1. Section 4.1 questions from textbook

21. \[
\sum_{m=0}^{4} \frac{1}{2^{m}} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}
\]
\[
= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}
\]
\[
= \frac{16}{16} + \frac{8}{16} + \frac{4}{16} + \frac{2}{16} + \frac{1}{16}
\]
\[
= \frac{31}{16}
\]

30. \((1^3 - 1) + (2^3 - 1) + (3^3 - 1) + (4^3 - 1) = \sum_{i=1}^{4} (i^3 - 1)\).

34. \((1 - r) \cdot (1 - r^2) \cdot (1 - r^3) \cdot (1 - r^4) = \prod_{i=1}^{4} (1 - r^i)\).

49. \[
\frac{5!}{7!} = \frac{5!}{7 \cdot 6 \cdot 5!} = \frac{1}{7 \cdot 6} = \frac{1}{42}.
\]

Section 4.2 questions from textbook.

15. **Proof** Let \(P(n)\) be the statement

\[
\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \ldots \cdot \left(1 - \frac{1}{n^2}\right) = \frac{n + 1}{2n},
\]

or equivalently

\[
\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n + 1}{2n}.
\]

Then \(P(2)\) is the statement

\[
\left(1 - \frac{1}{2^2}\right) = \frac{2 + 1}{2 \cdot 2}.
\]

\(P(k)\) is the statement

\[
\prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) = \frac{k + 1}{2k}.
\]

\(P(k + 1)\) is the statement

\[
\prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) = \frac{k + 2}{2(k + 1)}.
\]

Now \(P(2)\) is true since \(\left(1 - \frac{1}{2^2}\right) = \frac{3}{4}\) and \(\frac{2 + 1}{2 \cdot 2} = \frac{3}{4}\).
Assume that \( P(k) \) is true and use that to show that \( P(k+1) \) is true.

L.H.S. of \( P(k+1) \)  
\[
\prod_{i=2}^{k+1} \left( 1 - \frac{1}{i^2} \right) \\
= \left[ \prod_{i=2}^{k} \left( 1 - \frac{1}{i^2} \right) \right] \cdot \left( 1 - \frac{1}{(k+1)^2} \right) \\
= \left( \frac{k+1}{2k} \right) \cdot \left( 1 - \frac{1}{(k+1)^2} \right) \quad \text{(since we assumed \( P(k) \) is true)} \\
= \left( \frac{k+1}{2k} \right) \cdot \left( \frac{k(k+2)}{(k+1)^2} \right) \quad \text{(by the hint on the assignment sheet)} \\
= \frac{(k+1)k(k+2)}{2k(k+1)^2} \\
= \frac{k(k+2)}{2(k+1)} \\
= \frac{k+2}{2(k+1)} \\
= \text{R.H.S. of } P(k+1).
\]

Thus by mathematical induction, for all integers \( n \geq 2 \),
\[
\left( 1 - \frac{1}{2^2} \right) \cdot \left( 1 - \frac{1}{3^2} \right) \cdot \ldots \cdot \left( 1 - \frac{1}{n^2} \right) = \frac{n+1}{2n}.
\]

Section 4.3 questions from textbook.

20b. **Proof** Let \( P(n) \) be the statement \( n! > n^2 \).

Then \( P(4) \) is the statement \( 4! > 4^2 \).

\( P(k) \) is the statement \( k! > k^2 \).

\( P(k+1) \) is the statement \( (k+1)! > (k+1)^2 \).

\( P(4) \) is true since \( 4! = 24 \) and \( 4^2 = 16 \) and \( 24 > 16 \).

Assume that \( P(k) \) is true and use that to show that \( P(k+1) \) is true.

L.H.S. of \( P(k+1) \)  
\[
= (k+1)! \\
= (k+1)k! \\
> (k+1)k^2 \quad \text{(since we assumed } P(k) \text{ is true)} \\
> (k+1)(k+1) \quad \text{(since } k^2 > k+1 \text{ for } k \geq 4) \\
= (k+1)^2 \\
= \text{R.H.S. of } P(k+1).
\]

Thus, by mathematical induction, for all integers \( n \geq 4 \), \( n! > n^2 \).
24. **Proof** Let \( P(n) \) be the statement \( d_n = \frac{2}{n!} \).

Then \( P(1) \) is the statement \( d_1 = \frac{2}{1!} \).

\( P(k) \) is the statement \( d_k = \frac{2}{k!} \).

\( P(k + 1) \) is the statement \( d_{k+1} = \frac{2}{(k + 1)!} \).

\( P(1) \) is true since the value of \( d_1 = 2 \) was given in the question.

Assume that \( P(k) \) is true and use that to show that \( P(k + 1) \) is true.

\[
\text{L.H.S of } P(k + 1) = d_{k+1} = \frac{d_k}{k + 1} \quad \text{by the formula given in the question}
\]

\[
= d_k \cdot \frac{1}{k + 1}
\]

\[
= \frac{2}{k!} \cdot \frac{1}{k + 1} \quad \text{(since we assumed } P(k) \text{ is true)}
\]

\[
= \frac{2}{k!(k + 1)}
\]

\[
= \frac{2}{(k + 1)!}
\]

\[
= \text{R.H.S of } P(k + 1)
\]

Thus, by mathematical induction, for all integers \( n \geq 1 \), \( d_n = \frac{2}{n!} \).

2. (a) The first person has to shake hands with \((n - 1)\) people, the second person has to shake hands with \((n - 2)\) people (since they already shook hands with the first person), the third person has to shake hands with \((n - 3)\) people, the fourth person has to shake hands with \((n - 4)\) people, etc., until the \((n - 1)\)th person only has to shake hands with the \(n\)th person. Thus the total number of handshakes is

\[
(n - 1) + (n - 2) + (n - 3) + \ldots + 3 + 2 + 1.
\]

(b) In summation notation

\[
1 + 2 + 3 + \ldots + (n - 3) + (n - 2) + (n - 1) = \sum_{i=1}^{n-1} i.
\]
(c) **Proof** Let \( P(n) \) be the statement \( \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} \).

Then \( P(2) \) is the statement \( \sum_{i=1}^{1} i = \frac{2 \cdot 1}{2} \).

\( P(k) \) is the statement \( \sum_{i=1}^{k-1} i = \frac{k(k-1)}{2} \).

\( P(k+1) \) is the statement \( \sum_{i=1}^{k} i = \frac{(k+1)k}{2} \).

\( P(2) \) is true since \( \sum_{i=1}^{1} i = 1 \) and \( \frac{2 \cdot 1}{2} = 1 \).

Assume that \( P(k) \) is true and show that \( P(k+1) \) is true.

\[
\text{L.H.S. of } P(k+1) = \sum_{i=1}^{k} i = \sum_{i=1}^{k-1} i + k = \frac{k(k-1)}{2} + k \quad \text{(since we assumed \( P(k) \) is true)}
\]
\[
= \frac{k^2 - k}{2} + \frac{2k}{2}
\]
\[
= \frac{k^2 + k}{2}
\]
\[
= \frac{(k+1)k}{2}
\]
\[
= \text{R.H.S. of } P(k+1)
\]

Thus, by mathematical induction, for all integers \( n \geq 2 \),

\[
\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}.
\]

(d) If each pair of teams played exactly once, then there would be

\[1 + 2 + 3 + \ldots + (n - 2) + (n - 1) = \frac{n(n-1)}{2}\] games,

but each pair of teams plays exactly twice so there are \( 2 \cdot \frac{n(n-1)}{2} = n(n-1) \) games. With 14 teams, there would be \( 14 \cdot 13 = 182 \) games played in a complete season.
3. Section 5.1 questions from textbook.

6c. No, \( \{2\} \) is not an element of \( \{1, 2\} \).

6d. Yes, \( \{3\} \) is an element of \( \{1, \{2\}, \{3\}\} \).

9a. \( A \cup B = \{ x \in \mathbb{R} \mid -2 \leq x < 3 \} \).

9b. \( A \cap B = \{ x \in \mathbb{R} \mid -1 < x \leq 1 \} \).

13c. Yes, \( T \subseteq S \). Every integer that is divisible by 6 is also divisible by 3.

15d, 15e. The shaded areas corresponds to \( A - (B \cup C) \) and \( (A \cup B)^c \).

Section 5.3 questions from textbook.

38. Every integer is a member of exactly one of the sets \( A_0, A_1, A_2 \) or \( A_3 \), and the four sets are mutually disjoint. If we apply the quotient-remainder theorem to an arbitrary integer \( n \) with \( d = 4 \), then the possible remainders are zero (so \( n \) would be in the set \( A_0 \)), one (so \( n \) would be in the set \( A_1 \)), two (so \( n \) would be in the set \( A_2 \)), or three (so \( n \) would be in the set \( A_3 \)). Hence these four sets form a partition of the integers.

41b. The set \( X \times Y = \{(a, x), (b, x), (a, y), (b, y)\} \). Since \( X \times Y \) has 4 elements, the power set of \( X \times Y \) will have \( 2^4 = 16 \) elements. The set \( \mathcal{P}(X \times Y) \) is

\[
\{\emptyset, \{(a, x), \{(b, x)\}, \{(a, y)\}, \{(b, y)\}, \{(a, x), (b, x)\}, \{(a, x), (a, y)\}, \{(a, x), (b, y)\}, \{(b, x), (a, y)\}, \{(a, y), (b, y)\}, \{(a, x), (b, x), (a, y)\}, \{(a, x), (b, y), (a, y)\}, \{(a, x), (a, y), (b, y)\}, \{(b, x), (a, y), (b, y)\}, \{(a, x), (b, x), (a, y), (b, y)\}\}
\]
4. Let \( A(x) \) represent the statement \( x \in A \) and \( B(x) \) represent the statement \( x \in B \). Then the statement we are asked to prove is

\[
\forall x \in \text{a universal set}, (A(x) \rightarrow B(x)) \rightarrow ((A(x) \land B(x)) \rightarrow B(x)).
\]

Thus we use a truth table to investigate the statement form

\[
(a \rightarrow b) \rightarrow ((a \land b) \rightarrow b).
\]

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Since this statement form is a tautology, the property

\[
\text{if } A \subseteq B, \text{ then } (A \cap B) \subseteq B \text{ is true.}
\]

5. The following three Venn diagrams show \((A - B), (C - B)\) and \((A - B) \cap (C - B)\).

\[
\begin{align*}
\text{A } - \text{ B} & \quad \text{C } - \text{ B} \quad \text{(A } - \text{ B) } \cap \text{ (C } - \text{ B)}
\end{align*}
\]

The following two Venn diagrams show \((A \cap C)\) and \((A \cap C) - B\).

\[
\begin{align*}
\text{A } \cap \text{ C} \quad \text{(A } \cap \text{ C) } - \text{ B}
\end{align*}
\]

The Venn diagrams illustrate the fact that

\[
(A - B) \cap (C - B) = (A \cap C) - B.
\]

6. Here \( A = \{\emptyset \} \) and \( B = \{A\} = \{\{\emptyset \}\} \).

- The empty set is a member of \( A \).
- The empty set is not a member of \( B \).
- The empty set is a subset of \( A \). (The empty set is a subset of every set.)
- The empty set is a subset of \( B \).
- \( A \) is a not a subset of \( B \).
- \( A \) is a member of \( B \).