1. Section 4.1 questions from textbook

21.

$$\begin{aligned}\sum_{m=0}^{4}\frac{1}{2^m} &= \frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}\\ &= \frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\\ &= \frac{16}{16}+\frac{8}{16}+\frac{4}{16}+\frac{2}{16}+\frac{1}{16}\\ &= \frac{31}{16}.\end{aligned}$$

30. $(1^3-1)+(2^3-1)+(3^3-1)+(4^3-1)=\sum_{i=1}^{4}(i^3-1)$.

34. $(1-r)\cdot(1-r^2)\cdot(1-r^3)\cdot(1-r^4)=\prod_{i=1}^{4}(1-r^i)$.

49. $\frac{5!}{7!}=\frac{5!}{7\cdot 6\cdot 5!}=\frac{1}{7\cdot 6}=\frac{1}{42}$.

Section 4.2 questions from textbook.

15. Proof Let $P(n)$ be the statement

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n},$$

or equivalently

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}.$$

Then $P(2)$ is the statement $\left(1-\frac{1}{2^2}\right)=\frac{2+1}{2\cdot 2}$.

$P(k)$ is the statement $\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)=\frac{k+1}{2k}$.

$P(k+1)$ is the statement $\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right)=\frac{k+2}{2(k+1)}$.

Now $P(2)$ is true since $\left(1-\frac{1}{2^2}\right)=\frac{3}{4}$ and $\frac{2+1}{2\cdot 2}=\frac{3}{4}$.

Assume that $P(k)$ is true and use that to show that $P(k+1)$ is true.

$$\begin{aligned}
\text{L.H.S. of } P(k+1) &= \prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right)\\
&= \left[\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)\right]\cdot\left(1-\frac{1}{(k+1)^2}\right)\\
&= \left(\frac{k+1}{2k}\right)\cdot\left(1-\frac{1}{(k+1)^2}\right) \quad \text{(since we assumed } P(k) \text{ is true)}\\
&= \left(\frac{k+1}{2k}\right)\cdot\left(\frac{k(k+2)}{(k+1)^2}\right) \quad \text{(by the hint on the assignment sheet)}\\
&= \frac{(k+1)k(k+2)}{2k(k+1)^2}\\
&= \frac{k(k+2)}{2k(k+1)}\\
&= \frac{k+2}{2(k+1)}\\
&= \text{R.H.S of } P(k+1).
\end{aligned}$$

Thus by mathematical induction, for all integers $n \geq 2$,

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

Section 4.3 questions from textbook.

20b. Proof Let $P(n)$ be the statement $n! > n^2$.

Then $P(4)$ is the statement $4! > 4^2$.
$P(k)$ is the statement $k! > k^2$.
$P(k+1)$ is the statement $(k+1)! > (k+1)^2$.

$P(4)$ is true since $4! = 24$ and $4^2 = 16$ and $24 > 16$.

Assume that $P(k)$ is true and use that to show that $P(k+1)$ is true.

$$\begin{aligned}
\text{L.H.S. of } P(k+1) &= (k+1)!\\
&= (k+1)k!\\
&> (k+1)k^2 \quad \text{(since we assumed } P(k) \text{ is true)}\\
&> (k+1)(k+1) \quad \text{(since } k^2 > k+1 \text{ for } k \geq 4)\\
&= (k+1)^2\\
&= \text{R.H.S. of } P(k+1).
\end{aligned}$$

Thus, by mathematical induction, for all integers $n \geq 4$, $n! > n^2$.

24. Proof Let $P(n)$ be the statement $d_n = \frac{2}{n!}$.

Then $P(1)$ is the statement $d_1 = \frac{2}{1!}$.

$P(k)$ is the statement $d_k = \frac{2}{k!}$.

$P(k+1)$ is the statement $d_{k+1} = \frac{2}{(k+1)!}$.

$P(1)$ is true since the value of $d_1 = 2$ was given in the question.

Assume that $P(k)$ is true and use that to show that $P(k+1)$ is true.

$$\begin{aligned}
\text{L.H.S of } P(k+1) &= d_{k+1} \\
&= \frac{d_k}{k+1} \quad \text{(by the formula given in the question)} \\
&= d_k \cdot \frac{1}{k+1} \\
&= \frac{2}{k!} \cdot \frac{1}{k+1} \quad \text{(since we assumed } P(k) \text{ is true)} \\
&= \frac{2}{k!(k+1)} \\
&= \frac{2}{(k+1)!} \\
&= \text{R.H.S of } P(k+1)
\end{aligned}$$

Thus, by mathematical induction, for all integers $n \geq 1$, $d_n = \frac{2}{n!}$.

2. (a) The first person has to shake hands with $(n-1)$ people, the second person has to shake hands with $(n-2)$ people (since they already shook hands with the first person), the third person has to shake hands with $(n-3)$ people, the fourth person has to shake hands with $(n-4)$ people, etc.., until the $(n-1)$th person only has to shake hands with the nth person. Thus the total number of handshakes is

$$(n-1) + (n-2) + (n-3) + \ldots 3 + 2 + 1.$$

(b) In summation notation

$$1 + 2 + 3 + \ldots + (n-3) + (n-2) + (n-1) = \sum_{i=1}^{n-1} i.$$

(c) Proof Let $P(n)$ be the statement $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$.

Then $P(2)$ is the statement $\sum_{i=1}^{1} i = \frac{2 \cdot 1}{2}$.

$P(k)$ is the statement $\sum_{i=1}^{k-1} i = \frac{k(k-1)}{2}$.

$P(k+1)$ is the statement $\sum_{i=1}^{k} i = \frac{(k+1)k}{2}$.

$P(2)$ is true since $\sum_{i=1}^{1} i = 1$ and $\frac{2 \cdot 1}{2} = 1$.

Assume that $P(k)$ is true and show that $P(k+1)$ is true.

$$\begin{aligned}
\text{L.H.S. of } P(k+1) &= \sum_{i=1}^{k} i \\
&= \sum_{i=1}^{k-1} i + k \\
&= \frac{k(k-1)}{2} + k \quad \text{(since we assumed } P(k) \text{ is true)} \\
&= \frac{k^2 - k}{2} + \frac{2k}{2} \\
&= \frac{k^2 + k}{2} \\
&= \frac{(k+1)k}{2} \\
&= \text{R.H.S. of } P(k+1)
\end{aligned}$$

Thus, by mathematical induction, for all integers $n \geq 2$,

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}.$$

(d) If each pair of teams played exactly once, then there would be

$$1 + 2 + 3 + \ldots + (n-2) + (n-1) = \frac{n(n-1)}{2} \text{ games,}$$

but each pair of teams plays exactly twice so there are $2 \cdot \frac{n(n-1)}{2} = n(n-1)$ games. With 14 teams, there would be $14 \cdot 13 = 182$ games played in a complete season.

3. Section 5.1 questions from textbook.

6c. No, $\{2\}$ is not an element of $\{1, 2\}$.

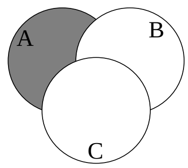
6d. Yes, $\{3\}$ is an element of $\{1, \{2\}, \{3\}\}$.

9a. $A \cup B = \{x \in \mathbb{R} \mid -2 \leq x < 3\}$.

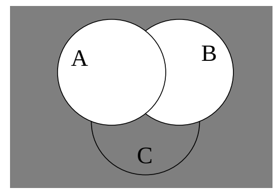
9b. $A \cap B = \{x \in \mathbb{R} \mid -1 < x \leq 1\}$.

13c. Yes, $T \subseteq S$. Every integer that is divisible by 6 is also divisible by 3.

15d, 15e. The shaded areas corresponds to $A - (B \cup C)$ and $(A \cup B)^c$.



A − (B ∪ C)



(A ∪ B)^{C}

Section 5.3 questions from textbook.

38. Every integer is a member of exactly one of the sets A_0, A_1, A_2 or A_3, and the four sets are mutually disjoint. If we apply the quotient-remainder theorem to an arbitrary integer n with $d = 4$, then the possible remainders are zero (so n would be in the set A_0), one (so n would be in the set A_1), two (so n would be in the set A_2), or three (so n would be in the set A_3). Hence these four sets form a partition of the integers.

41b. The set $X \times Y = \{(a, x), (b, x), (a, y), (b, y)\}$. Since $X \times Y$ has 4 elements, the power set of $X \times Y$ will have $2^4 = 16$ elements. The set $\mathcal{P}(X \times Y)$ is

$$
\begin{aligned}
&\{\emptyset, \\
&\{(a, x)\}, \{(b, x)\}, \{(a, y)\}, \{(b, y)\}, \\
&\{(a, x), (b, x)\}, \{(a, x), (a, y)\}, \{(a, x), (b, y)\}, \{(b, x), (a, y)\}, \{(b, x), (b, y)\}, \{(a, y), (b, y)\}, \\
&\{(a, x), (b, x), (a, y)\}, \{(a, x), (b, x), (b, y)\}, \{(a, x), (a, y), (b, y)\}, \{(b, x), (a, y), (b, y)\}, \\
&\{(a, x), (b, x), (a, y), (b, y)\}\}
\end{aligned}
$$

4. Let $A(x)$ represent the statement $x \in A$ and $B(x)$ represent the statment $x \in B$. Then the statement we are asked to prove is

$$\forall x \in \text{a universal set}, (A(x) \to B(x)) \to ((A(x) \wedge B(x)) \to B(x)).$$

Thus we use a truth table to investigate the statement form

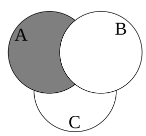
$$(a \to b) \to ((a \wedge b) \to b).$$

a	b	$a \to b$	$a \wedge b$	$(a \wedge b) \to b$	$(a \to b) \to ((a \wedge b) \to b)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	F	T	T

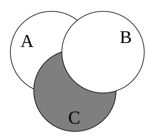
Since this statement form is a tautology, the property

$$\text{if } A \subseteq B, \text{ then } (A \cap B) \subseteq B \quad \text{is true.}$$

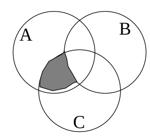
5. The following three Venn diagrams show $(A - B)$, $(C - B)$ and $(A - B) \cap (C - B)$.



A − B

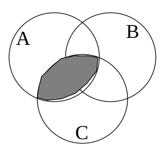


C − B

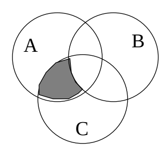


(A − B) ∩ (C − B)

The following two Venn diagrams show $(A \cap C)$ and $(A \cap C) - B$.



A ∩ C



(A ∩ C) − B

The Venn diagrams illustrate the fact that

$$(A - B) \cap (C - B) = (A \cap C) - B.$$

6. Here $A = \{\emptyset\}$ and $B = \{A\} = \{\{\emptyset\}\}$.

- The empty set is a member of A.
- The empty set is not a member of B.
- The empty set is a subset of A. (The empty set is a subset of every set.)
- The empty set is a subset of B.
- A is a not a subset of B.
- A is a member of B.