1. Section 3.1 questions from textbook

7. The perfect square 25 can be written as the sum of the perfect squares 9 and 16. Hence there exists a perfect square that can be written as the sum of two other perfect squares.

26. $\forall m, n \in \mathbb{Z}$, if m is even and n is odd, then $m+n$ is odd.
Proof: Suppose that m is an even integer and n is an odd integer. Then by the definitions of even and odd we know that

$$m = 2r \text{ for some integer } r \quad \text{and} \quad n = 2s+1 \text{ for some integer } s.$$

Then

$$m+n = 2r+2s+1 = 2(r+s)+1.$$

Since r and s are integers, $r+s$ is an integer, and so $m+n = 2k+1$ for some integer k. Hence $m+n$ is odd.

41. The statement "If m and n are positive integers and mn is a perfect square, then m and n are perfect squares" is false.
Let $m=2$ and $n=8$. Then $mn=16$ is a perfect square, but 2 and 8 are not perfect squares.

Section 3.2 questions from textbook.

15. $\forall r, s \in \mathbb{R}$, if r and s are rational, then $r-s$ is rational.
Proof: Suppose that r and s are rational numbers. Then, by the definition of rational, we have

$$r = \frac{a}{b} \text{ for } a, b \in \mathbb{Z}, b \neq 0 \quad \text{and} \quad s = \frac{c}{d} \text{ for } c, d \in \mathbb{Z}, d \neq 0.$$

Then

$$r - s = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}.$$

Since $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$, we know that $ad-bc$ and bd are both integers, and $bd \neq 0$. Thus $r-s$ is a rational number.

Section 3.3 questions from textbook.

8. 4 is a factor of $6a \cdot 10b$, since

$$6a \cdot 10b = 60ab = 4(15ab).$$

15. $\forall a, b, c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$, then $a \mid (b - c)$.
Proof: Let a, b, c be integers. Suppose that $a \mid b$ and $a \mid c$. Then

$$b = ra \text{ for some integer } r \quad \text{and} \quad c = sa \text{ for some integer } s.$$

Thus

$$b - c = ra - sa = (r - s)a.$$

Since r and s are integers, $r - s$ is also an integer, and so $b - c = ka$ for some integer k. Thus $a \mid (b - c)$.

22. The statement "$\forall a, b, c \in \mathbb{Z}$, if a is a factor of c, then ab is a factor of c" is false. Let $a = 2$, $b = 3$ and $c = 8$. Then a is a factor of c, but ab is not a factor of c, since $2 \mid 8$ but $6 \not\mid 8$.

31b. The unique factorization of 4851 is $3^2 \cdot 7^2 \cdot 11$.

$$4851 = 3 \cdot 1617 = 3^2 \cdot 539 = 3^2 \cdot 7 \cdot 77 = 3^2 \cdot 7^2 \cdot 11$$

Section 3.4 questions from textbook.

6. $-37 = 9 \cdot (-5) + 8$, so $q = -5$ and $r = 8$.

10b. $207 \bmod 4 = 3$, since $207 = 4 \cdot 51 + 3$.

23a. $\forall m, n \in \mathbb{Z}$, $m + n$ and $m - n$ are either both odd or both even.
Proof: The four cases for the parity of two integers m and n are:

- both are even;
- both are odd;
- m is even and n is odd;
- m is odd and n is even.

Case 1 Suppose that m and n are both even. Then

$$m = 2r \text{ for some integer } r \quad \text{and} \quad n = 2s \text{ for some integer } s.$$

Then $\quad m + n = 2r + 2s = 2(r + s) \quad$ and $\quad m - n = 2r - 2s = 2(r - s)$.
Since r and s are integers, $r + s$ and $r - s$ are also integers, so $m + n = 2a$ for some integer a and $m - n = 2b$ for some integer b. Thus, in this case, $m + n$ and $m - n$ are both even.

Case 2 Suppose that m and n are both odd. Then

$$m = 2r + 1 \text{ for some integer } r \quad \text{and} \quad n = 2s + 1 \text{ for some integer } s.$$

Then $\quad m + n = (2r + 1) + (2s + 1) = 2r + 2s + 2 = 2(r + s + 1) \quad$ and
$m - n = (2r + 1) - (2s + 1) = 2r - 2s = 2(r - s)$.

Since r and s are integers, $r+s+1$ and $r-s$ are also integers, so $m+n=2a$ for some integer a and $m-n=2b$ for some integer b. Thus, in this case, $m+n$ and $m-n$ are both even.

Case 3 Suppose that m is even and n is odd. Then

$$m = 2r \text{ for some integer } r \quad \text{and} \quad n = 2s+1 \text{ for some integer } s.$$

Then $\quad m+n = (2r)+(2s+1) = 2r+2s+1 = 2(r+s)+1 \quad$ and
$m-n = (2r)-(2s+1) = 2r-2s-1 = 2r-2s-2+1 = 2(r-s-1)+1$.
Since r and s are integers, $r+s$ and $r-s-1$ are also integers, so $m+n=2a+1$ for some integer a and $m-n=2b+1$ for some integer b. Thus, in this case, $m+n$ and $m-n$ are both odd.

Case 4 Suppose that m is odd and n is even. Then

$$m = 2r+1 \text{ for some integer } r \quad \text{and} \quad n = 2s \text{ for some integer } s.$$

Then $\quad m+n = (2r+1)+(2s) = 2r+2s+1 = 2(r+s)+1 \quad$ and
$m-n = (2r+1)-(2s) = 2r-2s+1 = 2(r-s)+1$.
Since r and s are integers, $r+s$ and $r-s$ are also integers, so $m+n=2a+1$ for some integer a and $m-n=2b+1$ for some integer b. Thus, in this case, $m+n$ and $m-n$ are both odd.

In each case, $m+n$ and $m-n$ are either both odd or both even.

Section 3.5 questions from textbook.

10a(ii). Let $n = 2025$. Then

$$\left(2025 + \left\lfloor \frac{2024}{4} \right\rfloor - \left\lfloor \frac{2024}{100} \right\rfloor + \left\lfloor \frac{2024}{400} \right\rfloor\right) \bmod 7 = (2025{+}506{-}20{+}5) \bmod 7 = 2516 \bmod 7.$$

2516 mod 7 = 3, so in the year 2025, January 1 will be a Wednesday.

13a. $\forall n, d \in \mathbb{Z}$ with $d \neq 0$, if $d \mid n$, then $n = \left\lfloor \frac{n}{d} \right\rfloor \cdot d$.

Proof: Let n and d be integers with $d \neq 0$. Suppose that $d \mid n$. Then

$$n = d \cdot k \text{ for some integer } k.$$

Thus

$$\left\lfloor \frac{n}{d} \right\rfloor \cdot d = \left\lfloor \frac{dk}{d} \right\rfloor \cdot d = \lfloor k \rfloor \cdot d = dk.$$

But $dk = n$, and hence $n = \left\lfloor \frac{n}{d} \right\rfloor \cdot d$.

Section 3.6 questions from textbook.

8. $\forall x, y \in \mathbb{R}$, if $x + y < 50$, then $x < 25$ or $y < 25$.
Proof: The contrapositive form of this statement is:

$\forall x, y \in \mathbb{R}$, if $x \geq 25$ and $y \geq 25$, then $x + y \geq 25$.

Suppose that x and y are real numbers and that $x \geq 25$ and $y \geq 25$. By adding together two numbers that are each greater than 25, we have

$$x + y \geq 50.$$

Thus if $x \geq 25$ and $y \geq 25$ then $x + y \geq 50$. By contraposition, we have shown that if $x + y < 50$, then either $x < 25$ or $y < 25$.

22. $\forall a, b \in \mathbb{Q}$ with $b \neq 0$, if r is an irrational number, then $a + br$ is an irrational number.
Proof: Suppose not. That is, suppose that there exist rational numbers a and b with $b \neq 0$ and an irrational number r such that $a + br$ is rational.
Since a and b are rational numbers and $b \neq 0$, we know that

$$a = \frac{c}{d} \text{ and } b = \frac{e}{f} \text{ for some integers } c, d, e \text{ and } f, \text{ where } d, e, f \neq 0.$$

Also, since $a + br$ is rational, we know that

$$a + br = \frac{g}{h} \text{ for some integers } g \text{ and } h, \text{ where } h \neq 0.$$

Thus

$$\begin{aligned} \frac{c}{d} + \frac{e}{f} \cdot r &= \frac{g}{h} \\ \frac{e}{f} \cdot r &= \frac{g}{h} - \frac{c}{d} \\ \frac{e}{f} \cdot r &= \frac{gd - ch}{hd} \\ r &= \frac{f(gd - ch)}{ehd} \end{aligned}$$

Now since c, d, e, f, g, h are all integers and $d, e, f, h \neq 0$, $f(gd - ch)$ and ehd are also integers, and $ehd \neq 0$. Thus r is a rational number, which contradicts our assumption. Thus it is impossible for the negation of the statement to be true, so we conclude that if a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

2. A reason for why each proof is incorrect is given below, followed by a discussion of whether the statement is true or false.

(a) This "proof" only shows that the statement is true for the integers 3, 4, 5, 6. To show that this statement is true, we must show it is true for all possible sets of four consecutive integers.

[Note that this statement is in fact true. Consider the four consecutive integers x, $x+1$, $x+2$, $x+3$. Then

$$x(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x.$$

Also $(x^2+3x+1)^2 = x^4 + 6x^3 + 11x^2 + 6x + 1$. Therefore, the product of any four consecutive integers is one less than a perfect square.]

(b) The "proof" shows that for all integers a, b, c, if $a \mid b$, then $a \mid bc$, which is not what was asked for. The statement has $p \to q$ and the "proof" does $q \to p$.

[Note that this statement is in fact false. Consider the integers $a = 15$, $b = 3$, $c = 10$. Then $a \mid bc$ since $15 \mid 30$, but $a \not\mid b$ since $15 \not\mid 3$.]

(c) This "proof" is wrong since it starts off by assuming that $m - n$ is even (which is what we are asked to prove).

[Note that this statement is in fact true. A correct proof is as follows. Suppose that m and n are odd integers. Since m and n are both odd, we have

$$m = 2s + 1 \text{ for some integer } s, \quad \text{and} \quad n = 2t + 1 \text{ for some integer } t.$$

Thus

$$m - n = (2s+1) - (2t+1) = 2s - 2t = 2(s-t).$$

Since s and t are integers, $s - t$ is an integer, thus $m - n = 2k$ for some integer k. Hence the difference of any two odd integers is even.]

3. Use the Euclidean Algorithm to calculate the greatest common divisor of each of the following pairs of numbers.

(a) We apply the Euclidean algorithm to 63 and 49.

$$\begin{aligned} 63 &= 49 \cdot 1 + 14 \\ 49 &= 14 \cdot 3 + 7 \\ 14 &= 7 \cdot 2 + 0 \end{aligned}$$

Hence $\gcd(63, 49) = 7$.

(b) We apply the Euclidean algorithm to 238 and 14.

$$238 = 14 \cdot 17 + 0$$

Since $14 \mid 238$, we have $\gcd(238, 14) = 14$.

(c) We apply the Euclidean algorithm to 1550 and 250.

$$\begin{aligned} 1550 &= 250 \cdot 6 + 50 \\ 250 &= 50 \cdot 5 + 0 \end{aligned}$$

Hence $\gcd(1550, 250) = 50$.

4. A solution to the linear Diophantine equation $1550c + 250d = 46500$ will only exist if $\gcd(1550, 250) \mid 46500$. Since $50 \mid 46500$, a solution exists. We use part (c) from the previous question to get $50 = 1550 \cdot 1 + 250 \cdot (-6)$. Then multiply both sides of the equation by $46500 \div 50 = 930$ to get

$$46500 = 1550 \cdot 930 + 250 \cdot (-5580).$$

Hence $c = 930$, $d = -5580$ is a solution to the linear Diophantine equation $1550c + 250d = 46500$.

5. A point with integer co-ordinates that lies on the line $63x + 47y = 4$ is a solution to the linear Diophantine equation $63x + 47y = 4$. Such a solution only exists if $\gcd(63, 49) \mid 4$. Since $7 \not| \, 4$, a solution does not exist. Therefore there is no point with integer co-ordinates that lies on the line $63x + 49y = 4$.

6. (Bonus Question)
A solution to the linear Diophantine equation $1550c + 250d = 46500$ in which both c and d are positive, will give values for the numbers of computers (c) and desks (d) the company could buy to spend exactly \$46500.00. From question 4 we know that one solution to this linear Diophantine equation is $c = 930$ and $d = -5580$. We apply Theorem 3.10.1 with

$$c_0 = 930, d_0 = -5530, a = 1550, b = 250, \gcd(a, b) = 50,$$

to get the general solution for c and d. Hence

$$c = 930 + \frac{250}{50}t = 930 + 5t \quad \text{and} \quad d = -5580 - \frac{1550}{50}t = -5580 - 31t$$

where $t \in \mathbb{Z}$ is the general solution to the linear Diophantine equation. To ensure that $c > 0$ and $d > 0$ we need a value of t for which

$$930 + 5t > 0 \quad \text{and} \quad -5580 - 31t > 0.$$

The inequality $930 + 5t > 0$ gives $t > -186$, and the inequality $-5580 - 31t > 0$ gives $t < -180$. Thus positive solutions for c and d occur when $t \in \{-181, -182, -183, -184, -185\}$. The possible values for x and y are given in the following table.

t	-181	-182	-183	-184	-185
c	25	20	15	10	5
d	31	62	93	124	155

In order to have a balance of desks and computers, the company should buy 25 computers and 31 desks.