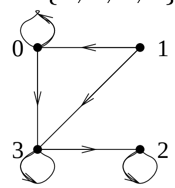
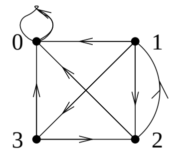
1. Section 10.5 questions from textbook

1. (c) The relation R_3 is antisymmetric.
For any distinct elements $a, b \in \{0, 1, 2, 3\}$, if $(a, b) \in R_3$ then $(b, a) \notin R_3$.



(d) The relation R_4 is not antisymmetric since $(1, 2) \in R_4$ and $(2, 1) \in R_4$.



7. The relation R is relexive, since it is true that $\forall m \in \mathbb{Z},\ m\ R\ m$.
If $m \in \mathbb{Z}$ then every prime factor of m is also a prime factor of m.

The relation R is antisymmetric, since it is true that

$$\forall \text{ distinct } m, n \in \mathbb{Z},\ \text{if } m\ R\ n,\ \text{then } n \text{ is not related to } m \text{ by } R.$$

Suppose that $m, n \in \mathbb{Z}$ and $m\ R\ n$. Then every prime factor of m is a prime factor of n, but m and n are not equal, so there is a prime factor of n which is not a prime factor of m. Therefore, n is not related to m by R.

The relation R is transitive, since it is true that

$$\forall m, n, p \in \mathbb{Z},\ \text{if } m\ R\ n \text{ and } n\ R\ p,\ \text{then } m\ R\ p.$$

Suppose that $m, n, p \in \mathbb{Z}$ and $m\ R\ n$ and $n\ R\ p$. Then every prime factor of m is a prime factor of n and every prime factor of n is a prime factor of p, so every prime factor of m must be a prime factor of p. Thus, $m\ R\ p$.

Since R is reflexive, antisymmetric and transitive, R is a partial order.

Section 7.1 questions from textbook.

2. (a) The domain of g is $\{1, 3, 5\}$ and the co-domain of g is $\{s, t, u, v\}$.
(b) $g(1) = t$, $g(3) = t$, $g(5) = t$.
(c) The range of g is $\{t\}$.
(d) The inverse image of t is $\{1, 3, 5\}$. The inverse image of u is $\emptyset$.
(e) $g = \{(1, t), (3, t), (5, t)\}$.

3. **(c)** The arrow diagram does not represent a function, because the element 4 in the domain has two different images in the co-domain.

(d) The arrow diagram does represent a function.

12. **(b)** $F(\emptyset) = 0$.

(d) $F(\{2,3,4,5\}) = 0$.

29. The mapping h is not well-defined, so it is not a function. For example,

$$\frac{1}{2} = \frac{2}{4} \text{ but } h\left(\frac{1}{2}\right) = \frac{1}{2} \text{ and } h\left(\frac{2}{4}\right) = \frac{4}{4} = 1.$$

So the rational number $\frac{1}{2}$ has more than one image.

Section 7.3 questions from textbook.

4. **(a)** H is not one-to-one, because $H(b) = H(c) = y$ but $b \neq c$.
H is not onto, because there is no element in the domain that maps to the element x in the co-domain.

(b) K is one-to-one, since each element in the co-domain has at most one arrow pointing to it.
K is not onto, because there is no element in the domain that maps to the element z in the co-domain.

5. **(b)** One function $g : X \to Z$ that is onto but not one-to-one is

$$g = \{(1,1), (2,2), (3,2)\}.$$

(c) One function $h : X \to X$ that is neither one-to-one nor onto is

$$h = \{(1,1), (2,1), (3,1)\}.$$

9. **(a)(i)** g is one-to-one. Suppose that n_1 and n_2 are two integers such that $g(n_1) = g(n_2)$. Then $3n_1 - 2 = 3n_2 - 2$ and so it follows that $n_1 = n_2$.

(ii) g is not onto. Consider the integer 2 which is in the co-domain of g. There is no integer n such that $g(n) = 2$, since $3n - 2 = 2$ would require $n = \frac{4}{3}$ which is not an integer.

(b) G is onto. Suppose y is a real number. Then there exists a real number x such that $G(x) = y$, namely $x = \dfrac{y+2}{3}$.

36 G is both one-to-one and onto so it is a one-to-one correspondence. The inverse function is $G^{-1} : \mathbb{R} \to \mathbb{R}$ where $G^{-1}(y) = \dfrac{y+2}{3}$.

Section 7.4 questions from textbook.

6. **(a)** Let X be the set of seven integers and Y be the set of possible remainders upon division by 6, thus $Y = \{0, 1, 2, 3, 4, 5\}$. Then $n(X) = 7$ and $n(Y) = 6$, so by the pigeonhole principle, there must be two elements of X that map to the same element in Y, so in any set of seven integers there must be two integers that have the same remainder when divided by 6.

(b) Let X be the set of seven integers and Y be the set of possible remainders upon division by 8, thus $Y = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then $n(X) = 7$ and $n(Y) = 8$, so it is possible to find seven integers (for example $\{1, 2, 3, 4, 5, 6, 7\}$) such that no two integers in the set have the same remainder when divided by 8.

13. You must pick at least seven individual boots from the pile to guarantee getting a matched pair. This is because in the worst case you would pick six boots, none of which matched, before you picked one that matched a boot you already had.

28. Let X be the set of people and Y be the set of possible birthdays. Thus $n(X) = 2000$ and $n(Y) = 366$ (there are at most 366 days in a year). Since $n(X) > 5 \cdot n(Y)$, the generalised pigeonhole principle says that there must be an element in Y which is the image of at least 6 elements in X. Thus, there must be at least 5 people that have the same birthday.

Section 7.5 questions from textbook.

7. $(G \circ F)(2) = G(F(2)) = G\left(\frac{4}{3}\right) = \left\lfloor \frac{4}{3} \right\rfloor = 1.$
$(G \circ F)(-3) = G(F(-3)) = G\left(\frac{9}{3}\right) = G(3) = \lfloor 3 \rfloor = 3.$
$(G \circ F)(5) = G(F(5)) = G\left(\frac{25}{3}\right) = \left\lfloor \frac{25}{3} \right\rfloor = 8.$

9. $(G \circ G^{-1})(x) = G(G^{-1}(x)) = G(\sqrt{x}) = (\sqrt{x})^2 = x.$
$(G^{-1} \circ G)(x) = G^{-1}(G(x)) = G^{-1}(x^2) = \sqrt{x^2} = x.$
Both compositions give the identity function.

21. $(g \circ f)(x) = g(f(x)) = g(x + 2) = -(x + 2) = -x - 2.$

To find $(g \circ f)^{-1}$, given a real number y, we need to find the element x such that $(g \circ f)(x) = y$. So, $-x - 2 = y$ and thus $x = -y - 2$. Hence $(g \circ f)^{-1}(y) = -y - 2$.

To find $(g)^{-1}$, given a real number y, we need to find the element x such that $g(x) = y$. So, $-x = y$ and thus $x = -y$. Hence $g^{-1}(y) = -y$.

To find $(f)^{-1}$, given a real number y, we need to find the element x such that $f(x) = y$. So, $x + 2 = y$ and thus $x = y - 2$. Hence $f^{-1}(y) = y - 2$.

Thus $(f^{-1} \circ g^{-1})(y) = f^{-1}(g^{-1}(y)) = f^{-1}(-y) = -y - 2$.

Hence, in this case $(g \circ f)^{-1}$ is equal to $f^{-1} \circ g^{-1}$.

2. Call the finite-state atomaton P.

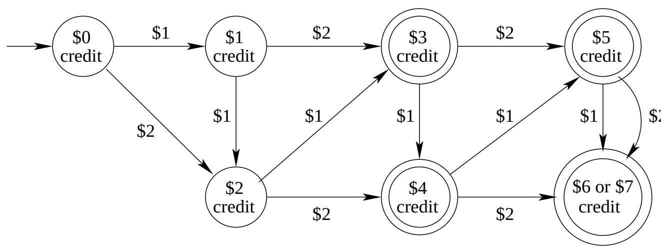
(a) The states of P are
{\$0 credit, \$1 credit, \$2 credit, \$3 credit, \$4 credit, \$5 credit, \$6 or more credit}.

(b) The input symbols of P are \$1 and \$2.

(c) The initial state of P is \$0 credit.

(d) The accepting states of P are {\$3 credit, \$4 credit, \$5 credit, \$6 or more credit}.

(e) A transition diagram for P is shown.



(f) The language accepted by this finite-state automaton is any string of \$1 and \$2 symbols that sum to 3, 4, 5, 6 or 7 dollars.

3. (a) The Cayley table for $(\{1, 2, 3, 4\}, \times_5)$ is as follows.

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(b) To show this is an abelian group we must show that it is closed, associative, commutative, that it has an identity element, and that each element has an inverse.

The entries in the body of the Cayley table are from the set $\{1, 2, 3, 4\}$, so it is closed.

We may assume that this operation is associative.

This operation is commutative, since the body of the Cayley table shows that $a \times_5 b = b \times_5 a$.

The identity element is 1. For all $a \in \{1, 2, 3, 4\}$, $a \times_5 1 = a = 1 \times_5 a$.

Each element in the set has an inverse: $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$, $4^{-1} = 4$.

(c) This set is not closed under the operation $\times_4$. For example, $2 \times_4 2 = 0$, which is not in the set $\{1, 2, 3\}$. Thus $(\{1, 2, 3\}, \times_4)$ is not a group.

4. (a) To show that this is a subgroup, we must show that it is closed, contains the identity of $\mathbb{Z}_{12}$ and that each element has its inverse in the subgroup.

The easiest way to show this is to construct a Cayley table for the subgroup.

$+_{12}$	0	3	6	9
0	0	3	6	9
3	3	6	9	0
6	6	9	0	3
9	9	0	3	6

From the Cayley table, it is clear that this set is closed under the operation $+_{12}$.

The element 0 is the identity since $0+_{12}a = a = a+_{12}0$ for all elements $a \in \{0, 3, 6, 9\}$.

Each element has its inverse in the set: $0^{-1} = 0$, $3^{-1} = 9$, $6^{-1} = 6$ and $9^{-1} = 3$.

Thus $(\{0, 3, 6, 9\}, +_{12})$ is a subgroup of $(\mathbb{Z}_{12}, +_{12})$.

(b) The cyclic subgroup generated by 4 is $(\{0, 4, 8\}, +_{12})$ since

$$4 = 4, \quad 4 +_{12} 4 = 8, \quad 4 +_{12} 4 +_{12} 4 = 0.$$

(c) This set is not closed under the operation $+_{12}$. For example, $5 +_{12} 10 = 3$ which is not listed in the set. Hence, $(\{0, 5, 10\}, +_{12})$ is not a subgroup of $(\mathbb{Z}_{12}, +_{12})$.

5. (Bonus) Define a function $f : \mathbb{Z} \to 3\mathbb{Z}$ where $f(n) = 3n$.

First prove that the function f is one-to-one. Suppose that there exist integers n_1 and n_2 such that $f(n_1) = f(n_2)$. Then $3n_1 = 3n_2$ and so we must have $n_1 = n_2$. Hence f is one-to-one.

Now prove that the function f is onto. Suppose there is an integer y in the co-domain $3\mathbb{Z}$. We must find the integer x in the domain such that $f(x) = y$. Consider $x = \frac{y}{3}$. Then x is an integer since $y \in 3\mathbb{Z}$ so y is divisible by 3. Hence $f(x) = f(\frac{y}{3}) = y$. So f is onto.

Therefore f is a one-to-one correspondence.

Hence $\mathbb{Z}$ and $3\mathbb{Z}$ have the same cardinality, and since $\mathbb{Z}$ is countable, $3\mathbb{Z}$ must also be countable.