The notation
$$\log_a b$$
means "to what power do we raise a to obtain the value b?" Hence, the statement $\log_a b = c$ is equivalent to the statement $a^c = b$.

- For example, $\log_5 25 = 2$, $\quad \log_7 1 = 0 \quad$ and $\quad \log_9 3 = \frac{1}{2}$.

- The notation $\log_a b$ is read as the logarithm of b with base a or as the logarithm with base a of b.

Example *4.2.1* Evaluate the following:

(a) $\log_4 64$ **(b)** $\log_5 \frac{1}{25}$ **(c)** $\log_{27} 3$

- To solve the equation $2^x = 7000$ we need to evaluate $x = \log_2 7000$. Since $2^{12} = 4096$ and $2^{13} = 8192$, the value of x will be between 12 and 13. To find an approximate value for x, we can use a calculator.

- Most calculators can only evaluate logarithms having one of two particular bases, 10 or e.

- The number $e = 2.71828\ldots$ is a very important irrational number, and $\log_e b$ is called the *natural logarithm* of b.

- When we write log without a base, the base is understood to be 10 and when we write ln we mean a natural logarithm.

  $$\log 100 = \log_{10} 100 = 2 \ \text{ and } \ \ln 100 = \log_e 100 \approx 4.605.$$

- In order to use a calculator to solve the equation $2^x = 7000$, we apply the change of base rule. Provided that $a > 0$, $a \neq 1$ and $b > 0$,

  $$\log_a b = \frac{\log b}{\log a} = \frac{\ln b}{\ln a}.$$

- Thus, by converting a single logarithm with base 2 into a quotient of natural logarithms, we find that

  $$x = \log_2 7000 = \frac{\ln 7000}{\ln 2} \approx 12.773.$$

Example *4.2.2* Solve for x:

(a) $5^x = 32$ **(b)** $9^{x+1} = 100$

- Now consider the equation $(-3)^x = -27$. Clearly, this equation has the solution $x = 3$. However, we cannot use the method of Example 4.2.2 to determine this. Very strange things happen if you use a non-positive number for the base of a logarithm and also if you try to evaluate a logarithm of a non-positive number. For the purposes of this course, the notation $\log_a b$ is only defined if $a > 0$ and $b > 0$.

- There are some useful rules which allow us to manipulate expressions involving logarithms.

  If $a > 0$, $a \neq 1$, $x, y > 0$ and r is any real number, then

$$\log_a(x^r) = r \log_a x$$
$$\log_a(xy) = \log_a x + \log_a y$$
$$\log_a\left(\tfrac{x}{y}\right) = \log_a x - \log_a y$$

- The change of base formula arises from the equation $a^x = b$ by taking the logarithm of both sides of the equation and then applying the first law from above.

$$
\begin{aligned}
a^x &= b \\
\log(a^x) &= \log b \\
x \log a &= \log b \\
x &= \frac{\log b}{\log a}
\end{aligned}
$$

4.3   Arithmetic and geometric sequences

- An *arithmetic sequence* is a sequence in which the difference between any two successive terms in a constant.

- The difference between successive terms in an arithmetic sequence is called the *common difference.*

- For example, the A.S. with first term 2 and common difference 3 starts with the five terms

$$2, \ 5, \ 8, \ 11, \ 14.$$

  The A.S. with first term 10 and common difference -5 starts with the five terms

$$10, \ 5, \ 0, \ -5, \ -10.$$

Example *4.3.1*   Determine whether each of the following could be the first four terms of an arithmetic sequence.

a) $1, \ 2\frac{1}{2}, \ 4, \ 5\frac{1}{4}.$

b) $5, \ 2, \ -1, \ -4.$

c) $3, \ 4 + 2x, \ 5 + 4x, \ 6 + 6x.$

- If we know the values of the first term and the common difference of an arithmetic sequence, we can determine a recursive definition or a closed form for the sequence.

- Let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic sequence with first term a and common difference d. Then

$$a_{n+1} = a_n + d \quad (n = 1, 2, \ldots) \quad \text{and } a_1 = a$$

  is a recursive definition of the A.S.

- Let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic sequence with first term a and common difference d. Then

$$
\begin{aligned}
a_1 &= a \\
a_2 &= a + d & \text{so } a_2 &= a + 1d \\
a_3 &= a + d + d & \text{so } a_3 &= a + 2d \\
a_4 &= a + d + d + d & \text{so } a_4 &= a + 3d \\
a_5 &= a + d + d + d + d & \text{so } a_5 &= a + 4d
\end{aligned}
$$

  Thus,

$$a_n = a + (n - 1)d, \quad (n = 1, 2, \ldots),$$

  is a closed form for the A.S.

Example *4.3.2* The first three terms of an A.S. are $5, -2, -9$. Determine the 100th term of the sequence.

Example *4.3.3* The third term of an A.S is 7 and the twelfth term is 10. Determine a closed form for the sequence.

- A *geometric sequence* is a sequence in which the ratio between any two successive terms is a constant, that is, every term is equal to the previous term multiplied by a particular number.

- The ratio between successive terms in a geometric sequence is called the *common ratio.*

- For example, the G.S. with first term 2 and common ratio 3 starts with the five terms

$$2, \ 6, \ 18, \ 54, \ 162,$$

since $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = \frac{162}{54} = 3$.
The G.S. with first term 20 and common ratio $-\frac{1}{2}$ starts with the five terms

$$20, \ -10, \ 5, \ -2\frac{1}{2}, \ 1\frac{1}{4}.$$

Example *4.3.4* Determine whether each of the following could be the first four terms of a geometric sequence.

a) 27, 9, 3, -1.

b) 2, x, $\frac{1}{2}x^2$, $\frac{1}{4}x^3$.

- If we know the values of the first term and the common ratio of a geometric sequence, we can determine a recursive definition or a closed form for the sequence.

- Let $\{a_n\}_{n=1}^{\infty}$ be a geometric sequence with first term a and common ratio r. Then

$$a_{n+1} = a_n \times r \quad (n = 1, 2, \ldots) \quad \text{and } a_1 = a$$

  is a recursive definition of the G.S.

- Let $\{a_n\}_{n=1}^{\infty}$ be a geometric sequence with first term a and common ratio r. Then

$$
\begin{aligned}
a_1 &= a \\
a_2 &= a \times r & \text{so } a_2 &= ar^1 \\
a_3 &= a \times r \times r & \text{so } a_3 &= ar^2 \\
a_4 &= a \times r \times r \times r & \text{so } a_4 &= ar^3
\end{aligned}
$$

  Thus,

$$a_n = ar^{n-1} \quad (n = 1, 2, \ldots)$$

  is a closed form for the G.S.

Example *4.3.5*   The first three terms of a G.S. are $-\frac{1}{3}, 1, -3$. Determine the 10th term of the sequence.

Example *4.3.6*   The first three terms of a G.S are $4, 8, 16$. Which term would be the first to exceed one million?

4.4 The Fibonacci sequence

- The *Fibonacci sequence* is the sequence $\{F_n\}$ where

$$F_{n+2} = F_{n+1} + F_n \quad (n = 1, 2, 3, \dots) \quad F_1 = 1, \ F_2 = 1.$$

- The sequence is named after an Italian mathematician Leonardo of Pisa (1170-1250) who was better known by the name of Fibonacci.

- The sequence arises from a problem posed by Fibonacci in 1202: Start with one pair of rabbits, and suppose that

  - the initial pair of rabbits takes one month to mature, they produce a new pair at the end of the second month, and at the end of each subsequent month;

  - each new pair becomes adult after one month and produces its first new pair at the end of its second month and a new pair at the end of each subsequent month;

  - no rabbits die.

  How many pairs of rabbits are there after a year?

Example *4.4.1* Determine the number of pairs of rabbits at the beginning of each month for the first 12 months.

- A closed form for the Fibonacci sequence is

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

- The *golden ratio* is the name given to the number

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618.$$

Example *4.4.2* The ratio of successive terms of the Fibonacci sequence tends to a fixed value. Show that

$$\frac{F_{n+1}}{F_n} \to \phi \quad \text{as } n \to \infty.$$

- The golden ratio and terms of the Fibonacci sequence occur in many places.

- In art and architecture, often the pictures or buildings that look most pleasing involve the golden ratio. For example, the French architect Le Corbusier used the golden ratio in the design of some of his buildings, and Leonardo da Vinci used the golden ratio in some of his paintings.

- Composers such as Bartok and Debussy are known to have used the golden ratio in composing their music.

- Fibonacci numbers occur in the structure of plants. For example, the number of petals on many flowers is $3, 5, 8, 13, 21$ or 34, and the successive leaves along a stem are often rotated around the stem by the fraction $1/\phi$ of a full turn.

- The most studied occurence of Fibonacci numbers in nature is the sunflower. Looking at the head of a sunflower, the seeds form spirals, and it is common for the numbers of such spirals to be consecutive Fibonacci numbers. This may be a mathematical consequence of nature selecting an approximation to the golden ratio in order to pack sunflower seeds as efficiently as possible.

4.5   Arithmetic and geometric series

- We are often interested in the sum of the terms of a sequence.

- A short-hand way of writing the sum of the terms of a sequence uses the Greek capital letter 'sigma', written Σ.

- *Sigma notation* or *summation notation* is usually of the form $\displaystyle\sum_{lowerbound}^{upperbound}$ *expression*.

- Some examples of sigma notation (and what it means) are:

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5$$

$$\sum_{j=2}^{50} (3 + j^2) = (3 + 2^2) + (3 + 3^2) + \cdots + (3 + 50^2)$$

- The variable used in the lower bound (and usually in the expression) is called a dummy variable, because it doesn't matter what letter you choose. Thus,

$$\sum_{i=1}^{10} i^2 = \sum_{j=1}^{10} j^2.$$

- Note that the same sum can be written in slightly different ways. For example,

$$\sum_{i=1}^{10} 2i + 1 = \sum_{i=2}^{11} 2i - 1.$$

Example _4.5.1_   Write each of the following as an expanded sum.

a) $\displaystyle\sum_{i=3}^{7} x^i$

b) $\displaystyle\sum_{j=1}^{n} (-1)^j a_j$

c) $\displaystyle\sum_{i=1}^{4} 3$

Example _4.5.2_   Write each of the following in sigma notation.

a) $6 + 8 + 10 + 12 + 14 + 16$

b) $1 - 4 + 9 - 16 + 25 - 36 + 49 - 64$

c) $x^2 + 2x^4 + 3x^6 + 4x^8 + 5x^{10} + 6x^{12} + 7x^{14}$

- A *series* is the sum of the terms of a sequence. If $\{a_n\}_{n=1}^{N}$ is a sequence, then $\sum_{n=1}^{N} a_n$ is the corresponding series.

- We often use the notation S_n to denote the sum of the first n terms of a sequence. If n is less than the total number of terms in the sequence, then S_n is a called a *partial sum* of the sequence.

- We can graph the terms of a sequence $\{a_n\}$ with n on the horizontal axis and a_n on the vertical axis. If we draw a rectangle of width 1 and height a_n for each term a_n, then the partial sum corrsponds to the area given by the sum of the areas of the rectangles (where a rectangle below the horizontal axis has negative area).

(a) $a_n = 2n + 1$
     $(n = 1, 2, \ldots)$
     $S_6 = 48.$
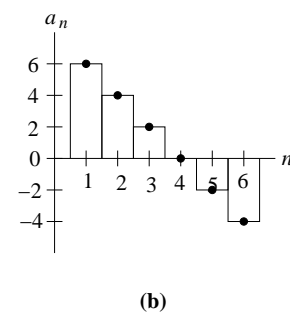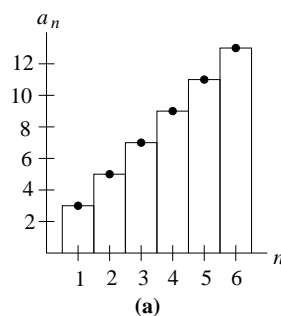
(b) $a_n = 8 - 2n$
     $(n = 1, 2, \ldots)$
     $S_6 = 6.$

(c) $a_n = 3^n$
     $(n = 1, 2, \ldots)$
     $S_5 = 363.$
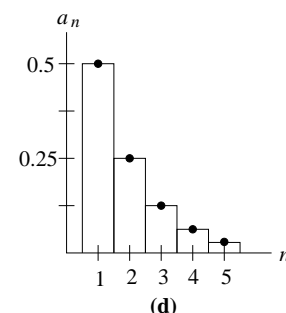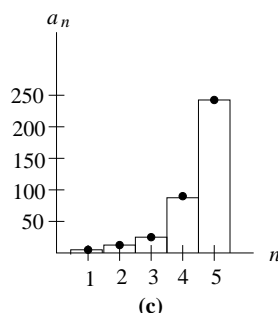
(d) $a_n = \left(\frac{1}{2}\right)^n$
     $(n = 1, 2, \ldots)$
     $S_5 = 0.96875.$

(a)

(b)

(c)

(d)

Arithmetic series

- If $\{a_j\}_{j=1}^{\infty}$ is an A.S. with first term a and common difference d, then

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j = \frac{n}{2}(2a + (n-1)d).$$

Proof Let S_n be the sum of the first n terms of the A.S. Thus

$$S_n = a + (a+d) + \cdots + (a + (n-2)d) + (a + (n-1)d).$$

Reversing the order of the series (which won't change the sum), we get

$$S_n = (a + (n-1)d) + (a + (n-2)d) + \cdots + (a+d) + a.$$

Adding these two equations together, we see that

$$
\begin{aligned}
2S_n & = (a + a + (n-1)d) + (a + d + a + (n-2)d) + \\
& \quad \cdots + (a + (n-2)d + a + d) + (a + (n-1)d + a) \\
\\
& = (2a + (n-1)d) + (2a + (n-1)d) + \\
& \quad \cdots + (2a + (n-1)d) + (2a + (n-1)d) \\
\\
& = n(2a + (n-1)d).
\end{aligned}
$$

Therefore,

$$S_n = \frac{n}{2}(2a + (n-1)d).$$

Example *4.5.3* Calculate the sum of the first n natural numbers.

Example *4.5.4* An arithmetic sequence has first term 3 and common difference $2\frac{1}{3}$. If the corresponding arithmetic series is equal to 3250, calculate the number of terms in the sequence.

Geometric series

- If $\{a_j\}_{j=1}^{\infty}$ is a geometric sequence with first term a and common ratio r, then

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}.$$

Proof Let S_n be the sum of the first n terms of the G.S. Thus

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}.$$

Multiplying both sides of this equation by r, we get

$$
\begin{aligned}
rS_n &= r(a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}) \\
&= ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n
\end{aligned}
$$

Subtracting the first equation from this one, we see that

$$
\begin{aligned}
rS_n - S_n &= (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n) \\
&\quad -(a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}) \\
&= ar^n - a.
\end{aligned}
$$

Therefore, $(r - 1)S_n = a(r^n - 1)$, so

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

By multiplying the numerator and denominator of this formula by -1, we obtain the alternative formula

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Example *4.5.5*   A geometric sequence has first three terms
$100, 94, 88.36$. Calculate the fewest number of terms needed for
the corresponding series to exceed 1000.

Infinite series

- Consider cutting a circular cake in half and eating one piece, then cutting the remaining piece in half and eating one of those pieces, and repeating this procedure over and over. After n steps, how much cake will you have eaten? Let S_n be the amount of cake you have eaten after n steps. Thus

$$S_1 = \tfrac{1}{2}, \quad S_2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}, \quad S_3 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{7}{8}, \ldots$$

The sizes of the pieces of cake you eat at each step of this process form the terms of an infinite geometric sequence $\{a_n\}$ with first term $\tfrac{1}{2}$ and common ratio $\tfrac{1}{2}$, so

$$a_n = \left(\tfrac{1}{2}\right) \times \left(\tfrac{1}{2}\right)^{n-1} = \left(\tfrac{1}{2}\right)^n.$$

S_n is the partial sum of the infinite geometric series. Thus,

$$S_n = \frac{\tfrac{1}{2}(1 - (\tfrac{1}{2})^n)}{1 - \tfrac{1}{2}} = \frac{\tfrac{1}{2} - (\tfrac{1}{2})^{n+1}}{\tfrac{1}{2}} = 1 - (\tfrac{1}{2})^n = 1 - \tfrac{1}{2^n}.$$

As n gets larger and larger, the partial sum gets closer and closer to 1, which we write as

$$\text{when } n \to \infty, \quad S_n \to 1.$$

In this case we say that the infinite geometric series *converges* with limit 1, and write

$$\sum_{i=1}^{\infty} a_n = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^n = 1.$$

- In order for an infinite series to converge, we need the terms of the corresponding sequence to approach zero. However, that is not enough to ensure that a series converges. For an infinite series to converge, we need the sequence of partial sums to approach a particular value.

- An infinite series that does not converge is said to *diverge*.

- An infinite arithmetic series will diverge since we are always adding a fixed amount.

- Diagram (a) illustrates the first few terms of the sequence $a_n = 1 + 2n$ ($n = 1, 2, \ldots$). The partial sum will get bigger and bigger as n increases, so the infinite arithmetic series will diverge.

- Diagram (b) illustrates the first few terms of the sequence $a_n = 8 - 2n$ ($n = 1, 2, \ldots$). The partial sums form the sequence

$$S_1 = 6, \ S_2 = 10, \ S_3 = 12, \ S_4 = 12, \ S_5 = 10, \ , S_6 = 6, \ \ldots.$$

As we continue to add a negative number, the partial sum will get bigger and bigger (in the negative sense) as n increases, so the infinite arithmetic series will diverge.
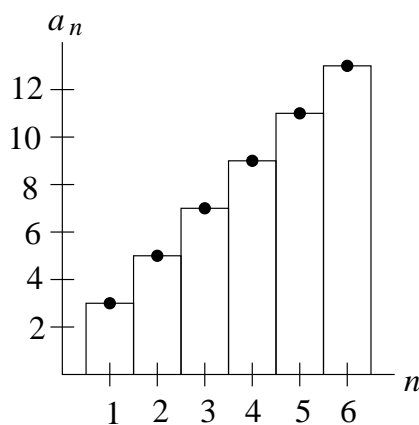
(a)
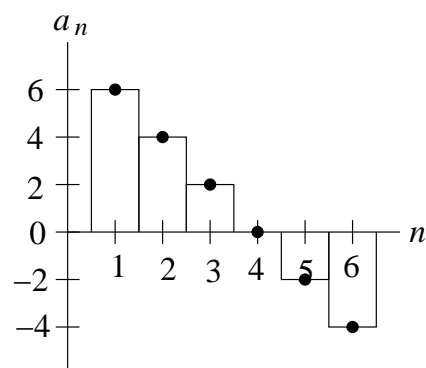
(b)

- An infinite geometric series will converge if the common ratio r satisfies $|r| < 1$. An infinite geometric series will diverge if the common ratio r satisfies $|r| \geq 1$.

- Diagram (a) shows the first few terms of a G.S. with $r = 3$. The corresponding infinite series will diverge.

- Diagram (b) shows the first few terms of a G.S with $r = \frac{1}{2}$. The corresponding infinite series will converge.

- Diagram (c) shows the first few terms of a G.S. with $r = -\frac{1}{2}$. The corresponding infinite series will converge.
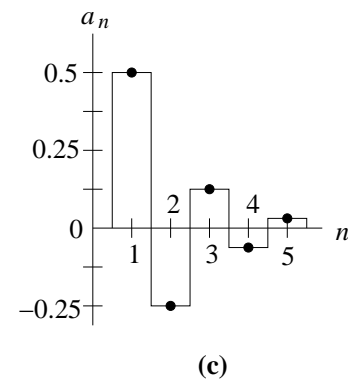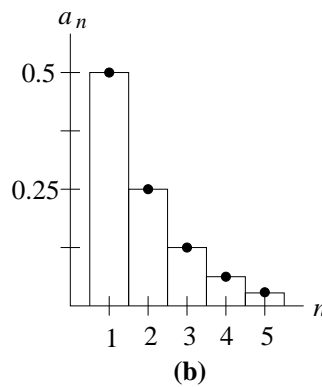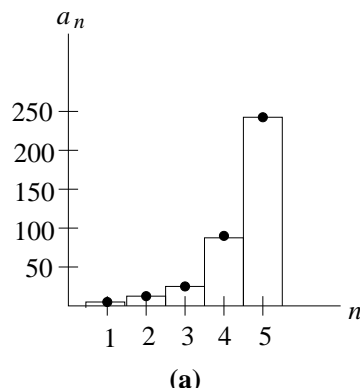
- An infinite geometric series $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$ where $|r| < 1$, converges and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

Proof The partial sum of this series is $S_n = \dfrac{a(1-r^n)}{1-r}$.
As $n \to \infty$, $\quad r^n \to 0$, so

$$S_n \to \frac{a}{1-r}.$$

- We could determine the sum of an infinite G.S. as follows.

$$
\begin{aligned}
\sum_{i=1}^{\infty} ar^{n-1} &= a + ar + ar^2 + ar^3 + \ldots \\
r\sum_{i=1}^{\infty} ar^{n-1} &= ar + ar^2 + ar^3 + \ldots \\
\hline
(1-r)\sum_{i=1}^{\infty} ar^{n-1} &= a
\end{aligned}
$$

Thus $\displaystyle\sum_{i=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

Example *4.5.6* A rubber ball is dropped from a height of 5 metres. Each time it hits the ground it bounces back up to one half the height from which it fell. Calculate the total distance the ball travels before it comes to rest.

4.6 Finite differences and telescoping sums

In this section we look at two methods to find a closed form for a given sequence.

Method of finite differences

- Arithmetic sequences are easy to work with because the difference between any two consecutive terms is a constant.

- We call the difference between successive terms of a sequence the *first difference*.

- If the first differences are all equal, then we have an arithmetic sequence and we know that the closed form is a linear polynomial in n, such as $2n + 3$, $5n - 1$ or $3n$.

- If the first differences are not all equal, we can look at the *second differences*, which are defined to be the differences between successive first differences.

- If the second differences are all equal, then we have a sequence whose closed form is a quadratic polynomial in n, such as $2n^2 + 3n - 1$, $n^2 + 4$ or $\frac{1}{2}(n^2 + n)$.

- Similarly, the third differences are defined to be the differences between successive second differences. If the third differences are all equal, then the sequence has a closed form which is a cubic polynomial in n.

Example *4.6.1*   Determine the first, second and third
differences for the first 6 terms of the sequence $\{n^3 + 2n\}_{n=1}^{\infty}$.

- In general, the ith differences are defined to be the
  differences between sucessive $(i-1)^{\text{th}}$ differences. If the ith
  differences are all equal, then the sequence has a closed
  form which is a polynomial in n, with highest power n^i.

- If you know that the closed form of a particular sequence
  will be a polynomial, you can set up a system of
  simultaneous linear equations and solve them to find the
  closed form.

Example *4.6.2*   Use finite differences to find a closed form for the sequence $1, 3, 6, 10, 15, 21, 28$.

(These numbers are the first seven terms in the sequence of *triangular numbers*.)

Telescoping sums

- A *telescoping sum* is a sum of similar expressions in which many terms pair up to cancel out.

- A telescoping sum can sometimes be used to determine a closed form for a sequence from its recursive definition.

Example *4.6.3* Use a telescoping sum to find a closed form for the sequence of triangular numbers, defined recursively by

$$a_n = a_{n-1} + n \quad (n = 2, 3, \ldots) \quad \text{where } a_1 = 1.$$

4.7   Applications of sequences

Simple interest

- When a bank account is paying simple interest, the interest is calculated as a percentage of the principal deposited into the account; interest is not paid on any interest previously deposited into your account.

Example *4.7.1*   On January 1 1999, you deposited $1000 into a savings account that pays simple interest of 7% per year. On each sucessive January 1, you add another $1000 to the savings account. After n years, how much money is in the account.

Compound Interest

- When a bank account earns compound interest, interest is calculated on the total amount in the account, including any interest previously deposited into the account.

- An amount A is invested at an interest rate of r per period, compounded at the end of each period. Let V_n be the value of the account after n periods. Thus

$$
\begin{aligned}
V_0 &= A \\
V_1 &= A + Ar = A(1 + r) \\
V_2 &= A(1 + r) + A(1 + r)r = A(1 + r)^2 \\
&\vdots \\
V_n &= A(1 + r)^{n-1} + A(1 + r)^{n-1}r = A(1 + r)^n,
\end{aligned}
$$

so $V_n = V_0(1 + r)^n = A(1 + r)^n$.

Example *4.7.2*   Suppose you deposit \$10,000 in a term deposit that pays interest at 5% per year, compounded monthly. What is the value of your term deposit after 5 years.

Population growth

- In any given year, the population of any particular group of organisms can change for a variety of reasons. In setting up a population model, it is useful to identify additions to the population (births and immigration) and deletions from the population (deaths and emigration).

- For now, we restrict our attention to situations where the only change in a population is due to births and deaths.

- Let P_n be the population in year n. If the population has a yearly birth rate of $b\%$ and death rate of $d\%$, then

$$P_{n+1} = P_n + \frac{b}{100}P_n - \frac{d}{100}P_n = P_n(1 + \frac{b-d}{100}).$$

- If we are given an initial population, then the above recurrence relation gives us a way to calculate the population change over time.

- Let P_0 be the initial population, and let $\frac{b-d}{100} = k$ the population growth rate. Then

$$
\begin{array}{rcl}
P_1 & = & P_0(1+k) \\
P_2 & = & P_1(1+k) = P_0(1+k)^2 \\
P_3 & = & P_2(1+k) = P_0(1+k)^3 \\
& \vdots & \\
P_n & = & P_{n-1}(1+k) = P_0(1+k)^n
\end{array}
$$

- Thus, for an initial population P_0 and growth rate k, the population after n years is $P_n = P_0(1+k)^n$.

- This type of population model is called the *exponential population model*. It assumes that the growth rate remains constant, which is not often the case in reality.

Example *4.7.3* The population of a town on January 1, 2003 was 13,690. The birth rate is estimated at 20% and the death rate at 18%. Assuming there is no immigration or emigration from this town, estimate the population on January 1, 2013.

Example *4.7.4* A colony of penguins in Antarctica numbered 26 in the year 1959 and 160 in the year 1969. Assuming that this population can be modelled accurately by the exponential population model, determine the annual growth rate of this population. Then predict what the population was in 1979.

4.8　Mathematical Induction

- To prove results about series we often use a mathematical proof technique called *mathematical induction.*

- Before explaining the technique in general, consider the following example.

Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively as

$$a_1 \;=\; 2 \tag{1}$$

$$a_{k+1} \;=\; 2 \times a_k \text{ for } k = 1, 2, 3, \ldots . \tag{2}$$

We could use the formula for the general term of a geometric sequence to determine that $a_n = 2^n$, but now we show that $a_n = 2^n$ using a different type of argument.

It follows from Equation 2 that, for any natural number k,

$$\text{if } a_k = 2^k, \text{ then } a_{k+1} = 2 \times a_k = 2 \times 2^k = 2^{k+1}.$$

Since we know that $a_1 = 2$, this statement (with $k = 1$) tells us that $a_2 = 4$. The next step uses the deduction that $a_2 = 4$ and the above statement with $k = 2$, to deduce that $a_3 = 8$. Continuing in this way, we deduce that $a_4 = 16$, $a_5 = 32$, and so on.

Formally, let $P(n)$ be the claim that $a_n = 2^n$. Then we know that

1. $P(1)$ is true;

2. if $P(k)$ is true, then $P(k+1)$ is true for $k = 1, 2, 3, \ldots .$

From these two facts, we can deduce that $P(2)$ is true, then that $P(3)$ is true, and so on. In general, we can deduce that $P(n)$ is true for all natural numbers n.

Thus $a_n = 2^n$ for all natural numbers n.

Mathematical induction

Let $P(n)$ be a claim about a natural number n, where n is greater than or equal to a particular natural number n_0. If

1. $P(n_0)$ is true, and

2. if $P(k)$ is true, then $P(k+1)$ is also true, for all natural numbers $k \geq n_0$,

then $P(n)$ is true for all $n \geq n_0$.

Example *4.8.1* Prove that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, for all $n \geq 1$.

Example *4.8.2* Prove that $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, for all $n \geq 1$.

Example *4.8.3*   Show that 7 divides $8^n - 1$ for all $n \geq 1$.

Example *4.8.3*   Show that 7 divides $8^n - 1$ for all $n \geq 1$.

5 Real and complex numbers

- Number systems evolved from a basic need to count and to measure. As the need arose to solve more sophisticated problems, less intuitive (but very useful) number systems were introduced.

- You are probably already familiar with the real numbers. This section starts with a review of the number systems that make up the real numbers.

- Complex numbers evolved from the need to solve quadratic equations that have no solutions in the real number system. To do this, the symbol i was introduced, where $i^2 = -1$. Although this seems artificial (and the term 'imaginary numbers' is sometimes used for complex numbers), this breakthrough had a huge impact and complex numbers are widely used in mathematics, physics and engineering.

- We will look at several representations of complex numbers and investigate some important theorems.

- Topics in this section are:
  - Real numbers
  - Absolute value
  - Introduction to complex numbers
  - Complex numbers in polar form
  - Powers of complex numbers
  - Fundamental theorem of algebra

5.1   Real numbers

A real number can be represented by a decimal expansion (finite or infinite). The set of real numbers, denoted $\mathbb{R}$, contains several important number systems.

- The *natural numbers*, denoted $\mathbb{N}$, are the counting numbers $\{1, 2, 3, 4, \dots\}$.

- The *integers*, denoted $\mathbb{Z}$, are the positive natural numbers, the negative natural numbers and zero.

- The *rational numbers*, denoted $\mathbb{Q}$, are the real numbers that can be represented in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. In decimal form, each rational number is a terminating or repeating decimal.

- The *irrational numbers*, denoted $\bar{\mathbb{Q}}$, are the real numbers that are not rational. In decimal form, each irrational number is a non-terminating, non-repeating decimal.

The following diagram shows how these number systems are related.
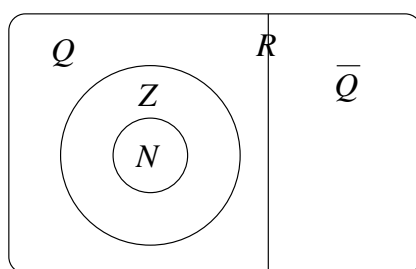
The set of real numbers can be represented by a number line.
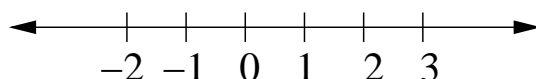
Surds

- A *surd* is an irrational number which is represented by a root sign. For example, $\sqrt{2}$, $\sqrt{7}$ and $\sqrt[3]{15}$ are surds, but $\sqrt{9}$ and $\sqrt[3]{125}$ are not surds.

Example *5.1.1* Simplify the following expressions.
a) $5\sqrt{3} + 2\sqrt{12} - \sqrt{18}$

b) $5\sqrt{3a^3b^2}\sqrt{12a}$ where a and b are positive real numbers.

- Rationalising the denominator is the process of multiplying the numerator and denominator of a fraction by an appropriate number so that the result does not have a surd in the denominator.

Example *5.1.2* Simplify the expression $\dfrac{\sqrt{6} + 3\sqrt{2}}{3 + \sqrt{5}}$.

Interval notation

- The real numbers are ordered, so we can describe one real number as being less than, equal to, or greater than, another real number.

- A subset of consecutive real numbers can be represented graphically on the number line, or using interval notation.

- The set of all $x \in \mathbb{R}$ such that $a \leq x \leq b$, written $\{x \in \mathbb{R} \mid a \leq x \leq b\}$, is illustrated below. In interval notation, this *closed interval* is denoted $[a, b]$.
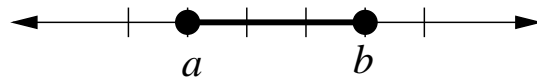
- The set of all $x \in \mathbb{R}$ such that $a < x < b$, written $\{x \in \mathbb{R} \mid a < x < b\}$, is illustrated below. In interval notation, this *open interval* is denoted (a, b).
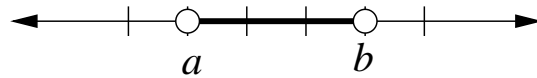
- The sets $\{x \in \mathbb{R} \mid a \leq x < b\}$ and $\{x \in \mathbb{R} \mid a < x \leq b\}$ are illustrated below. In interval notation, these *half-open, half-closed intervals* are denoted $[a, b)$ and $(a, b]$, respectively.
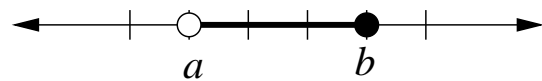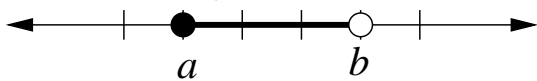
- Infinity is not a real number! However we use the symbol for infinity, ∞, to describe intervals of real numbers that extend forever.

$$\{x \in \mathbb{R} \mid x \geq a\} = [a, \infty) \qquad \{x \in \mathbb{R} \mid x > a\} = (a, \infty)$$
$$\{x \in \mathbb{R} \mid x \leq a\} = (-\infty, a] \qquad \{x \in \mathbb{R} \mid x < a\} = (-\infty, a)$$
$$\mathbb{R} = (-\infty, \infty)$$

You never have a closed interval with the symbol ∞ in it!

Inequalities

Example *5.1.3* Determine the values of x for which $\dfrac{3}{x} < 6$.

Example *5.1.4* Determine the values of x for which $(x-2)(x+1) \geq 0$.

Example *5.1.5* Determine the values of x for which
$$\frac{x-3}{x+2} \le 0.$$

5.2   Absolute value

- The *absolute value*, or *modulus*, of a real number x is given by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

  So, for every real number x, $|x| \geq 0$.

- The distance between two real numbers a and b is

$$|b - a| = |a - b| = \begin{cases} b - a & \text{if } b \geq a, \\ a - b & \text{if } b < a. \end{cases}$$

  Thus $|x|$ gives the distance between x and 0.

Properties of the absolute value

- For every real number x, $|x|^2 = x^2$.

  Proof If $x \geq 0$, then $|x| = x$, so $|x|^2 = x^2$.
  If $x < 0$, then $|x| = -x$ so $|x|^2 = (-x)(-x) = x^2$.

- For every pair of real numbers x and y, $|xy| = |x| \cdot |y|$.

  Proof If $x \geq 0$ and $y \geq 0$ then $|xy| = xy$ and $|x| \cdot |y| = xy$.
  If $x < 0$ and $y < 0$ then $|xy| = xy$ and
  $|x| \cdot |y| = (-x)(-y) = xy$.
  If $x \geq 0$ and $y < 0$ then $|xy| = -(xy)$ and
  $|x| \cdot |y| = x(-y) = -xy$. Similarly for $x < 0$ and $y \geq 0$.

- For every pair of real numbers x and y, $|x + y| \leq |x| + |y|$.
  (This property is called the triangle inequality.)

Example *5.2.1*   Solve the following equations.

a) $|4 - 3x| = 3$

b) $|y - 3| = 3y + 8$

c) $|a - 2| = |2a + 5|$

Example *5.2.2* Solve the following inequalities and illustrate the solutions on a number line.

a) $|2x - 3| < 2$

b) $\dfrac{1}{|x + 2|} \leq 3$

Example 5.2.3 Solve $\left| \dfrac{y-2}{y+3} \right| < 2$ and illustrate the solution on a number line.

5.3 Introduction to complex numbers

- By the 16th century, it was known that the quadratic equation

$$ax^2 + bx + c = 0, \quad \text{where } a, b, c \in \mathbb{R}$$

  has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

  provided that $a \neq 0$ and $b^2 - 4ac \geq 0$.

- The quantity $b^2 - 4ac$ is called the *discriminant* of the quadratic equation. A quadratic equation (with real coefficients) having a negative discriminant has no solution over the real numbers.

- In the next century, the first steps were taken to enlarge the real number system and create a new number system called the *complex numbers*, denoted $\mathbb{C}$, in which all quadratic equations have solutions.

- The idea that allowed the formation of the complex numbers was the introduction of the symbol i, which satisfies $i^2 = -1$.

- Eventually engineers and scientists discovered uses for complex numbers and they are now used extensively in many applications, including electric circuits and electromagnetic waves.

- A *complex number* is an expression of the form $a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. The set of such numbers is called the complex numbers and is denoted $\mathbb{C}$.

- If $z = a + bi \in \mathbb{C}$, then we call a the *real part* of z and b the *imaginary part* of z. We write this as $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$.

- The representation of a complex number in the form $a + bi$ is called the *Cartesian form* of the complex number.

Example 5.3.1   Let $z = -3 + 4i$. Write down $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

- Two complex numbers $a + bi$ and $c + di$ are *equal* if and only if $a = c$ and $b = d$.

- Let $z = a + bi$ and $w = c + di$ be two complex numbers.

Addition

$$
\begin{aligned}
z + w &= (a + bi) + (c + di) \\
&= (a + c) + (b + d)i
\end{aligned}
$$

So $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$
and $\mathrm{Im}(z + w) = \mathrm{Im}(z) + \mathrm{Im}(w)$.

Multiplication

$$
\begin{aligned}
zw &= (a + bi)(c + di) \\
&= ac + adi + bci + bdi^2 \\
&= ac + adi + bci - bd \\
&= (ac - bd) + (ad + bc)i.
\end{aligned}
$$

Example 5.3.2   Let $z = -3 + 2i$ and $w = 4 - i$. Write each of the following in Cartesian form.

a) $3z + 2w$

b) wz

c) iw

Example 5.3.3   Solve for $x \in \mathbb{C}$ where $x^2 + 2x + 2 = 0$.

- Let $z = a + bi \in \mathbb{C}$. The *complex conjugate* of z, denoted $\bar{z}$, is $\bar{z} = a - bi$.

- The complex conjugate is useful because the product $z\bar{z}$ gives a real number.

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$$

- The complex conjugate allows us to divide by a non-zero complex number.

$$
\begin{aligned}
\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\
&= \frac{(a + bi)(c - di)}{c^2 + d^2} \\
&= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i
\end{aligned}
$$

Example 5.3.4   Write $\dfrac{2 + i}{2 - i}$ in Cartesian form.

- The *modulus* of the complex number $z = a + bi$ is denoted $|z|$. It is defined to be

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

Properties of the modulus and complex conjugate

Let $z = a + bi$ and $w = c + di$ be complex numbers.

- $|z| = |\bar{z}|$

- $|z|^2 = z\bar{z}$

- $\overline{z + w} = \bar{z} + \bar{w}$

- $\overline{zw} = \bar{z}\,\bar{w}$

- $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\bar{z}}{\bar{w}}$

- $\bar{z} = z$ if and only if $z \in \mathbb{R}$

- $\bar{\bar{z}} = z$

Example 5.3.5   Find real numbers x and y such that

$$(2x + 1)i - y = \frac{x + yi}{1 + i}.$$

Geometric representation of complex numbers

- The complex number $z = a + bi$ can be represented by a point (a, b) in the *complex plane.* To represent the complex plane we use a pair of axes where the horizontal axis is the Real axis and the vertical axis is the Imaginary axis.

- The illustration of complex numbers in the complex plane is often called an *Argand diagram.*
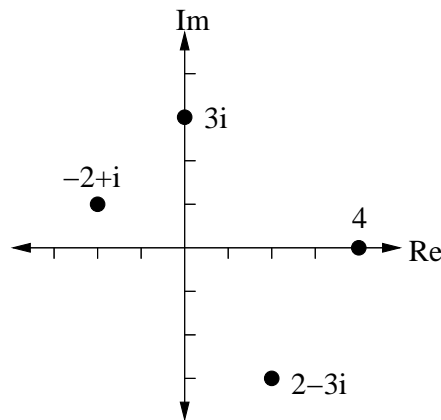
- If we represent each complex number by its position vector, then the sum of two complex numbers corresponds to vector addition.

- The point corresponding to $\bar{z}$ is the reflection of the point corresponding to z in the Real axis.

- The modulus of a complex number z is the length of the position vector corresponding to z.
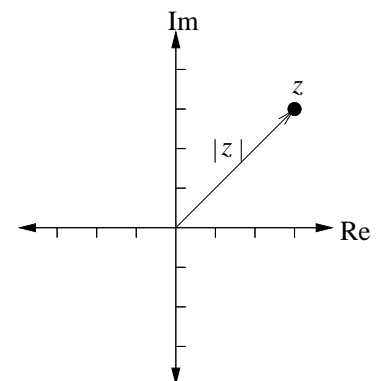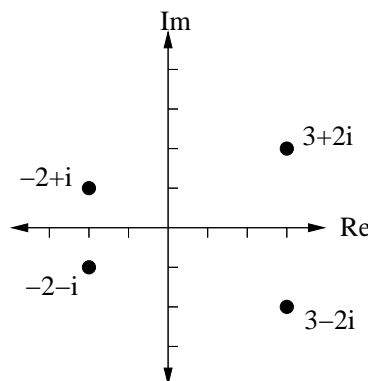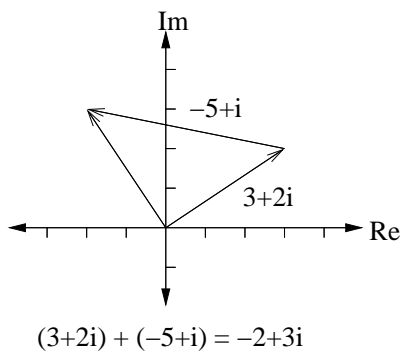
5.4 Complex numbers in polar form

Suppose that the complex number $z = a + bi$ is represented as a point in the complex plane.

- The modulus of z gives the distance from z to the origin.

- The *argument* of z, $\arg(z)$, is the angle that the position vector of z makes with the positive Real axis. Note that there can be many different arguments for a non-zero complex number z, since the angle θ is equivalent to the angle θ plus or minus any multiple of 2π radians.

- Note that positive angles are in an anti-clockwise direction and negative angles are in a clockwise direction from the positive Real axis.

- We call the argument of z that lies in the interval $(-\pi, \pi]$ the *principal argument* of z, denoted $\mathrm{Arg}(z)$.

- If $z = a + bi$ then $\arg(z) = \theta$ where $\tan\theta = \frac{b}{a}$. Always draw a sketch to help you choose the correct value for θ.

Example *5.4.1* Let $z = -4 + 4i$. Determine $|z|$ and $\mathrm{Arg}(z)$.

- Consider the representation of $z = a + bi$ in the complex plane below, which shows that $a = |z| \cos \theta$ and $b = |z| \sin \theta$.
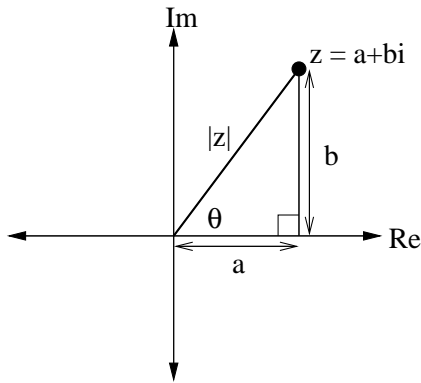
$$
\begin{aligned}
z &= |z| \cos \theta + |z| \sin \theta i \\
&= |z|(\cos \theta + i \sin \theta).
\end{aligned}
$$

- The complex number z with modulus $|z| = r$ and principal argument θ can be written in *polar form* as

$$z = r(\cos \theta + i \sin \theta).$$

The *polar coordinates* of z are (r, θ).

- The abbreviation $\operatorname{cis} \theta$ is often used for $\cos \theta + i \sin \theta$.

Example *5.4.2* **a)** Express the complex number $z = -1 - i$ in polar form.

b) Express the complex number $z = 3 \operatorname{cis} \frac{2\pi}{3}$ in Cartesian form.

- Let $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$.

  Multiplication in polar form

  $$z_1 z_2 = r_1 r_2 \operatorname{cis} (\theta_1 + \theta_2)$$

  Division in polar form

  $$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis} (\theta_1 - \theta_2)$$

Example *5.4.3*   Let $z = 10 \operatorname{cis} \frac{5\pi}{6}$ and $w = 6 \operatorname{cis} \frac{\pi}{3}$.

a) Calculate zw and express your answer in polar form.

b) Calculate $\dfrac{z}{w}$ and express your answer in polar form.

- Recall that the number $e = 2.71828\ldots$ is an irrational number. It can be defined as $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$.

- The following important result is known as *Euler's formula*:

$$\cos \theta + i \sin \theta = e^{i\theta}.$$

  This result was introduced by the Swiss mathematician Euler in 1748. The proof is beyond the scope of this course.

- Every non-zero complex number $z = a + bi$ with polar coordinates (r, θ) can be written as

$$z = r \operatorname{cis} \theta \ \text{ (polar form) } \text{ and } \ z = re^{i\theta} \ \text{ (exponential form).}$$

- Some people call both of the above forms the polar form of z, since they are both based on the polar coordinates of z.

Example *5.4.4*   Write the complex number $z = 1 - \sqrt{3}i$ in exponential form.

Example 5.4.5   Let $z = re^{i\theta}$. Write $\bar{z}$ in exponential form.

Example 5.4.6   Write -1 in exponential form.

It is wonderful that two irrational numbers and the square root of a negative number combine in this way to give -1.

- The exponential form of a complex number can be used to show that multiplication by i can be interpreted geometrically as a counter-clockwise rotation through $\pi/2$ radians about the origin.

  If $z = re^{i\theta}$, then

$$
\begin{aligned}
iz &= e^{i\frac{\pi}{2}} \times re^{i\theta} \\
&= re^{i\frac{\pi}{2}}e^{i\theta} \\
&= re^{i(\frac{\pi}{2}+\theta)}.
\end{aligned}
$$

- The complex numbers that can be written as $z = e^{i\theta}$ for some θ are the complex numbers with modulus 1. Their corresponding points in the complex plane make up a circle of radius one, centered at the origin. Multiplication of any complex number z by the complex number $e^{i\theta}$ can be interpreted geometrically as a counter-clockwise rotation through θ about the origin.

  If $z = re^{i\phi}$, then

$$
\begin{aligned}
e^{i\theta}z &= e^{i\theta} \times re^{i\phi} \\
&= re^{i\theta}e^{i\phi} \\
&= re^{i(\theta+\phi)}.
\end{aligned}
$$

$$i\,(3+2i) = -2+3i$$

$$z = (-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\,i)\,(4+2i)$$
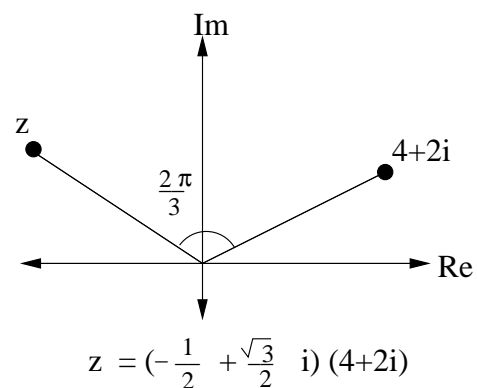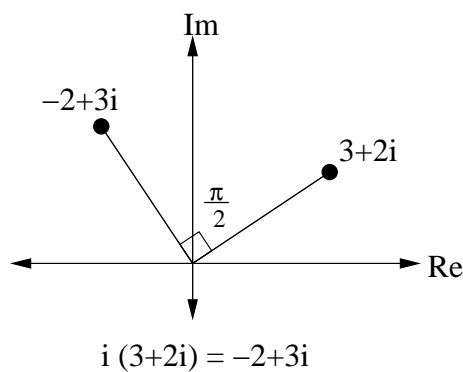
5.5   Powers of complex numbers

- *De Moivre's Theorem* states that if $z = r(\cos\theta + i\sin\theta)$ and n is a natural number, then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

  This theorem was named after the French mathematician Abraham de Moivre (1667–1754).

  Proof: Let $z = r(\cos\theta + i\sin\theta)$. Thus $z = re^{i\theta}$, and

$$
\begin{aligned}
z^n &= (re^{i\theta})^n \\
&= r^n e^{i\theta n} \\
&= r^n(\cos(n\theta) + i\sin(n\theta)).
\end{aligned}
$$

- Notice that the proof of de Moivre's theorem is also true for $n \in \mathbb{R}$.

- De Moivre's Theorem allows us to calculate powers of complex numbers with ease.

Example *5.5.1*   Evaluate $(1 - \sqrt{3}i)^6$.

- De Moivre's Theorem also allows us to calculate the nth roots of any complex number, where $n \in \mathbb{N}$.

- The nth *roots* of the complex number w are the values of z such that $z^n = w$. There are exactly n such values of z.

- To solve the equation $z^n = w$ for some $w \in \mathbb{C}$ and $n \in \mathbb{N}$.

  1. Express w in polar form as $w = r \operatorname{cis} \theta$.

  2. Since the angle θ is equivalent to the angle $(\theta + 2k\pi)$ for any $k \in \mathbb{Z}$, we know that
  $$r \operatorname{cis} \theta = r \operatorname{cis} (\theta + 2k\pi), \text{ where } k \in \mathbb{Z}.$$

  3. Thus $z = (r \operatorname{cis} (\theta + 2k\pi))^{\frac{1}{n}}$, and by the extension of de Moivre's theorem
  $$z = r^{\frac{1}{n}} \operatorname{cis} \left(\frac{\theta + 2k\pi}{n} \right)$$
  for $k \in \mathbb{Z}$.

Example *5.5.2*   Determine the four values of z for which $z^4 = 1$. (These are called the fourth roots of unity.)

Example *5.5.3* Determine the five values of w for which $w^5 = -2 - 2i$, and draw them on an Argand diagram.

5.6   The fundamental theorem of algebra

- A *polynomial* in the variable z is an expression of the form

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0,$$

  where $n \in \mathbb{N}$ is the *degree* (highest power) of the polynomial, $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are the coefficients of z, and $a_n \neq 0$.

- If $P(z)$ is a polynomial, then $P(z) = 0$ is a *polynomial equation.*

Example *5.6.1*   Determine which of these are polynomials and which are polynomial equations.

$$2x^3 - 10x = 1, \qquad 3z^4 + z^{-2} + 5, \qquad (3 + 2i)z^4 - 5iz^2 + 10.$$

- *Fundamental Theorem of Algebra* (FTA) Every polynomial equation with real or complex coefficients has a solution in the complex numbers.

- The first proof of this theorem was given by Gauss in his doctoral thesis in 1799. None of the known proofs are easy, so we won't prove it in this course.

- *Factor Theorem* Let $P(z)$ be a polynomial and let $a \in \mathbb{C}$. Then $P(a) = 0$ if and only if $(z - a)$ is a factor of $P(z)$.

Example 5.6.2   Let $P(z) = z^3 - 7z^2 + 6z + 14$. Determine which of $(z + 1)$ or $(z - 2)$ is a factor of $P(z)$.

- Let $P(z) = 0$ be a polynomial equation. The FTA guarantees that there is a solution to $P(z) = 0$, suppose it is $z = a$. Then the factor theorem tells us that $(z - a)$ is a factor of $P(z)$ and so we can write $P(z) = (z - a)Q(z)$ where $Q(z)$ is a polynomial equation whose degree is one less than the degree of $P(z)$. Now the FTA guarantees that we can find a solution to the polynomial equation $Q(z) = 0$, so we can repeat this procedure.

- From Example 5.6.2 above, we know that $(z + 1)$ is a factor of $P(z)$. Thus
$$P(z) = (z + 1)Q(z),$$
where the degree of the polynomial $Q(z)$ is two. Let the coefficients of $Q(z)$ be a, b and c.

$$
\begin{aligned}
z^3 - 7z^2 + 6z + 14 &= (z + 1)(az^2 + bz + c) \\
&= az^3 + az^2 + bz^2 + bz + cz + c \\
&= az^3 + (a + b)z^2 + (b + c)z + c
\end{aligned}
$$

Equating coefficients on each side we see that

$$a = 1, \quad b = -8 \quad \text{and} \quad c = 14.$$

Hence

$$z^3 - 7z^2 + 6z + 14 = (z + 1)(z^2 - 8z + 14).$$

- If we start with a polynomial equation $P(z) = 0$ where $P(z)$ has degree n, then we can continue taking out linear factors of the form $(z - a)$ until we have found n linear factors.

- From Example 5.6.2 above,

$$z^3 - 7z^2 + 6z + 14 = (z + 1)(z^2 - 8z + 14).$$

We can use the quadratic formula to solve $z^2 - 8z + 14 = 0$, as

$$z = \frac{8 \pm \sqrt{64 - 4 \times 14}}{2} = 4 \pm \sqrt{2}.$$

Thus

$$z^3 - 7z^2 + 6z + 14 = (z + 1)(z - (4 + \sqrt{2}))(z - (4 - \sqrt{2})).$$

- **Important result** Let $P(z)$ be a polynomial of degree n, where $n \in \mathbb{N}$. Then the polynomial equation $P(z) = 0$ has n solutions.

- Note that the n solutions may not be distinct.

- Useful factorising results to remember.
  - Difference of squares: $x^2 - y^2 = (x - y)(x + y)$.
  - Perfect square: $x^2 + 2xy + y^2 = (x + y)(x + y)$
    $$x^2 - 2xy + y^2 = (x - y)(x - y).$$
  - Quadratic: $ax^2 + bx + c = (a_1 x + c_1)(a_2 x + c_2)$, where $a = a_1 a_2$, $c = c_1 c_2$ and $b = a_1 c_2 + a_2 c_1$.

Example 5.6.3 **a)** Write $z^4 - 81$ as the product of linear factors and hence find all solutions of $z^4 - 81 = 0$ (over $\mathbb{C}$).

b) Write $6z^4 + z^2 - 1$ as the product of linear factors and hence find all solutions of $6z^4 + z^2 - 1 = 0$ (over $\mathbb{C}$).

c) Write $z^2 + 2iz - 1$ as the product of linear factors and hence find all solutions of $z^2 + 2iz - 1 = 0$ (over $\mathbb{C}$).

- If a polynomial equation has real coefficients, its solutions (roots) are either real numbers or occur as pairs of conjugate complex numbers.

Example *5.6.4* $z = 5 + 5\sqrt{2}i$ is a solution of

$$z^3 - 8z^2 + 55z + 150 = 0.$$

Find all the other solutions of this equation.

6 Functions

- A function is a rule that associates a unique output to each input.

- Functions are used heavily in all areas of mathematics and in most applications of mathematics.

- Some examples of functions include:

  - A formula that converts temperature in Fahrenheit to temperature in Celcius, $c = \frac{5}{9}(f - 32)$, describes temperature in degrees centigrade as a function of temperature in degrees Fahrenheit.

  - A formula for the area of a circle in terms of its radius, $A = \pi r^2$, describes area as a function of radius.

  - A table of values that given the human population of the world for each year from 1900 to 2000 describes population as a function of year.

- Topics in this section are:

  - Introduction to functions
  - A collection of standard functions
  - Composition of functions
  - Inverse functions
  - Limits of functions

6.1   Introduction to functions

- A *function* is a rule that assigns to each element x in a set A exactly one element from a set B.

- The notation $f(x)$ is used to denote the element in B that arises from applying the function f to the element x, and we say that f maps x to $f(x)$. We call $f(x)$ the value of f at x, or the image of x under f.

- The set A is called the *domain* of the function. It is the set of all possible inputs to the function.

- The set of all possible values of the function is called the *range* of the function. The range is a subset of the set B.

Example *6.1.1*   Let f be the function that takes a real number as input and returns two plus the square of that real number as output.

a) Determine $f(2)$, $f(-3)$ and $f(x)$ for some real number x.

b) Determine the domain and range of f.

- The following example illustrates some common notation used to denote functions.

Example 6.1.2 Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x$ and let $g : \mathbb{Z} \to \mathbb{Z}$ where $g(x) = 2x$. Determine the domain and range for each function f and g.

- **Domain convention** If a function is given by a formula and the domain is not specified, the convention is that the domain is the set of all real numbers for which the formula defines a real number.

- Although we often use x as the variable in the rule of a function, this is not required. For example, if f is the function which computes the area of a circle given its radius then $f(r) = \pi r^2$, and we say that f is a function of r.

Example 6.1.3 Let g be the function defined by the rule $g(a) = \sqrt{a - 1}$. Determine the domain and range of g.

- We can represent a function f using an *arrow diagram*. We draw the set A on the left and the set B on the right, and draw an arrow from each $x \in A$ to its image $f(x) \in B$.

- If an arrow diagram illustrates a function, then there will be a single arrow coming out of each element on the left.

Example *6.1.4*   Let $A = \{-2, -1, 0, 1, 2\}$. Let $g : A \to \mathbb{Z}$ where $g(x) = x^2$. Draw the arrow diagram for g.

- The most common representation of a function is a graph with the domain of the function on the horizontal axis and its range on the vertical axis.

- The graph of a function f with domain A is given by the set of ordered pairs $\{(x, f(x))$ for all $x \in A\}$. If f is a function of x, we often let $y = f(x)$ and draw the graph of f on the x, y-axes.

Example *6.1.5*   A function $f : \mathbb{R} \to \mathbb{R}$ has the graph shown below. Determine $f(2)$ and the domain and range of f.
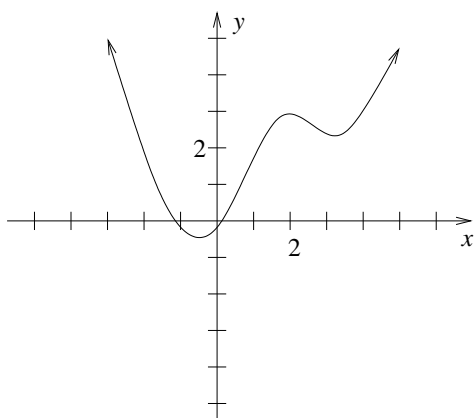
- An important property of a function f is that each x in the domain corresponds to a unique value $f(x)$. Thus, to check whether a graph represents a function, we can use the vertical line test.

- **Vertical line test** A curve in the x, y-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

Example 6.1.6 Determine which of the following graphs represent functions.
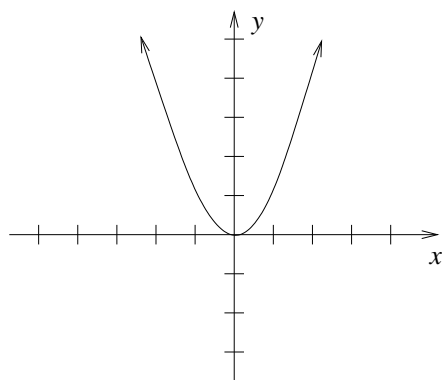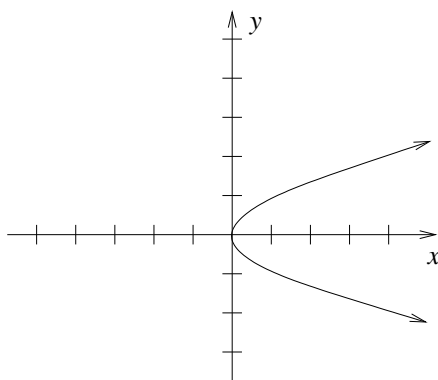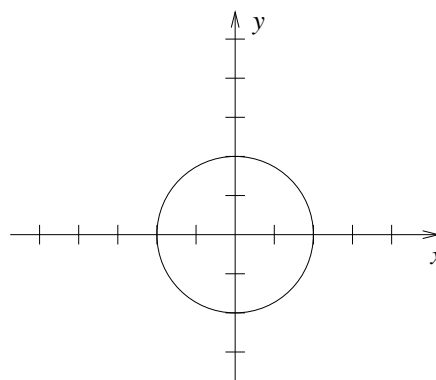
$$y = x^2 \qquad y^2 = x \qquad x^2 + y^2 = 4$$

Example 6.1.7 Determine the domain and range of each of the following functions, and sketch the graph of the function.

a) $f(x) = 5x - 3$
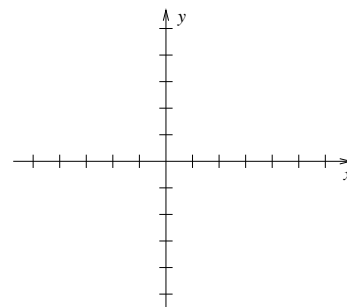
Example *6.1.7 (continued)*

b) $f(x) = x^2 - 2x + 1$
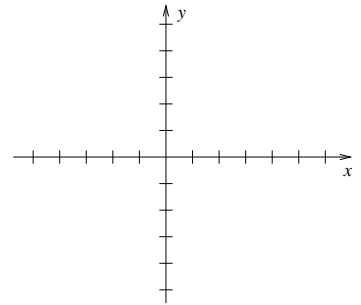
c) $f(x) = \sqrt{2x - 4}$
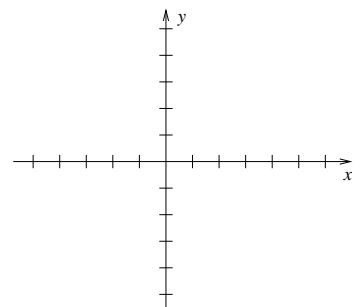
d) $f(x) = \dfrac{1}{x - 2}$
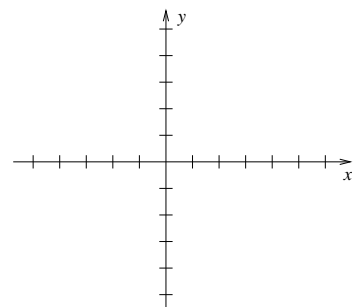
e) $f(x) = |3x| + 2$
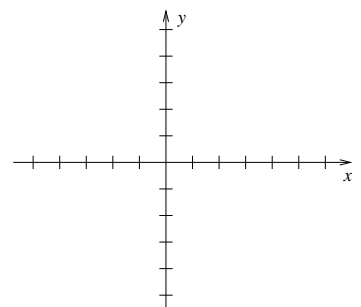
- A *piecewise defined function* is defined by different formulas in different parts of its domain.

Example *6.1.8* Draw the graph, and state the domain and range, of each of the following functions.

a) $f(x) = \begin{cases} x - 1 & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$
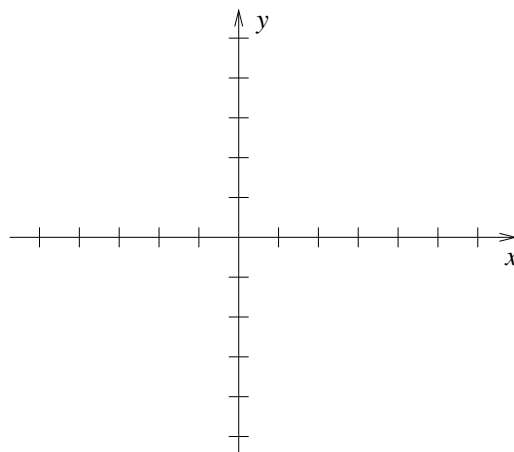
b) $g(x) = \begin{cases} -1 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } 2 \leq x < 3 \end{cases}$
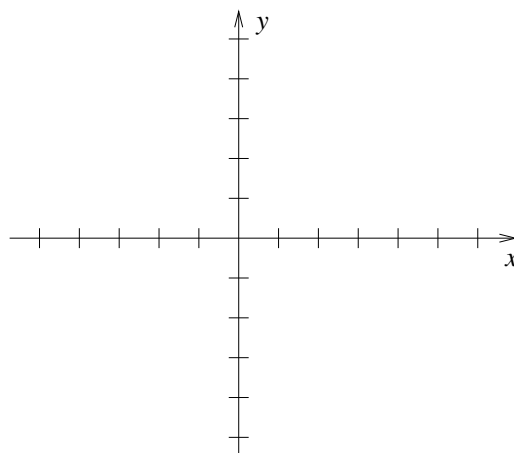
6.2 A collection of standard functions

Polynomial functions

- A *polynomial function* is a function whose rule is a polynomial. The general shape of the graph of a polynomial function depends on the degree (highest power) of the polynomial.

- A polynomial function f of degree 0 is defined by a formula of the form $f(x) = a$ where a is a real number. It is called a *constant function* and its graph is a horizontal line.

- A polynomial function f of degree 1 is defined by a formula of the form $f(x) = ax + b$ where a and b are real numbers $(a \neq 0)$. It is called a *linear function* and its graph is a line with slope a.
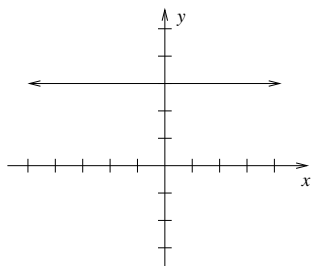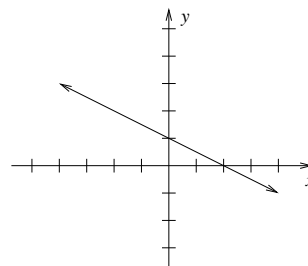
$$y = 3 \qquad\qquad y = -\tfrac{1}{2}x + 1$$

- A polynomial function f of degree 2 is defined by a formula of the form $f(x) = ax^2 + bx + c$ where a, b and c are real numbers $(a \neq 0)$. It is called a *quadratic function* and its graph is a parabola.
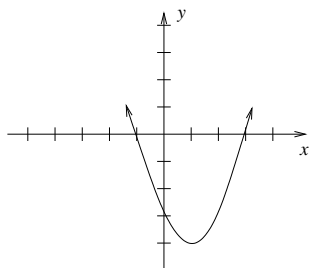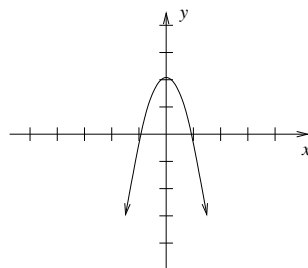
$$y = x^2 - 2x - 3 \qquad\qquad y = -2x^2 + 2$$

- A polynomial function f of degree 3 is defined by a formula of the form $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are real numbers ($a \neq 0$). It is called a *cubic function.*
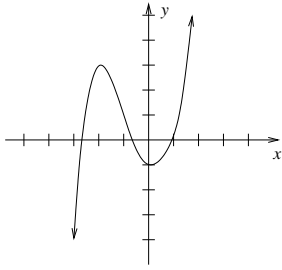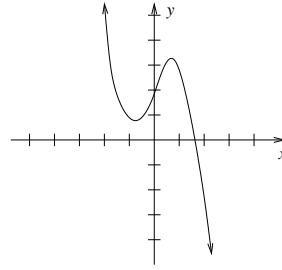
$$y = x^3 + 2x^2 - 2x - 1 \qquad\qquad y = -x^3 + 2x + 2$$

- A polynomial function f of even degree greater than two has a graph that is similar in shape to that of a quadratic function, but potentially having some bumps in the middle.
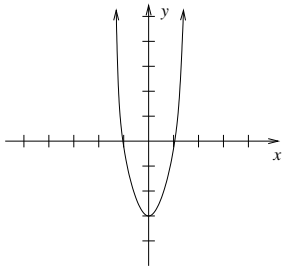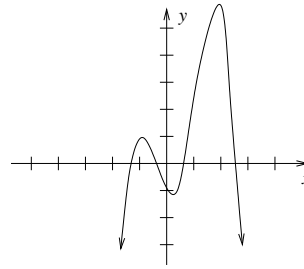
$$y = x^4 + 2x^2 - 3 \qquad\qquad y = -x^4 + 2x^3 + 3x^2 - 2x - 1$$

- A polynomial function f of odd degree greater than three has a graph that is similar in shape to that of a cubic function, but potentially having some more bumps in the middle.
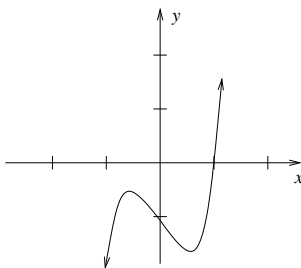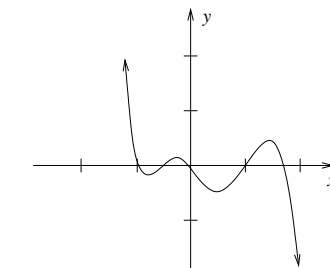
$$y = x^5 + 2x^3 - 2x - 1 \qquad\qquad y = -x^5 + x^4 + 2x^3 - x^2 - x$$

Power functions

- A *power function* f is defined by a formula of the form $f(x) = x^a$ where a is a real number.

- If a is a positive integer, then the power function $f(x) = x^a$ is a polynomial function of degree a (with only one term).
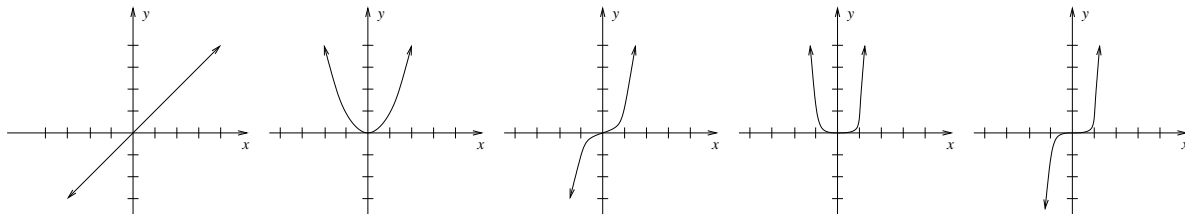
$$y = x \qquad y = x^2 \qquad y = x^3 \qquad y = x^4 \qquad y = x^5$$

- If $a = 1/n$ where n is a positive integer, then $f(x) = x^a = \sqrt[n]{x}$ is the nth root function. If n is even, then the domain of $f(x)$ is $[0, \infty)$ and the graph of $f(x) = \sqrt[n]{x}$ looks similar to that of $f(x) = \sqrt{x}$ shown below. If n is odd, then the domain of $f(x)$ is $\mathbb{R}$ and the graph of $f(x) = \sqrt[n]{x}$ looks similar to that of $f(x) = \sqrt[3]{x}$ shown below.
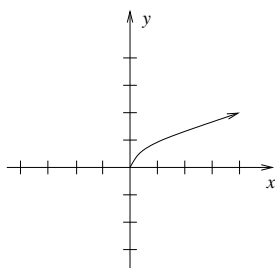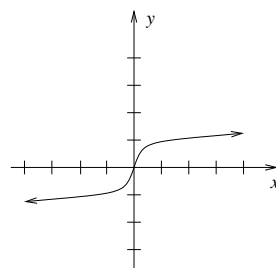
$$y = \sqrt{x} \qquad\qquad y = \sqrt[3]{x}$$

- If $a = -1$, then $f(x) = \frac{1}{x}$ is the *reciprocal function*. Its domain and range are both $\mathbb{R} \setminus \{0\}$ and its graph is a hyperbola.
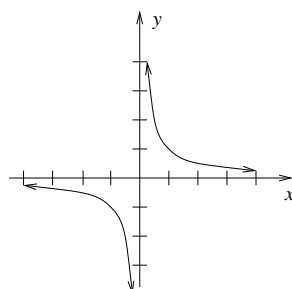
Absolute value functions

- An *absolute value function* is a function whose rule involves
  absolute value bars. An absolute value function can be
  written as a piecewise defined function.

Example *6.2.1*   Let f and g be functions where $f(x) = |x|$ and
$g(x) = |x^2 - 4|$. Write the rules of f and g as piecewise defined
functions, and sketch the graphs of f and g.
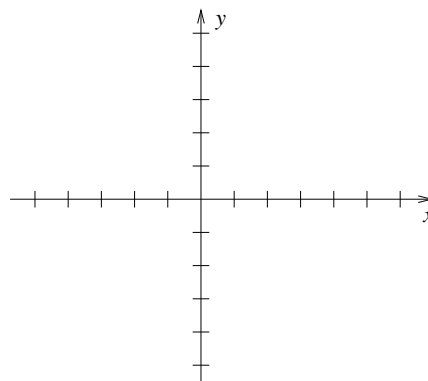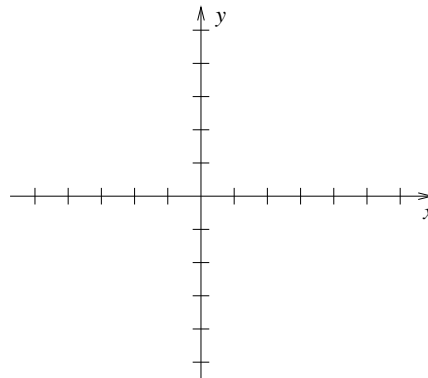
Rational functions

- An *asymptote* is a line that the graph of a function approaches arbitrarily closely, far from the origin.

- A *rational function f* is defined by a formula of the form $f(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomial functions.
  Unless otherwise specified, the domain of f consists of all real numbers x such that $q(x) \neq 0$.

- The graph of the rational function $f(x) = \dfrac{p(x)}{q(x)}$ may have vertical asymptotes at those values of x for which $q(x) = 0$.
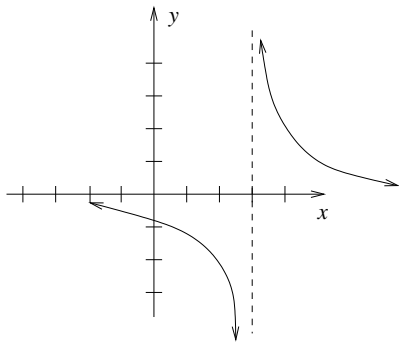
$$y = \frac{2}{x - 3}$$
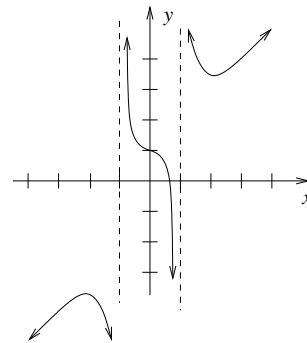
$$y = \frac{x^3 + 2x - 1}{x^2 - 1}$$
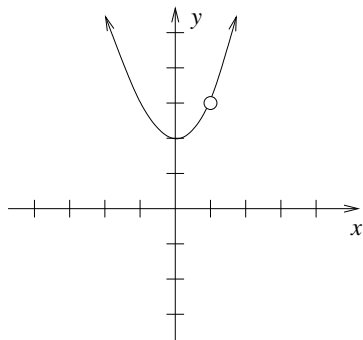
$$y = \frac{x^3 - x^2 + 2x - 2}{x - 1}$$

Exponential functions

- An *exponential function* is defined by a rule of the form $f(x) = a^x$ where a is a positive real number.

- When $a > 1$, the graph of $f(x) = a^x$ has the x-axis as a horizontal asymptote, passes through the point $(0, 1)$ and rises steeply to the right of the y-axis.

- When $a < 1$, the graph of $f(x) = a^x$ has the x-axis as a horizontal asymptote, passes through the point $(0, 1)$ and rises steeply to the left of the y-axis.
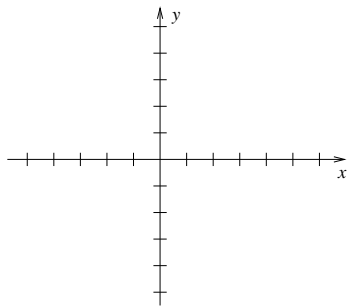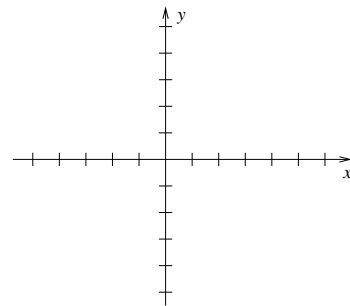
$$y = e^x \qquad\qquad\qquad\qquad y = \left(\tfrac{1}{2}\right)^x$$

Logarithmic functions

- A *logarithmic function* is defined by a rule of the form $f(x) = \log_a x$ where the base a is a positive real number.

- A logarithmic function has domain $(0, \infty)$ and its graph has the y-axis as a vertical asymptote, passes through the point $(1, 0)$ and rises slowly for $x > 1$.
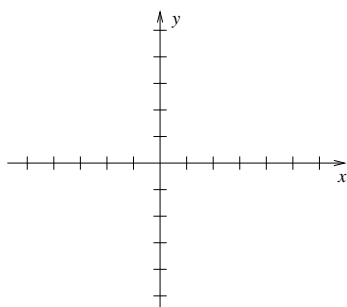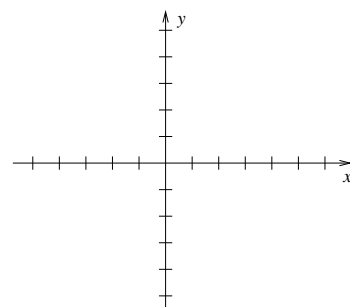
$$y = \ln x = \log_e x \qquad\qquad\qquad y = \log_{10} x$$

6.3 Composition of functions

- Often we apply one function and then apply another function to the result of the first function.

- Given any two functions f and g, we start with a number x in the domain of g and determine its image $g(x)$. If this number is in the domain of f, we can now apply f to the number $g(x)$ and obtain the value $f(g(x))$. This process is called the *composition* of f and g.

- Given two functions f and g, the function with input x and output $f(g(x))$ is called the *composite function* of f and g and is denoted $f \circ g$.

- The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

- To apply the function $f \circ g$, we must first apply g and then apply f.

Example 6.3.1   Let f and g be functions defined by the formulas $f(x) = \sqrt{x-5}$ amd $g(x) = x^2 - 4$. Determine the functions $f \circ g$ and $g \circ f$, and state their domains and ranges.

Example 6.3.2   Let $F(x) = 1 + 2\cos(x^2 - 1)$. Find functions f and g such that $F = f \circ g$. State the domain and range of F.

6.4 Inverse functions

- If f is a function, then each element x in the domain of f gives rise to a unique value $f(x)$ in the range of f. However, it may be the case that multiple elements in the domain gives rise to the same element in the range.

- Let f be the function defined by $f(x) = x^2$, so $f(2) = 4$ and $f(-2) = 4$. The question: 'Which value of x gives rise to the value 4?' does not have a unique answer. Consider the arrow diagram illustrating f on the domain $A = \{-2, -1, 0, 1, 2\}$ with range $B = \{0, 1, 4\}$.
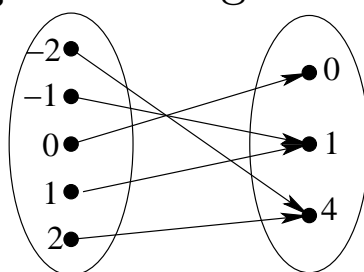
  The value 4 has two arrows pointing to it, so if we try to reverse the action of f, there is confusion over what to do with 4. Hence, there is no function that reverses the action of f to map elements in B back to their corresponding elements in A.

- A function f is called a *one-to-one function* if it never takes the same value twice; that is

$$f(x_1) \neq f(x_2) \quad \text{for } x_1 \neq x_2.$$

- Let f be a one-to-one function with domain A and range B. Then its *inverse function f^{-1}* has domain B and range A and is defined by

$$f^{-1}(b) = a \text{ if and only if } f(a) = b$$

  for any b in B.

Example *6.4.1* Let $f(x)$ be the function defined by $f(x) = x^2$. Determine a suitable domain for f so that f^{-1} exists.

- To determine the inverse of a one-to-one function f:

    1. Solve the equation $y = f(x)$ for x in terms of y.

    2. To express f^{-1} as a function of x, interchange x and y in the result of the previous step.

Example *6.4.2* Determine the inverse function of $f(x) = 2x + 3$ and sketch the graphs of f and f^{-1} on the same set of axes.
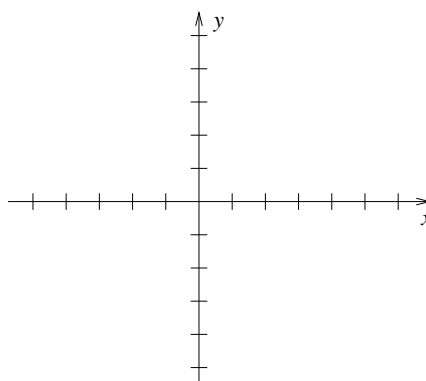
- If the point (a, b) lies on the graph of a function f, then $f(a) = b$. Thus, if f^{-1} exists, then $f^{-1}(b) = a$, and the point (b, a) lies on the graph of f^{-1}.

- The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

Example *6.4.3*   Sketch the graph of the function $f(x) = \sqrt{x-2}$ and its inverse function f^{-1}. Determine f^{-1} algebraically and verify that your sketch is correct.
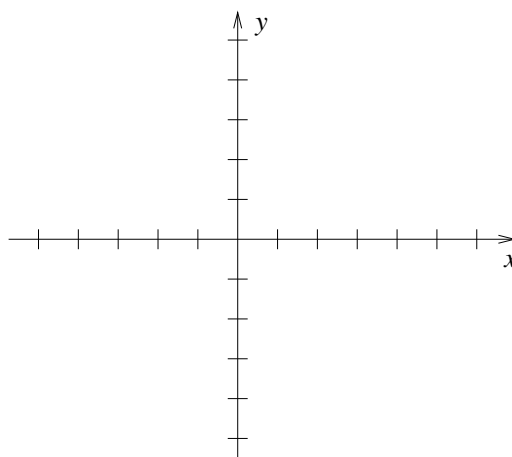
Inverse trigonometric functions

- The trigonometric functions sine, cosine and tangent take an angle as input and return the value of the trigonometric ratio of that angle. We often wish to do the reverse of this; that is, given the value of a trigonometric ratio, compute the corresponding angle. For this, we use the inverses of the trigonometric functions.

- With domain $\mathbb{R}$, none of the functions sine, cosine or tangent are one-to-one, so we must restrict the domains of the trigonometric functions.

- The function $f(x) = \sin x$ where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ is a one-to-one function. The function f^{-1} exists and is denoted $\sin^{-1}$ or arcsin. The inverse sine function has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
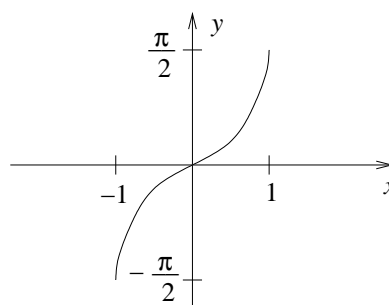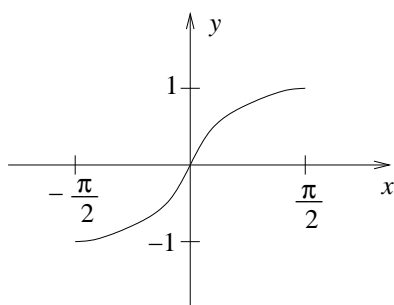
$$y = \sin x, \ -\tfrac{\pi}{2} \leq x \leq \tfrac{\pi}{2} \qquad y = \sin^{-1} x, \ -1 \leq x \leq 1$$

- The function $f(x) = \cos x$ where $0 \leq x \leq \pi$ is a one-to-one function. The function f^{-1} exists and is denoted $\cos^{-1}$ or arccos. The inverse cosine function has domain $[-1, 1]$ and range $[0, \pi]$.
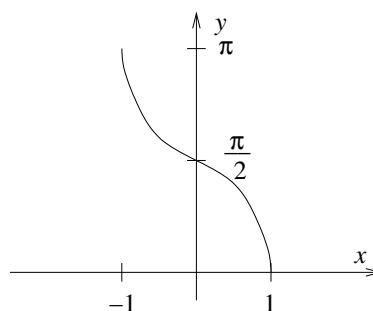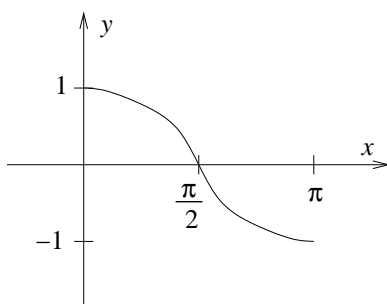
$$y = \cos x, \ 0 \leq x \leq \pi \qquad\qquad y = \cos^{-1} x, \ -1 \leq x \leq 1$$

- The function $f(x) = \tan x$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is a one-to-one function. The function f^{-1} exists and is denoted $\tan^{-1}$ or arctan. The inverse tangent function has domain $\mathbb{R}$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
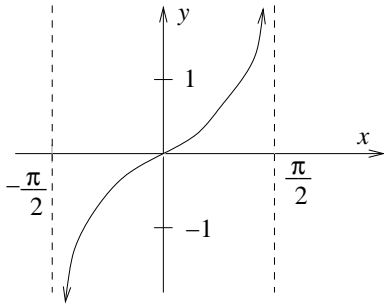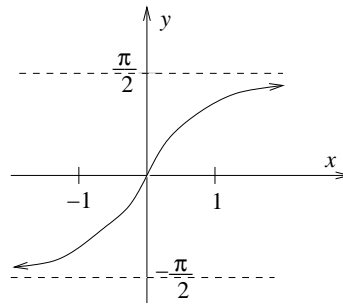
$$y = \tan x, \ -\frac{\pi}{2} < x < \frac{\pi}{2} \qquad y = \tan^{-1} x, \ x \in \mathbb{R}$$

Example *6.4.4* **a)** Evaluate $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$.

b) Evaluate $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

Example 6.4.5   **a)** Let $f(x) = \cos^{-1}(x^2 - 1)$. Determine the domain of f.

b) Show that $\cos(\tan^{-1} x) = \dfrac{1}{\sqrt{1 + x^2}}$.

6.5   Limits of functions

- When working with functions we often need to know what the function values are close to a particular value that may, or may not, be in the domain of the function. To obtain this type of information, we use limits, which describe function values for elements of the domain which are close to the particular value of interest.

- Consider the following example: Let f be the function defined by $f(x) = x^2 + 1$. What happens to the values of $f(x)$ as x approaches 2 from the left? What happens to the values of $f(x)$ as x approaches 2 from the right?

x	$f(x)$	x	$f(x)$
1.9	4.61	2.1	5.41
1.99	4.9601	2.01	5.0401
1.999	4.996001	2.001	5.004001
1.9999	4.99960001	2.0001	5.00040001
1.99999	4.99996	2.00001	5.00004
1.999999	4.999996	2.000001	5.000004

It appears that for x approaching 2 from the left $(x < 2)$, the function value is approaching 5 from below $(f(x) < 5)$, and for x approaching 2 from the right $(x > 2)$, the function value is approaching 5 from above $(f(x) > 5)$. From the table (and from the graph), we see that as x gets closer and closer to 2, $f(x)$ gets closer and closer to 5. We express this as 'the limit of the function $f(x) = x^2 + 1$ as x approaches 2 is equal to 5'. We write this as

$$\lim_{x \to 2}(x^2 + 1) = 5.$$

- Let f be a function defined on both sides of the value a, except possibly at a. We write $\lim\limits_{x \to a} f(x) = L$ and say 'the limit of $f(x)$ as x approaches a equals L' if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side) but not equal to a.

- Notice that we do not let $x = a$ when evaluating the limit. For all three of the following functions, we have
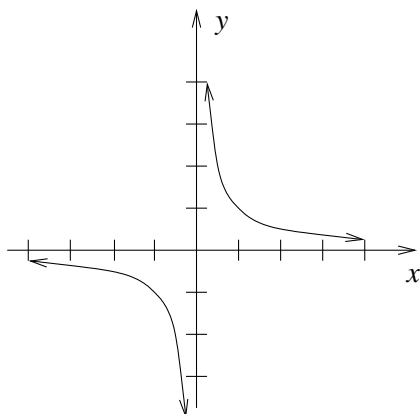
$$\lim_{x \to 2} f(x) = 5.$$
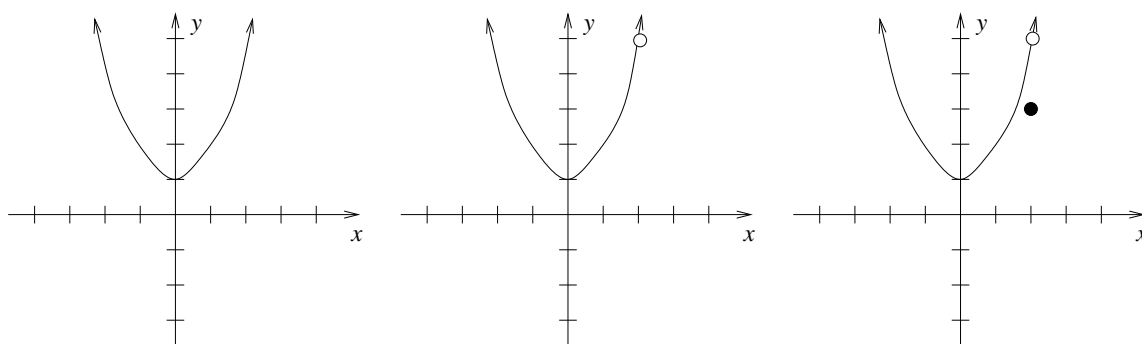
$$f(x) = x^2 + 1 \qquad f(x) = \frac{(x^2+1)(x-2)}{x-2} \qquad f(x) = \begin{cases} x^2 + 1 & x \neq 2 \\ 3 & x = 2 \end{cases}$$

Example 6.5.1 Let $f(x) = -x^3 + 2$. Determine $\lim\limits_{x \to 1} f(x)$.

Limits at infinity and infinite limits

- The notation $\lim\limits_{x\to\infty} f(x) = L$ means that as x gets larger and larger, $f(x)$ gets closer and closer to the value L.

- The notation $\lim\limits_{x\to-\infty} f(x) = L$ means that as x gets larger and larger in the negative sense, $f(x)$ gets closer and closer to the value L.

$$\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$$

$$\lim_{x\to-\infty}\left(\frac{1}{x}\right) = 0$$

- Let f be a function defined on both sides of a, except possibly at a itself. The notation $\lim\limits_{x\to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a (but not equal to a).

- Similarly, the notation $\lim\limits_{x\to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large in the negative sense by taking x sufficiently close to a (but not equal to a).

- In each of these cases, we say that the function f diverges as x approaches a.
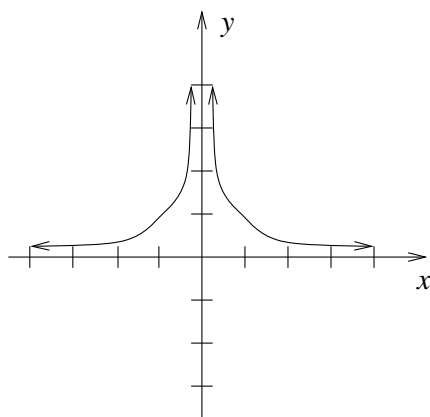
$$\lim_{x\to 0}\left(\frac{1}{x^2}\right) = \infty$$

One-sided limits

- Let f be a function defined on both sides of a, except possibly at a. If $\lim\limits_{x \to a} f(x) = L$ then the function values must approach L as x approaches a with $x < a$ AND also as x approaches a with $x > a$.

- If the graph of our function f has a break at $x = a$, $\lim\limits_{x \to a} f(x)$ may not exist, but we can talk about the one-sided limits as x approaches a from above and from below.

- The notation $\lim\limits_{x \to a^+} f(x) = L$ means that the values of $f(x)$ approach L as x approaches a with $x > a$.

- The notation $\lim\limits_{x \to a^-} f(x) = L$ means that the values of $f(x)$ approach L as x approaches a with $x < a$.
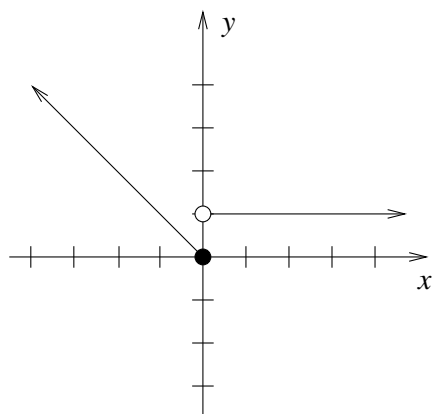
$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = 0$$

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

- For a function f, $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = L$.

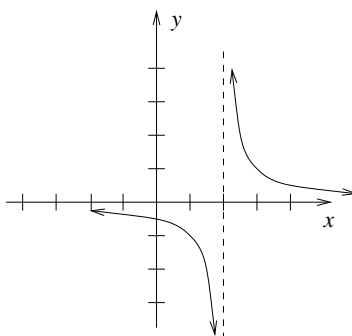**Example** 6.5.2   Use the graphs of the following functions to determine the given limits.

a) Let $f(x) = \frac{x^3 - x^2 + 2x - 2}{x - 1}$. Determine $\lim\limits_{x \to 1} f(x)$.
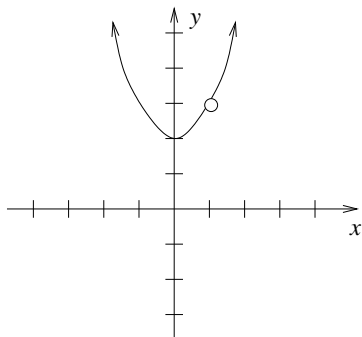
b) Let $f(x) = \begin{cases} x^2 + 2 & \text{if } x \geq 0 \\ x & \text{if } x < 0. \end{cases}$ Determine $\lim\limits_{x \to 0} f(x)$.
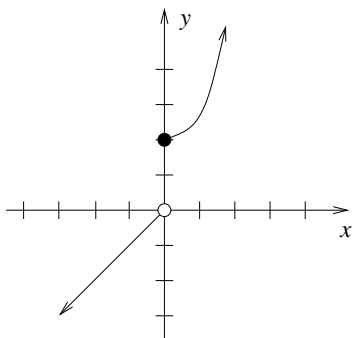
c) Let $f(x) = \frac{1}{x - 2}$. Determine $\lim\limits_{x \to 2} f(x)$.

- If you are asked to determine the limit of a messy looking function and you have no idea what the graph of that function looks like, you might be tempted to use your calculator to evaluate the function close to the value of interest. This approach could be dangerous.

- For example, consider the function $g(x) = \dfrac{\sqrt{x^2 + 4} - 2}{x^2}$ and the limit $\lim\limits_{x \to 0} g(x)$. Using a typical calculator, the following results were obtained. (Note that exact results depend on the calculator.)

x	$-x$	$g(x) = g(-x)$
0.1	-0.1	0.2498439
0.01	-0.01	0.2499984
0.001	-0.001	0.2500000
0.0001	-0.0001	0.2490000
0.00001	-0.00001	0.0000000
0.000001	-0.000001	0.0000000

From these results, you might be tempted to guess that $\lim\limits_{x \to 0} g(x) = 0$, which is incorrect.

- It can be very dangerous to use your calculator to estimate a limit. Rounding errors can cause small numbers to appear as zero on your calculator, so we need a more accuarate way to work out limits.

- If the function f can be evaluated at $x = a$ and f is a 'nice' function with no jumps, then usually $\lim\limits_{x \to a} f(x) = f(a)$.

  For example $\lim\limits_{x \to 2} x^3 = 2^3 = 8$.

- If the function f cannot be evaluated at $x = a$, then you may be able to use algebraic manipulation to rewrite f in a form that can be evaluated at $x = a$.

- For example, the function from Example 6.5.2 (a), $f(x) = \frac{x^3 - x^2 + 2x - 2}{x - 1}$ is not defined at $x = 1$, but provided that $x \neq 1$, we have $f(x) = \frac{(x^2 + 2)(x - 1)}{x - 1} = x^2 + 2$. Thus

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 2) = 1^2 + 2 = 3.$$

- Similarly, the function from the calculator discussion, $g(x) = \dfrac{\sqrt{x^2 + 4} - 2}{x^2}$ is not defined at $x = 0$, but provided that $x \neq 0$, we have

$$g(x) = \left(\frac{\sqrt{x^2 + 4} - 2}{x^2} \right) \left(\frac{\sqrt{x^2 + 4} + 2}{\sqrt{x^2 + 4} + 2} \right) = \frac{x^2}{x^2 (\sqrt{x^2 + 4} + 2)}.$$

Thus

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \left(\frac{1}{\sqrt{x^2 + 4} + 2} \right) = \frac{1}{\sqrt{0^2 + 4} + 2} = \frac{1}{4}.$$

- If f and g are functions such that $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x);$$

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x);$$

$$\lim_{x \to a} (f(x) g(x)) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x);$$

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0.$$

Continuous functions
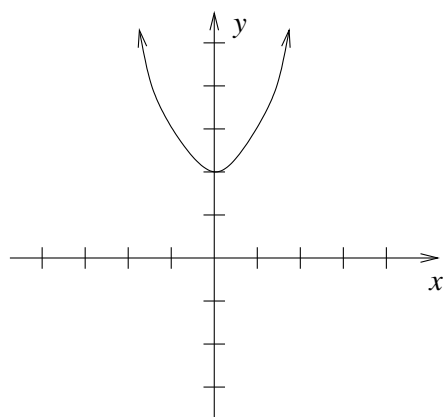
- A function f is *continuous* at a number a if
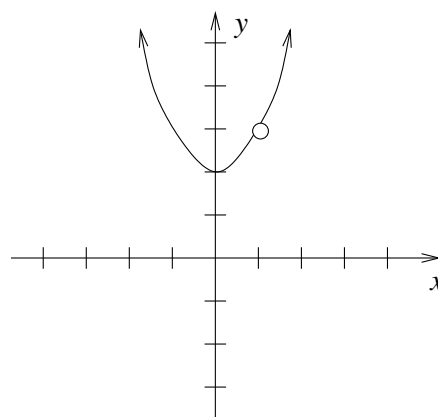
$$\lim_{x \to a} f(x) = f(a).$$

- A function f is *continuous on an interval* if it is contiuous at every number in the interval.
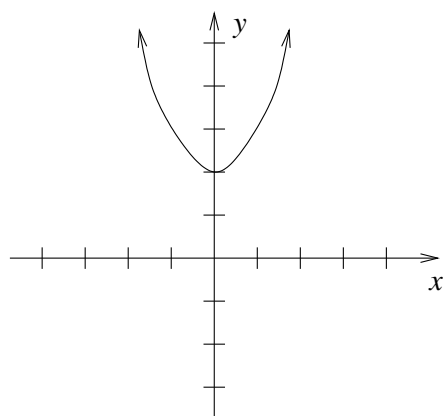
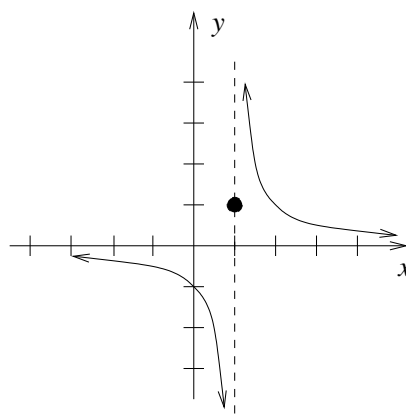***Example*** *6.5.3*   Determine which of the following functions are continuous on the interval $[-2, 2]$.

$$f(x) = x^2 + 2$$

$$f(x) = \frac{x^3 - x^2 + 2x - 2}{x - 1}$$

$$f(x) = \begin{cases} \frac{x^3 - x^2 + 2x - 2}{x - 1} & x \neq 1 \\ 3 & x = 1 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{x - 1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$

7   Differentiation

- In many applications of mathematics, it is very important to be able to calculate the gradient, or slope, of the graph of a function at a particular point.

- This gradient is called the derivative of the function at the point and it measures the rate of change of the function at that point.

- Derivatives of functions provide useful information about the graph of a function and also have important applications in many areas of science and economics.

- In this section we will look at the definition of the derivative of a function and review some rules that allow us to calculate the derivatives of many functions.

- We will apply our knowledge of derivatives to sketch the graphs of functions and to solve optimisation problems and problems involving rates of change.

- Topics in this section are:
    - Tangent lines
    - The derivative of a function
    - Differentiation rules
    - Critical points and curve sketching
   - Applications of differentiation

7.1 Tangent lines

- The slope of a line describes the rate of change of y with respect to x.
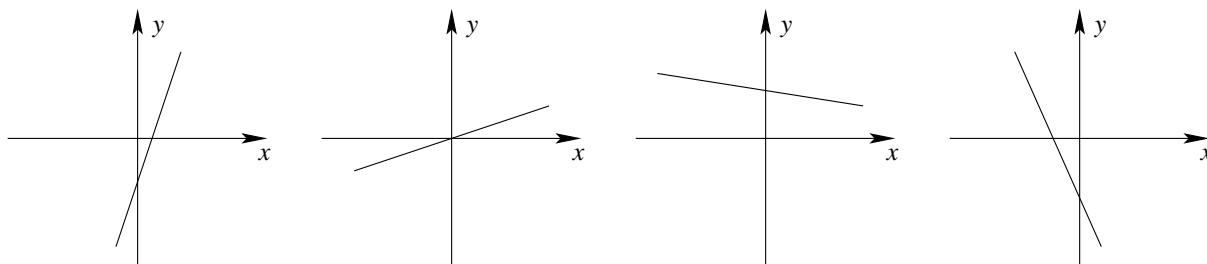
Rapid increase Slow increase Slow decrease Rapid decrease

- A *secant line* is a line that passes through two points on a curve.

- For a curve $y = f(x)$, we can use the slope of a secant line to describe the average rate of change of $f(x)$ with respect to x over a given interval of values for x.

- The slope of the secant line passing through the points $(a, f(a))$ and $(b, f(b))$ is $\dfrac{f(b) - f(a)}{b - a}$.

- Consider the family of secant lines that we obtain by letting b get closer and closer to a.
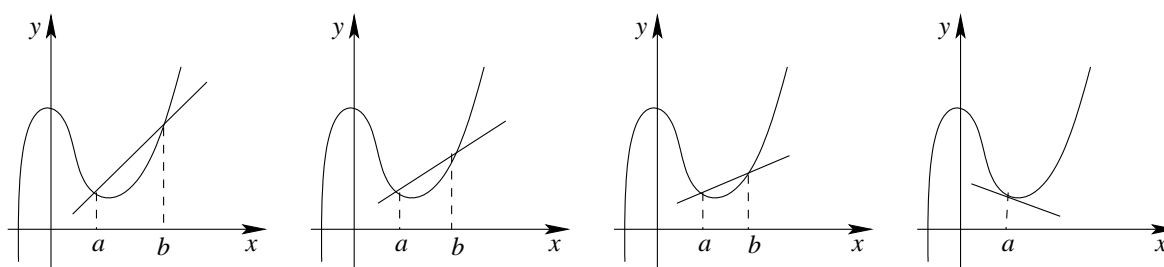
- The limit of these lines (as $b \to a$) is the line that just touches the curve at the point $(a, f(a))$. We call this line the *tangent line* to $f(x)$ at the point $x = a$.

- The slope of the tangent line to $f(x)$ at the point $x = a$ is $\displaystyle\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$.

- The slope of the tangent line at $x = a$ gives the instantaneous rate of change of $f(x)$ with respect to x at that point.

Example *7.1.1* We can guess the slope of the tangent line to the curve $f(x) = 2x^2 - x + 1$ at $x = 1$, by evaluating the slopes of the secant lines passing through $(1, f(1))$ and $(b, f(b))$ for values of b that approach 1.

b	1.5	1.1	1.01	1.001
$\dfrac{f(b) - f(1)}{b - 1}$	$\dfrac{4-2}{1.5-1}$ $= 4$	$\dfrac{2.32-2}{1.1-1}$ $= 3.2$	$\dfrac{2.0302-2}{1.01-1}$ $= 3.02$	$\dfrac{2.003002-2}{1.001-1}$ $= 3.002$

We might guess that the slope of the tangent line at $x = 1$ is 3.

Determine the slope of the tangent line to the curve $f(x) = 2x^2 - x + 1$ at $x = 1$ by evaluating the limit

$$\lim_{b \to 1} \frac{f(b) - f(1)}{b - 1}.$$

- Another way of writing the slope of the tangent line to $f(x)$ at $x = a$ is to let $b = a + h$. Then $b \to a$ is equivalent to $h \to 0$ and the slope of the tangent line to $f(x)$ at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{a + h - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example *7.1.2* **a)** Determine the slope of the tangent line to the curve $f(x) = x^3 - 2x$ at $x = 2$.

b) Use your answer from part (a) to determine the equation of the tangent line to the curve $f(x) = x^3 - 2x$ at $x = 2$.

7.2 The derivative of a function

- The *derivative* of a function f *at the number* a, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

  if this limit exists.

- Thus the tangent line to the curve $y = f(x)$ at $x = a$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

- The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Example *7.2.1* In a controlled laboratory experiment, the number of bacteria after t hours is described by a function $n(t)$ having the following graph.
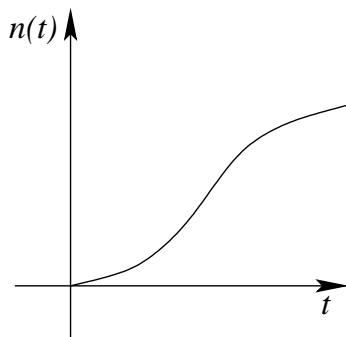
a) What does the derivative $n'(t)$ represent?

b) On the graph, locate the value of t that gives the largest value of the derivative.

- Given a function f, we can define a new function, called the *derivative of f*, and denoted f', by the rule that
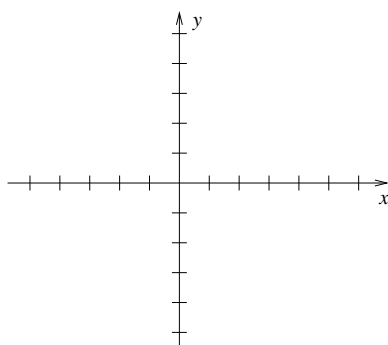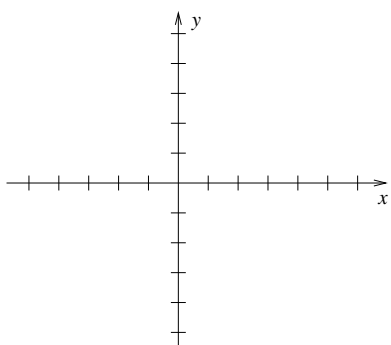
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

 The domain of f' is the set of all x in the domain of f such that $f'(x)$ exists.

- The derivative of f is also called the *derived function* of f. The process of determining the derivative of a function f is called *differentiation*.

Example *7.2.2*   Use the definition of the derivative to determine the derived function of f where $f(x) = \sqrt{x}$. Draw the graphs of f and f'.
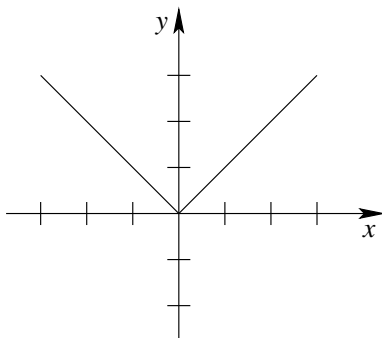
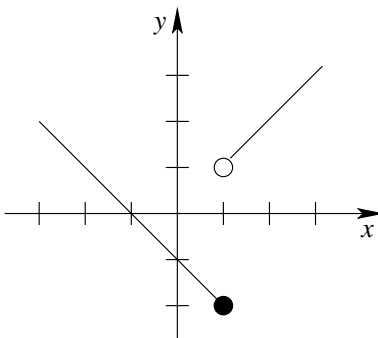**Example 7.2.3** The graph of a function f is shown below. Sketch (roughly) the graph of the derivative of f.
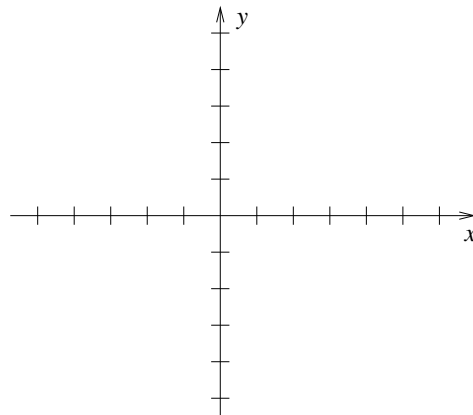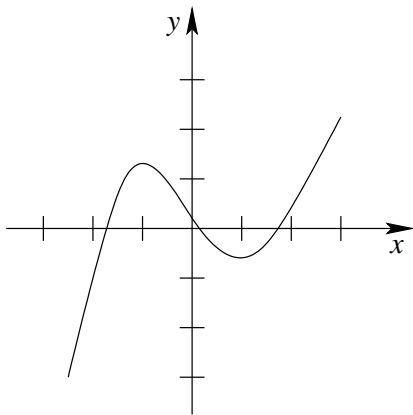
- If $y = f(x)$, then other notation commonly used for the derivative $f'(x)$ includes:

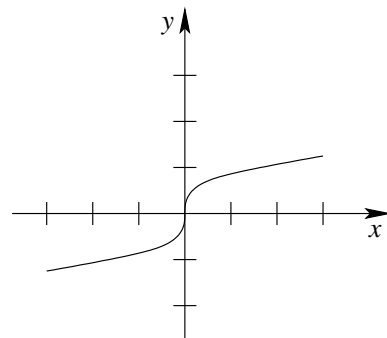$$y' \qquad \frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}f(x).$$

- A function f is said to be *differentiable* at a if $f'(a)$ exists. Note that a function may not be differentiable at all points in its domain.

not smooth
enough at $x = 0$

not continuous
at $x = 1$

too steep
at $x = 0$

7.3   Differentiation rules

- Determining derivatives from the definition can be time-consuming. Luckily there are some rules for differentiation that speed up the process. The proofs of these rules can be found in most calculus textbooks, but you do not need to know the proofs for this course.

- The derivative of a constant function is zero.

- If $f(x) = x^n$, then $f'(x) = nx^{n-1}$ for $n \in \mathbb{R}$.

- **Constant multiple rule** If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x).$$

- **Sum rule** If f and g are both differentiable functions, then

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

This rule can also be written as $(f + g)' = f' + g'$.

- **Product rule** If f and g are both differentiable functions, then

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x).$$

This rule can also be written as $(fg)' = fg' + gf'$.

- **Quotient rule** If f and g are both differentiable functions, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}.$$

This rule can also be written as $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$.

Example _7.3.1_ Determine the derivative of each of the following functions.

a) $f(x) = x^3 + 4x - 1$

b) $g(u) = \sqrt{u}(2u - u^2)$

c) $h(n) = \dfrac{4n - n^3 + 2}{3n^4}$

- **Chain rule** (also called the Composite function rule) If f and g are both differentiable functions, then

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

If $y = f(u)$ and $u = g(x)$, then this rule can be written as

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

- To apply the chain rule, think about starting with the outside function and working your way in.

Example 7.3.2 **a)** Find $h'(x)$ where $h(x) = (2x^3 - 3x + 1)^4$.

Example *7.3.2 (continued)*

b) Find $h'(x)$ where $h(x) = \sqrt{x^2 - 3x}$.

c) Find $h'(x)$ where $h(x) = (4x + (x^2 + 1)^3)^5$.

Derivatives of trigonometric functions

- Note that when we use the trigonometric functions, such as $f(x) = \sin x$, all angles are measured in radians.

- To determine the derivative of $f(x) = \sin x$ we require two special limits:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \qquad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

  The proofs of these limits can be found in most calculus textbooks, but we won't worry about the proofs in this course.

- If $f(x) = \sin x$ then $f'(x) = \cos x$.

Proof

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\[2mm]
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\[2mm]
&= \lim_{h \to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) \\[2mm]
&= \lim_{h \to 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) \\[2mm]
&= \lim_{h \to 0} \sin x \lim_{h \to 0} \left(\frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \cos x \lim_{h \to 0} \left(\frac{\sin h}{h} \right) \\[2mm]
&= \sin x \cdot 0 + \cos x \cdot 1 \\[2mm]
&= \cos x.
\end{aligned}
$$

- Using a similar method to the previous slide, we can show that if $f(x) = \cos x$ then $f'(x) = -\sin x$.

Example *7.3.3*   Use the quotient rule to determine $\dfrac{d}{dx}\tan x$.

- The derivatives of the trigonometric functions are summarised below.

$$\frac{d}{dx}(\sin x) = \cos x \qquad\qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad\qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad\qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of exponential and logarithmic functions

- If $f(x) = e^x$, then $f'(x) = e^x$.

- Thus, on the graph of $f(x) = e^x$, the slope of the tangent line at each point is equal to the function value at that point.

- If $f(x) = \ln x$, then $f'(x) = \dfrac{1}{x}$. Note that here $x > 0$ since the domain of $f(x) = \ln x$ is $x > 0$.

Example *7.3.4* Determine the derivative of each of the following functions.

a) $f(x) = xe^{x^2}$

b) $g(x) = \ln(x^3 - 2x)$

7.4　Critical points and curve sketching

- When solving practical problems, or when sketching the graph of a function, we often need to know when a function attains a maximum or minimum value.

- A function f has a *global maximum* at c if $f(c) \geq f(x)$ for all x in the domain of f. The number $f(c)$ is called the maximum value of f on its domain. A global maximum is also called an *absolute maximum.*

- A function f has a *global minimum* at c if $f(c) \leq f(x)$ for all x in the domain of f. The number $f(c)$ is called the minimum value of f on its domain. A global minimum is also called an *absolute minimum.*

- A function f has a *local maximum* at c if $f(c) \geq f(x)$ for all x near c.

- A function f has a *local minimum* at c if $f(c) \leq f(x)$ for all x near c.

***Example*** *7.4.1*　On the graph of the function f below, identify all global and local maxima and minima.
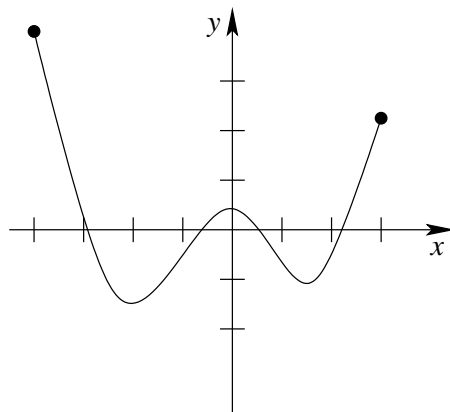
- If f has a local maximum or minimum at a, and if $f'(a)$ exists, then $f'(a) = 0$.

- The point $(a, f(a))$ is a *critical point* of the function f if $f'(a) = 0$ or if $f'(a)$ does not exist.

- Thus, all local maxima and minima are critical points. Note however, that not all critical points are local maxima or minima.
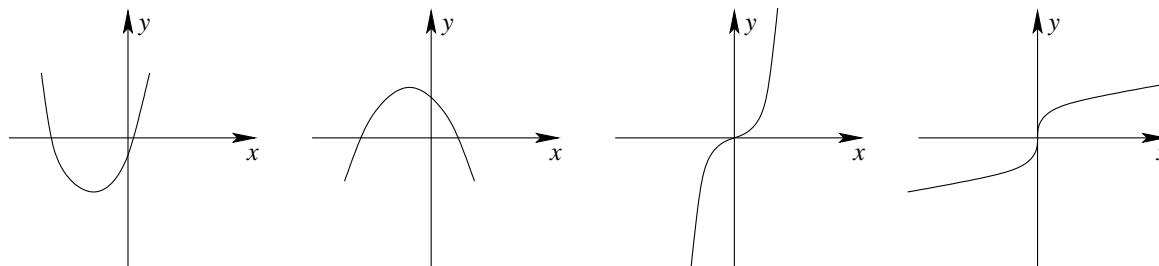
- To find any local maxima or minima of a function f, we solve the equation $f' = 0$. We can then classify any critical points we find using the information about the function near the critical point.

- A function f is strictly increasing on an interval $[a, b]$ if for all x_1 and x_2 in $[a, b]$, $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

- A function f is strictly decreasing on an interval $[a, b]$ if for all x_1 and x_2 in $[a, b]$, $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

- If $f'(x) > 0$ on an interval, then f is strictly increasing on that interval.

- If $f'(x) < 0$ on an interval, then f is strictly decreasing on that interval.

- If $f'(x) = 0$ on an interval, then f is constant on that interval.

- **First derivative test** Suppose that the function f has a critical point at $x = c$. Then
  - If f' changes sign from postive to negative at c, then f has a local maximum at c.
  - If f' changes sign from negative to positive at c, then f has a local minimum at c.
 - If f' does not change sign at c, then f has neither a local maximum nor a local minimum at c.

Example *7.4.2* Find all local maxima and minima of $f(x) = x^3 - 2x^2 + x + 1$ and classify them using the first derivative test. Use this information to sketch the graph of f.
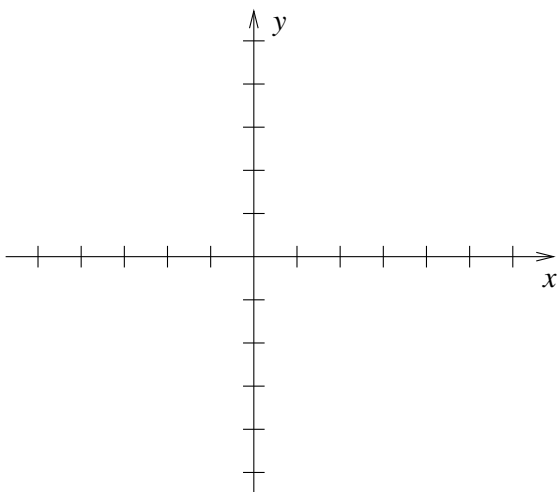
The second derivative

- The *second derivative* of a function f is the derivative of the derived function f'. The second derivative of f is denoted f''.

- The second derivative provides information about the concavity of the graph of a function.

- If the graph of f lies above all of its tangent lines on an interval, then it is *concave up* on that interval. If the graph of f lies below all of its tangent lines on an interval, then it is *concave down* on that interval.
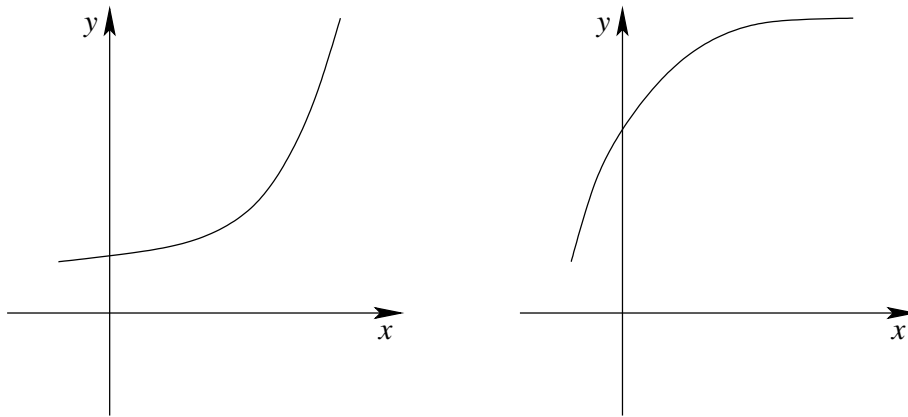
- If $f''(x) > 0$ for all x in an interval, then the graph of f is concave up on that interval. If $f''(x) < 0$ for all x in an interval, then the graph of f is concave down on that interval.

- **Second derivative test** Suppose f'' is a continuous function near a point c.

  - If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

  - If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

  - If $f''(c) = 0$, then this test is inconclusive.

Example 7.4.3 Analyse the curve $y = x^4 - 2x^3$ with respect to concavity, and local maxima and minima. Use this information to sketch the curve.
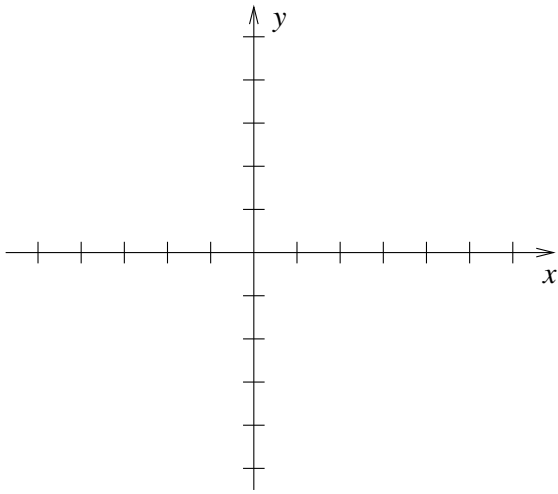
Curve sketching To sketch the curve of a function $y = f(x)$:

- Determine the domain of f.

- Determine the y-intercept of the graph by evaluating $f(0)$.

- If it is possible to solve the equation $f(x) = 0$, find the x-intercepts of the graph.

- Determine f' and identify the intervals on which f is increasing and the intervals on which f is decreasing.

- Find the critical points of f. Determine which critical points are local maxima or local minima (use first or second derivative test).

- Determine f'' and identify the intervals on which f is concave up and the intervals on which f is concave down.

- Sketch the graph.

Example *7.4.4* Sketch the graph of the function $f(x) = x^2 e^x$.

Example *7.4.4 (continued)* Extra space for your work.
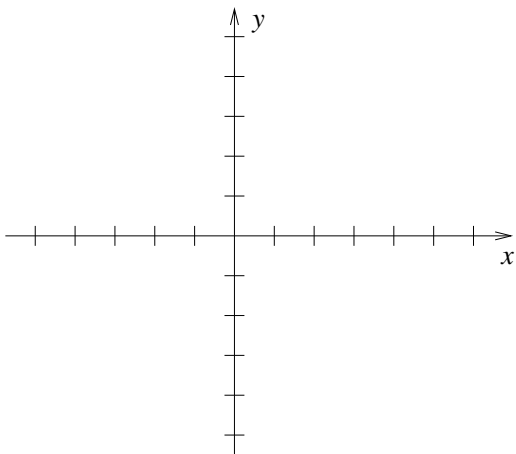
Example *7.4.4 (continued)* Extra space for your work.

7.5   Applications of differentiation

Optimisation problems

Example *7.5.1*   Find two numbers whose difference is 100 and whose product is a minimum.

Example *7.5.2*   A box with a square base and open top must have a volume of exactly 4000 cm^3, a height of at least 5 cm, and a base side length of at least 5 cm.

a) Find the dimensions of the box that minimise the amount of material used.

Example *7.5.2 (continued)* **b)** Find the dimensions of the box that maximise the amount of material used.

Rates of change

- Rates of change have important applications in many areas. The derivative of a function f with respect to a variable x gives the rate of change of f with respect to x.

- If $s(t)$ is a function of displacement with respect to time, then the derivative $s'(t)$ gives the velocity, since velocity is the rate of change of displacement with respect to time. Similarly, $s''(t)$ gives the acceleration, since acceleration is the rate of change of velocity with respect to time.

Example *7.5.3* A skier pushes off and heads directly down a ski slope with an initial velocity of 3 m/s. The distance the skier has travelled after t seconds is given by $s(t) = 3t + 2t^2$.

a) Determine the velocity of the skier after 2 seconds.

b) When the skier reaches a velocity of 15 m/s (that is 54 km/h), he makes a turn to slow down. How far has the skier travelled down the hill when he starts the turn?

- In economics, it is often important to study the functions that describe the cost of producing various quantities of a given product.

- Let $C(x)$ be the total cost to produce x units of a certain commodity. The *marginal cost* is the rate of change of the cost function with respect to the number of units produced.

- It can be shown that the marginal cost is approximately equal to the cost of producing one more unit of product.

$$C'(x) \approx C(x+1) - C(x)$$

Example *7.5.4* A company has estimated that the cost (in dollars) of producing x snowboards is

$$C(x) = 10000 + 18x + 0.02x^2.$$

a) Determine the marginal cost function and evaluate $C'(10)$.

b) Compare $C'(10)$ with the actual cost of producing the 11th snowboard.

---

8   Integration

- Situations often arise in which we know the derivative of a function and we need to determine the function itself. The reverse process of differentiation is called *antidifferentiation* or *integration*.

- We will start this section by looking at how to determine antiderivatives for some simple functions.

- We will then investigate how to approximate the area of a region under the graph of a function.

- An extremely important theorem, called the Fundamental Theorem of Calculus, tells us how the area under the graph of a function is related to an antiderivative of that function.

- Determining an antiderivative of a function is usually more challenging than determining the derivative of a function. We will take a brief look at the technique of integration by substitution, which can be thought of as the reverse of the chain rule for differentiation.

- Just as the process of differentiation has many practical applications, so does the process of antidifferentiation. We will end this section with a few practical problems.

- Topics in this section are:
    - Antiderivatives and indefinite integrals
    - The area under a curve
    - Definite integrals and the Fundamental Theorem of Calculus
    - Integration by substitution
    - Applications of integration

8.1 Antiderivatives and indefinite integrals

- A function F is called an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

- For example,

$$F(x) = 2x^3 \text{ is an antiderivative of } f(x) = 6x^2$$

  for all $x \in \mathbb{R}$ since $F'(x) = 6x^2 = f(x)$ for all $x \in \mathbb{R}$. However, $F(x) = 2x^3 + 10$ is also an antiderivative of $f(x) = 6x^2$ for all $x \in \mathbb{R}$.

- If F is an antiderivative of f on an interval I, then the *most general antiderivative* of f on I is

$$F(x) + C \text{ where } C \text{ is an arbitrary constant.}$$

Rules for antidifferentiation

- Let F be an antiderivative of f and let k be a constant. An antiderivative of $kf(x)$ is $kF(x)$.

- Let F be an antiderivative of f and let G be an antiderivative of g. Then $F + G$ is an antiderivative of $f + g$.

Common antiderivatives

Function	Antiderivative	Function	Antiderivative
x^a for $a \neq -1$	$\frac{1}{a+1} x^{a+1}$	$\sin x$	$-\cos x$
$\dfrac{1}{x}$ for $x > 0$	$\ln x$	$\cos x$	$\sin x$
e^x	e^x	$\sec^2 x$	$\tan x$

Example *8.1.1*   Find the most general antiderivative of each of the following functions.

a) $f(x) = 3x^2 - 7x + 2$

b) $f(x) = 4\sin x$

- If we are given a particular value of the antiderivative, we can use that information to decide which antiderivative to chose.

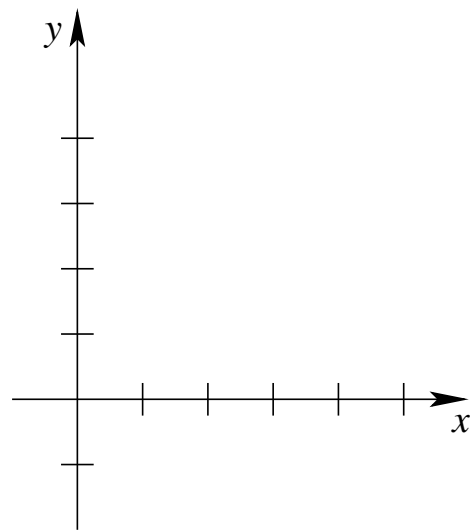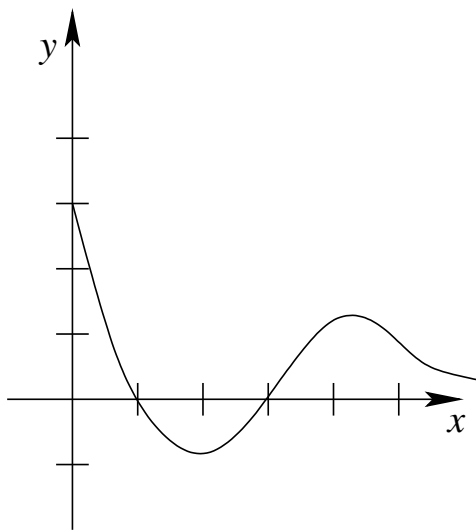**Example** *8.1.2*   Determine the antiderivative F of $f(x) = 4\sin x$ such that $F(0) = -5$.

Example *8.1.3* Suppose that f is a function satisfying
$f(1) = 4$ and $f'(x) = 3\sqrt{x} - \dfrac{1}{x^3}$. Determine $f(x)$.

Example *8.1.4* The graph of a function f is shown below.
Make a rough sketch of the antiderivative F of f which satisfies
$F(0) = 1$.

- We need a convenient notation for antiderivatives.

- $\int f(x)\,dx$ denotes the most general antiderivative of f, and we call this the *indefinite integral* of f.

- Thus, the indefinite integral of f is defined by

$$\int f(x)\,dx = F(x) + C$$

  where $F(x)$ is any antiderivative of f and C is an arbitrary constant, called the *constant of integration.*

- The indefinite integral of f gives a family of functions: the antiderivatives of f as the constant of integration varies.

- After evaluating an indefinite integral, you should always differentiate your result to check your answer.

Example *8.1.5*   Determine the following indefinite integrals.

a) $\int x^2(2+x)\,dx$

b) $\int \sqrt[5]{u}\,du$

8.2   The area under a curve

- Consider the problem of finding the area under the curve defined by $f(x) = -x^2 + 7x - 6$ between $x = 2$ and $x = 4$.
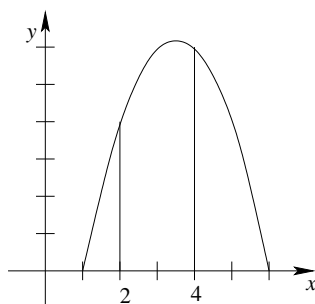
- How do we determine the area of a region with a curved side? We start by approximating the area with rectangles.
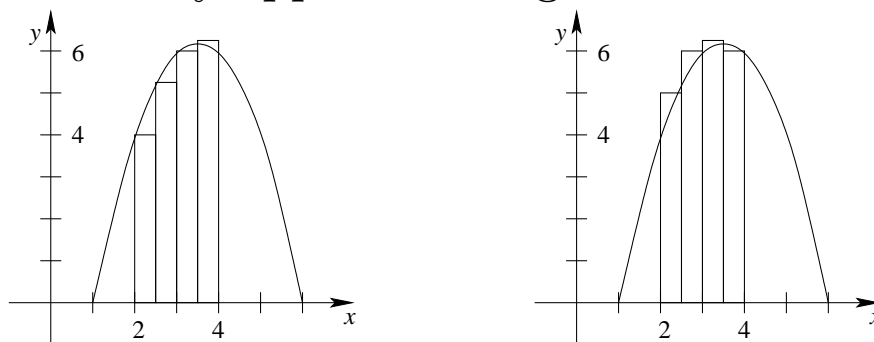
 Using the left side to define the height of the rectangles:

$$
\begin{aligned}
A_L &= \tfrac{1}{2}f(2) + \tfrac{1}{2}f(2.5) + \tfrac{1}{2}f(3) + \tfrac{1}{2}f(3.5) \\
&= \tfrac{1}{2}(4) + \tfrac{1}{2}(5.25) + \tfrac{1}{2}(6) + \tfrac{1}{2}(6.25) \\
&= 10.75
\end{aligned}
$$

  Using the right side to define the height of the rectangles:

$$
\begin{aligned}
A_R &= \tfrac{1}{2}f(2.5) + \tfrac{1}{2}f(3) + \tfrac{1}{2}f(3.5) + \tfrac{1}{2}f(4) \\
&= \tfrac{1}{2}(5.25) + \tfrac{1}{2}(6) + \tfrac{1}{2}(6.25) + \tfrac{1}{2}(6) \\
&= 11.75
\end{aligned}
$$

- The real area is between 10.75 and 11.75. By taking more rectangles, we can get a better approximation to the area.

- Now consider the general problem of determining the area under the curve defined by $y = f(x)$, between $x = a$ and $x = b$, where $f(x) \geq 0$ for $x \in [a, b]$.
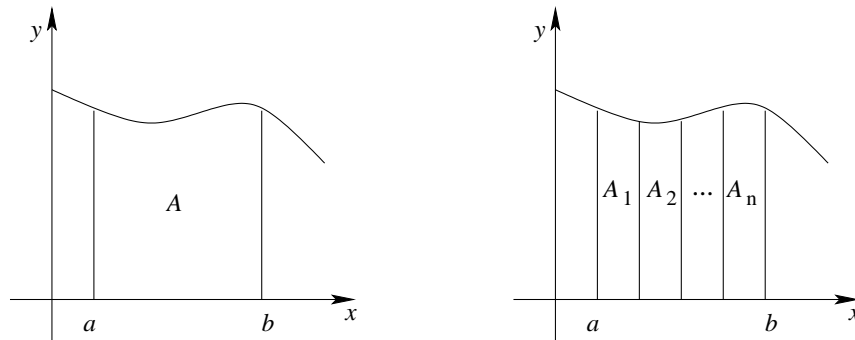
- We subdivide A into n strips $A_1, A_2, \ldots, A_n$ of equal width. The width of each strip is denoted Δx, so

$$\Delta x = \frac{b - a}{n}.$$

- The strips divide the interval $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ where $x_0 = a$ and $x_n = b$.

- We can then approximate the area A by summing the areas of n rectangles, where each rectangle approximates one strip.

- We need to decide on the height of each rectangle. We can do this by evaluating the function at any particular value within each interval. Let c_i be the value in the ith subinterval, $c_i \in [x_{i-1}, x_i]$, that defines the height of the ith rectangle. Then we approximate A by the sum

$$f(c_1) \, \Delta x + f(c_2) \, \Delta x + \cdots + f(c_n) \, \Delta x.$$

- This sum is called a *Riemann sum* and the points we choose to use to determine the height of each rectangle are called *sample points*.

- In practice we usually choose either the left-hand endpoints, or the right-hand endpoints, of each subinterval as the sample points.

- If we use the left-hand endpoints of each subinterval to determine the height of the corresponding rectangle, then the Riemann sum becomes

$$
\begin{aligned}
A_L &= f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x \\
&= \sum_{i=1}^{n} f(x_{i-1})\,\Delta x.
\end{aligned}
$$

- If we use the right-hand endpoints of each subinterval to determine the height of the corresponding rectangle, then the Riemann sum becomes

$$
\begin{aligned}
A_R &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\
&= \sum_{i=1}^{n} f(x_i)\,\Delta x.
\end{aligned}
$$

- It can be shown that the limit of a Riemann sum, as $n \to \infty$, is independent of the sample points chosen.

- We define the area A under the curve to be this limit. Thus

$$
A = \lim_{n \to \infty} A_L = \lim_{n \to \infty} A_R.
$$

Example *8.2.1* Determine the Riemann sum for
$f(x) = -x^2 + 7x - 6$, with $2 \le x \le 4$, having 10 subintervals and
taking the sample points to be the right endpoints.
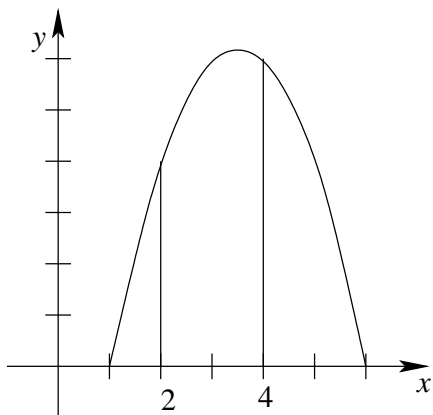
You can use the following facts:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

***Example*** *8.2.1 (continued)*   Extra space for your work.

Example *8.2.1 (continued)*   Extra space for your work.

- The limit of a Riemann sum of a function f over the interval $[a, b]$ gives the area under the curve f only if the curve lies above the x-axis for all $x \in [a, b]$.

- If we evaluate a Riemann sum of a function f over an interval on which f is not always positive, then the rectangles below the x-axis count as negative area.

Example *8.2.2* Let R_n be a Riemann sum of $f(x) = \sin x$ over the interval $[a, b]$ having n rectangles.
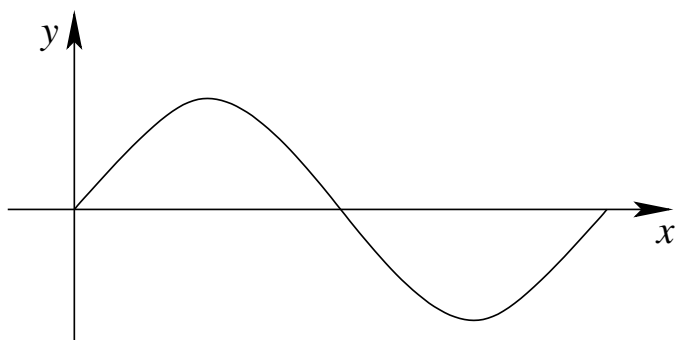
a) If $[a, b] = [0, 2\pi]$, will $\lim_{n \to \infty} R_n$ be positive, negative or zero.

b) If $[a, b] = [0, \frac{3\pi}{2}]$, will $\lim_{n \to \infty} R_n$ be positive, negative or zero.

c) If $[a, b] = [\frac{\pi}{2}, 2\pi]$, will $\lim_{n \to \infty} R_n$ be positive, negative or zero.

8.3 Definite integrals and the Fundamental Theorem of Calculus

- Let R_n be a Riemann sum for a continuous function f over the interval $a \le x \le b$, having n subintervals. Then the *definite integral* of f from a to b is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^n f(x_i)\, \Delta x.$$

- The definite integral is also called the *Riemann integral.*

- We call a and b the *limits of integration* where a the *lower limit* and b the *upper limit.*

- The integral sign $\int$ was introduced by Leibniz. It is an elongated S and was chosen because the integral represents the limit of a sum.

- Note that the definite integral is a number. Thus $\int_a^b f(x)\, dx = \int_a^b f(u)\, du = \int_a^b f(r)\, dr$.

Properties of the definite integral

- $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$

- $\int_a^a f(x)\, dx = 0$

- $\int_a^b c\, dx = c(b - a)$ where c is a constant

- $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

- $\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$ where c is a constant

- $\int_a^c f(x)\, dx + \int_c^b f(x)\, dx = \int_a^b f(x)\, dx$

The Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$.

- Thus, for functions for which we can determine an antiderivative, the time-consuming process of calculating definite integrals using sums and limits can be replaced by determining and evaluating an antiderivative.

- The notation $[F(x)]_a^b$ is often used for $F(b) - F(a)$.

- This is a simplified version of the FTC. If you continue your studies in mathematics, you will encounter it in more detail.

Example 8.3.1   Determine the area under the curve $f(x) = -x^2 + 7x - 6$ between $x = 2$ and $x = 4$.
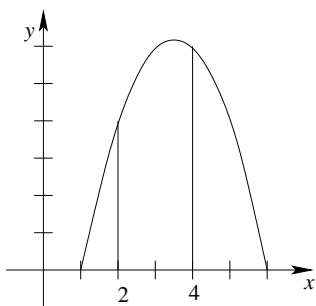
Example *8.3.2*   Refer back to Example 8.2.2.

a) Determine $\int_0^{2\pi} \sin x \, dx$.

b) Use the properties of the definite integral to determine the area enclosed by the sine curve and the x-axis over the interval $[0, 2\pi]$.

Example *8.3.3*   Determine $\displaystyle\int_{-1}^{1} \frac{1}{x^2} \, dx$.

8.4   Integration by substitution

- Recall the Chain rule for differentiation: If f and g are differentiable functions, then $(f \circ g)'(x) = f'(g(x))g'(x)$.

- Reversing the Chain rule gives the following integration formula:
$$\int f'(g(x))g'(x)\, dx = f(g(x)) + C.$$

Example 8.4.1   Determine $\int 2x e^{x^2}\, dx$.

- Sometime it is hard to see that an integrand has the form $f'(g(x))g'(x)$, so we need a systematic way to apply the above integration formula.

- If $y = f(x)$, where f is a differentiable function, then the *differentials dx* and *dy* are related by the equation $dy = f'(x)\, dx$.

- The geometric meaning of the differentials is that if ℓ is the tangent line to f at the point $P = (x, f(x))$, then the point $R = (x + dx, f(x) + dy)$ is also on the line ℓ.

- It can be shown that you can interpret the dx in an indefinite integral as a differential. Thus, if $u = g(x)$, then $du = g'(x)\, dx$ so
$$\int f'(g(x))g'(x)\, dx = \int f'(u)\, du.$$

Integration by substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

where F is an antiderivative of f.

Example *8.4.2*   Use integration by substitution to determine the following indefinite integrals.

a) $\int 2xe^{x^2}\,dx$

b) $\int 6x(3x^2 - 2)^4\,dx$

c) $\int 2x^2 \sin(4x^3 + 2)\,dx$

- It can be difficult to choose a good substitution. If the first function you choose doesn't seem to work, try something else. The key is to look for a function that appears in the integrand AND whose derivative also appears in the integrand (perhaps multiplied by a constant).

- Sometime there is more than one substitution that will work.

Example *8.4.3* Determine the following indefinite integrals by applying the indicated substitution.

a) $\displaystyle\int \sqrt{3x - 2}\, dx$ using $u = 3x - 2$

b) $\displaystyle\int \sqrt{3x - 2}\, dx$ using $u = \sqrt{3x - 2}$

- Since the function $f(x) = \ln x$ has domain $(0, \infty)$, we need to ensure that we never take the natural logarithm of a negative number or zero.

- If $f(x) = \ln x$ where $x > 0$, then

$$f'(x) = \frac{1}{x} \quad \text{and so} \quad \int \frac{1}{x} \, dx = \ln x + C.$$

  If $f(x) = \ln(-x)$ where $x < 0$, then we have

$$\frac{dy}{dx} = \frac{1}{-x} \times (-1) = \frac{1}{x} \quad \text{and so} \quad \int \frac{1}{x} \, dx = \ln(-x) + C.$$

  We often combine these cases and write

$$\int \frac{1}{x} \, dx = \ln|x| + C, \text{ for } x \neq 0.$$

- An integral of the form

$$\int \frac{g'(x)}{g(x)} \, dx$$

  can usually be treated by integration by substitution.

- Let $u = g(x)$, so $du = g'(x) \, dx$. Then

$$\int \frac{g'(x)}{g(x)} \, dx = \int \frac{1}{u} \, du = \ln|u| + C = \ln|g(x)| + C.$$

Example *8.4.4*   Determine the following indefinite integrals

a) $\int \dfrac{2x}{x^2 + 3}\, dx$

b) $\int \dfrac{6x^2 - 4x}{x^3 - x^2}\, dx$

c) $\int -\tan x\, dx$

Integration by substitution for definite integrals

- There are two ways to apply integration by substitution to a definite integral.

- The first way is to first determine the indefinite integral using integration by substitution and then use this to solve the definite integral.

Example *8.4.5*   Evaluate

$$\int_{-2}^{1} 18x(10 - 3x^2)^3 \, dx.$$

- The second way is to change the limits of integration when the substitution is performed.

- If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example *8.4.6* Evaluate the following definite integrals.

a) $\displaystyle\int_{-2}^{1} 18x(10 - 3x^2)^3\,dx$

Example *8.4.6 (continued)*

b) $\displaystyle\int_0^7 9\sqrt{4 + 3x}\,dx$

Proof of the substitution rule for definite integrals

We need to prove that if g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du. \qquad (1)$$

Let F be an antiderivative of f. Then by the substitution rule for indefinite integrals, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$.

By applying the FTC to the left-hand side of (1) above,

$$\int_a^b f(g(x))g'(x)\,dx = [F(g(x))]_a^b = F(g(b)) - F(g(a)).$$

By applying the FTC to the right-hand side of (1) above,

$$\int_{g(a)}^{g(b)} f(u)\,du = [F(u)]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).$$

Thus

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

8.5 Applications of integration

- When faced with a practical problem in which we know the rate at which a quantity is changing, integration can give us information about that quantity.

- For a particle moving in a straight line, an antiderivative of the acceleration function $a(t)$ is the velocity function $v(t)$ and an antiderivative of the velocity function $v(t)$ is the displacement function $s(t)$.

- Additional information about the velocity or displacement at a particular time can help us to choose the correct antiderivative.

Example 8.5.1 A balloon filled with water is thrown upward from Level 6 of the outside stairs of the Priestley building, with an initial speed of 10 m/s. The balloon is thrown from an initial height of 20 metres. Assume that the acceleration due to gravity is 9.8 m/s (downward).

a) Determine a function $v(t)$ describing the velocity of the balloon at t seconds.

b) Determine a function $s(t)$ describing the displacement of the balloon at t seconds.

c) When does the balloon reach its maximum height and what is this maximum height?

d) When does the balloon hit the ground and what is its velocity just before it hits the ground?

Example 8.5.1 (continued)

Example *8.5.2* The velocity of a car driving along a straight road is described by the following graph. By approximating this curve using straight line segments, estimate the total distance travelled by the car in 5 seconds.
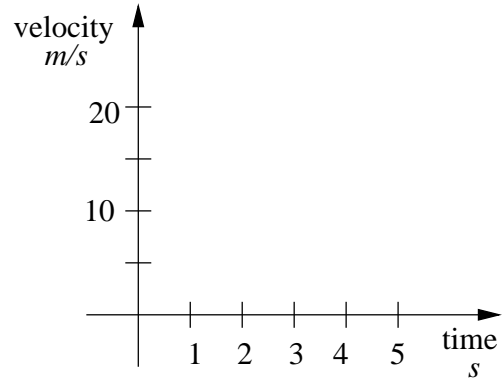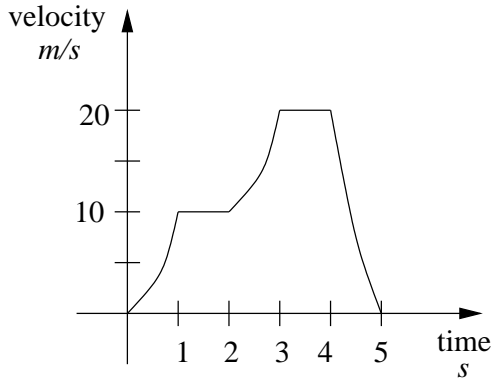
- It is important to distinguish between the displacement of an object at time t and the total distance travelled by the object at time t.

- When an object moves in a straight line and its velocity changes from positive to negative (or negative to positive) then the object is backtracking on ground already covered, so displacement is reduced but total distance continues to accumulate.

- Displacement is given by a definite integral but total distance travelled is given by the area enclosed by the velocity curve and the horizontal axis (time).

- If an object moving in a straight line has displacement function $s(t)$ and velocity function $v(t) = s'(t)$, then

$$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)$$

  is the net change of displacement (position) of the object during the time period from t_1 to t_2.

- If the object is moving in the same direction for all of the time interval t_1 to t_2, then this is also the total distance travelled during that time.

Example *8.5.3* A particle moves along a line with acceleration at time t of $a(t) = 2t + 2$ m/s^2. The particle has an initial velocity of -3 m/s.

a) Determine the displacement of the particle after 2 seconds.

b) Determine the total distance the particle travels in the first two seconds.

Example *8.5.4*   Water flows from the bottom of a storage tank at a rate of $r(t) = -6t + 180$ litres per minute, where $0 \leq t \leq 30$.

a) Find the amount of water that flows from the tank during the first 10 minutes.

b) If the tank is empty at time $t = 30$, determine how much water was in the tank at time $t = 0$.

9 Revision

1. **Matrices**

   - Arithmetic with matrices (addition, multiplication, ...)

   - Inverse of a 2×2 matrix

   - The determinant of a square matrix

   - Using matrices to solve systems of linear equations

2. **Vectors**

   - Three representations: geometric form, matrix form, component form

   - Arithmetic of vectors (addition, scalar multiplication, ...)

   - Trigonometry

3. **Applications of vectors**

   - Vectors in geometry

   - Forces

   - Displacement, velocity and momentum

   - Scalar product and vector product

4. **Sequences and series**

   - Arithmetic and geometric sequences and series

   - Finite differences and telescoping sums

   - Applications to compound interest and population growth

   - Mathematical induction

5. Real and complex numbers

- Solving inequalities and equations

- Arithmetic of complex numbers

- Polar and exponential form

- De Moivre's Theorem

- Fundamental Theorem of Algebra

6. Functions

- Domain and range

- Composition of functions

- Inverse functions

- Limits

7. Differentiation

- Tangent lines

- Definition of the derivative

- Differentiation rules

- Curve sketching

- Applications: optimisation and rate of change

8. Integration

- Antidifferentiation

- Area under a curve

- Definite integrals and the Fundamental Theorem of Calculus

- Integration by substitution

- Applications: rate of change

9.1   Revision examples

1. Determine whether or not the following system of linear equations has a unique solution.

$$
\begin{aligned}
x - y + 2z &= 3 \\
-3x + y + z &= -4 \\
4x - 6z &= 2
\end{aligned}
$$

2. Let $\mathbf{u} = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and let $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$. Determine the angle (in radians) between $\mathbf{u}$ and $\mathbf{v}$. Round your answer to two decimal place accuracy.

3. A pilot is steering a plane in the direction N60°E with an airspeed (speed in still air) of 250 km/h. There is a wind blowing from the direction N45°W at a speed of 50 km/h. Determine the resultant speed and direction of the plane. Give the speed to the nearest km/h and the direction as a bearing to the nearest degree.

4. Use mathematical induction to prove that, for all $n \geq 1$,

$$\sum_{i=1}^{n} 2i = n(n+1).$$

5. Solve the following inequality and give your solution in interval form and illustrate it on a number line.

$$\frac{|x+2|}{x-5} \leq 2$$

6. Find all solutions of $z^5 = 32$ over the complex numbers. Write your answers in polar form using the principal arguments.

7. Let $f(x) = \frac{1}{x}$ and let $g(x) = \sqrt{x+2}$.

a) State the domain and range of f and g.

b) Determine $(f \circ g)(x)$ and state its domain and range.

8. a) Use the definition of the derivative to determine the derived function of f where $f(x) = 3x^2 - x$.

b) Use your answer from a) to determine the equation of the tangent line to the curve $y = 3x^2 - x$ at $x = -1$.

9. Evaluate the following definite integrals.

a) $\int_2^4 \left(\frac{1}{x^2} + 2x^3 \right) \, dx$

b) $\int_{-1}^2 \frac{2x^3 - x}{x^4 - x^2 + 5} \, dx$

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