**1. (a)** To show that $(\mathbb{R}, +, \times)$ is a field we must show that $(\mathbb{R}, +)$ is an abelian group, $(\mathbb{R} - \{0\}, \times)$ is an abelian group, and that $\forall r, s, t \in \mathbb{R}$, $r \times (s + t) = r \times s + r \times t$.

$(\mathbb{R}, +)$ is an abelian group since:
the sum of any two real numbers is a real number;
addition is associative and commutative;
there exists an identity element, namely 0;
and for each $r \in \mathbb{R}$, there exists an inverse $-r$ such that $r + (-r) = 0 = (-r) + r$.

$(\mathbb{R} - \{0\}, \times)$ is an abelian group since:
the product of any two real numbers is a real number;
multiplication is associative and commutative;
there exists an identity element, namely 1;
and for each real number $r$, there exists an inverse $\frac{1}{r}$ such that $r \times \frac{1}{r} = 1 = \frac{1}{r} \times r$.

Multiplication distributes over addition so $\forall r, s, t \in \mathbb{R}$, $r \times (s + t) = r \times s + r \times t$.

**(b)** $(\mathbb{Z}_8, \oplus, \odot)$ is not a field. $(\mathbb{Z}_8, \oplus)$ is a group (with identity 0) but $(\mathbb{Z}_8 - \{0\}, \odot)$ is not a group.
Consider the elements 2 and 4 in $\mathbb{Z}_8 - \{0\}$. $2 \odot 4 = 8 = 0$ under multiplication modulo 8, so the set $\mathbb{Z}_8 - \{0\}$ is not closed under this binary operation.

**2. Section 6.1 questions from textbook.**

**8. (b)** Let $I$ represent that a person is ill and $W$ represent that a person is well. The sample space is

$$S = \{III, IIW, IWI, WII, IWW, WIW, WWI, WWW\}.$$

Let $E$ be the event that at least two of the people become ill. So $E = \{III, IIW, IWI, WII\}$. Thus the probability that at least two of the people become ill is $P(E) = \frac{4}{8} = \frac{1}{2}$.

**(c)** Let $E$ be the event that none of the three people becomes ill. So $E = \{WWW\}$. The probability that none of the three people becomes ill is $P(E) = \frac{1}{8}$.

**11. (a)** The positive three-digit integers are the integers from 100 to 999 inclusive. The smallest of these that is a multiple of 6 is $102 = 17 \cdot 6$ and the largest of these that is a multiple of 6 is $996 = 166 \cdot 6$. Thus the number of positive three-digit integers that are mutiples of 6 is the same as the number of integers from 17 and 166 inclusive. This is $166 - 17 + 1 = 150$.

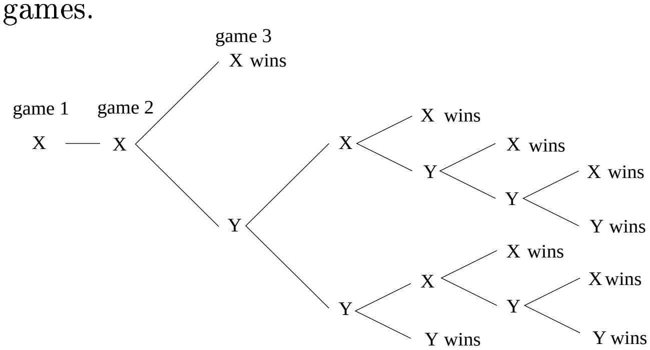
**(b)** Let $E$ be the event that a randomly chosen positive three-digit integer is a multiple of 6. The total number of positive three-digit integers is $999 - 100 + 1 = 900$, so

$$P(E) = \frac{150}{900} = \frac{1}{6}.$$

**18.** We require a list of 87 consecutive integes that ends with 326. Suppose the list starts with the integer $x$. Then $326 - x + 1 = 87$, so $x = 240$ is the smallest integer in the list.

**Section 6.2 questions from textbook.**

**5.** The following tree illustrates the nine ways the competition can be played if $X$ wins the first two games.




**15.** **(a)** There are 20 ways to choose each of the three numbers. Hence there are $20^3 = 8000$ different combinations.

**(b)** If no number is used twice then there are 20 choices for the first number, 19 for the second number and 18 for the third number. Hence there are $20 \cdot 19 \cdot 18 = 6840$ different combinations.

**30.** **(a)** There are $6! = 720$ ways to seat six people in a row.

**(b)** If the doctor sits in the aisle seat then the five remaining people must be seated in the other five seats. There are $5! = 120$ ways to do this.

**(c)** Each couple must have the husband on the left and the wife on the right, so effectively we are arranging three items (the three couples). There are $3! = 6$ ways to do this.

**37.** Here we require $n \geq 2$ since $P(n, r)$ is only defined for $n \geq r$.

$$P(n+1, 3) = (n+1) \cdot (n) \cdot (n-1) = n(n^2 - 1) = n^3 - n.$$

**Section 6.3 questions from textbook.**

**10.** **(a)** There are six distinct letters in the word DESIGN, so they can be arranged in $6! = 720$ ways.

**(b)** If the letters $GN$ appear consecutively in that order then there are $5! = 120$ ways to arrange the letters. If the letters $NG$ appear consecutively in that order then there are $5! = 120$ ways to arrange the letters. There two situations do not overlap, so there are $120 + 120 = 240$ ways to arrange the letters with $G$ and $N$ next to each other.

**22.** **(a)** The smallest of the numbers from 1 to 1000 that is a multiple of 2 is $2 = 1 \cdot 2$ and the largest of these that is a multiple of 2 is $1000 = 500 \cdot 2$. Thus there are 500 numbers that are multiples of 2.

The smallest of the numbers from 1 to 1000 that is a multiple of 9 is $9 = 1 \cdot 9$ and the largest of these that is a multiple of 9 is $999 = 111 \cdot 9$. Thus there are 111 numbers that are multiples of 9.

A number that is a multiple of both 2 and 9 is the same as a number that is a multiple of 18. The smallest of the numbers from 1 to 1000 that is a multiple of 18 is $18 = 1 \cdot 18$ and the largest of these that is a multiple of 18 is $990 = 55 \cdot 18$. Thus there are 55 numbers that are multiples of 18.

Hence, there are $500 + 111 - 55 = 556$ numbers from 1 to 1000 that are multiples of 2 or multiples of 9.

**(b)** The probability that a randomly chosen integer between 1 and 1000 inclusive is a multiple of 2 or a multiple of 9 is

$$P = \frac{556}{1000} = \frac{139}{250}.$$

**(c)** There are $1000 - 556 = 444$ integers from 1 through 1000 that are neither multiples of 2 nor multiples of 9.

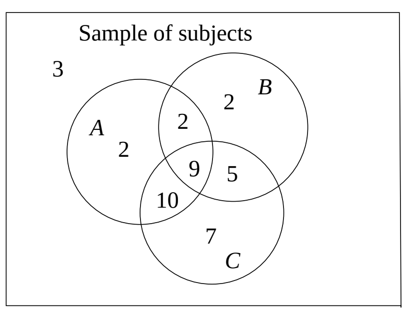
**24.** **(a)** Since 37 people reported relief from at least one of the druges, $40 - 37 = 3$ people got relief from none of the drugs.

**(b)** $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$, so

$$37 = 23 + 18 + 31 - 11 - 19 - 14 + n(A \cap B \cap C).$$

Thus the number of people that got relief from all three drugs is $n(A \cap B \cap C) = 9$.

**(c)** Each number indicates the number of people in that region of the Venn digram.




**(d)** Two people got relief from $A$ only.

**Section 6.4 questions from textbook.**

**7. (a)** We chose 7 people from 14. The number of ways to do this is

$$\binom{14}{7} = 3432.$$

**(b)** The number of teams containing four women and three men is

$$\binom{8}{4} \cdot \binom{6}{3} = 70 \cdot 20 = 1400.$$

The number of teams that contain at least one man can be calculated as the total number of teams minus the number of teams that contain no men. This is

$$3432 - \binom{8}{7} = 3432 - 8 = 3424.$$

Since there are only 6 men, and each team has 7 people, there are no teams that contain zero women. The number of teams that contain at most three women can be calculated as the number of teams that contain one women plus the number of teams that contain 2 women plus the number of teams that contain three women. This is

$$\binom{8}{1} \cdot \binom{6}{6} + \binom{8}{2} \cdot \binom{6}{5} + \binom{8}{3} \cdot \binom{6}{4} = 8 \cdot 1 + 28 \cdot 6 + 56 \cdot 15 = 1016.$$

**(c)** The two people cannot both be in the team. The number of teams that do not contain both of the people can be calculated as the total number of teams minus the number of teams that do contain both of the people. This is

$$3432 - \binom{12}{5} = 3432 - 792 = 2640.$$

**(d)** The number of teams that contain either both of the people or neither of the two people can be calculated as the number of teams that contain both of the people plus the number of teams that contain neither of them. This is

$$\binom{12}{5} + \binom{12}{7} = 792 + 792 = 1584.$$

**13.(b)** Choose 5 out of 10 tossings to be heads. The number of possible outcomes with exactly 5 heads is $\binom{10}{5} = 252$.

**(c)** The number of possible outcomes with at least 9 heads is the number of outcomes with 9 heads plus the number of outcomes with 10 heads. This is

$$\binom{10}{9} + \binom{10}{10} = 10 + 1 = 11.$$

**16. (a)** Choose 4 boards out of 40. The number of ways to do this is

$$\binom{40}{4} = 91390.$$

**(b)** The number of samples that contain at least one defective board is equal to the total number of samples minus the number of samples that contain no defective boards. This is

$$91390 - \binom{37}{4} = 91390 - 66045 = 25345.$$

**(c)** The probability that a randomly chosen sample of four contains at least one defective board is

$$P = \frac{25345}{91390} = \frac{5069}{18278} = 0.277 \text{ (to three decimal places)}.$$

**20 (a)** The word INTELLIGENCE contains 12 letters: 2 Is, 2Ns, 1 T, 3 Es, 2 Ls, 1 G, 1 C. The number of distinct arrangements of the letters in the word INTELLIGENCE is

$$\frac{12!}{2!\ 2!\ 1!\ 3!\ 2!\ 1!\ 1!} = 9\ 979\ 200.$$

**(b)** The number of distinct arrangements of the letters in the word INTELLIGENCE that start with a T and end with a G is

$$\frac{10!}{2!\ 2!\ 3!\ 2!\ 1!} = 75\ 600.$$

**(c)** You can think of this as arranging the letters 'INT', 'IG', E, E, E, L, L, N, C. The number of distinct arrangements of the letters in the word INTELLIGENCE that contain INT (in order) and IG (in order) is

$$\frac{9!}{3!\ 2!\ 1!\ 1!} = 30\ 240.$$

**3. (a)** A simplification is

$$\binom{y+1}{y-2} = \frac{(y+1)!}{(y-2)!\ 3!} = \frac{(y+1)(y)(y-1)}{3!} = \frac{y^3-y}{6}.$$

**(b)** Using Pascal's formula:

$$\binom{10}{4} + \binom{9}{4} + \binom{9}{5} = \binom{10}{4} + \binom{10}{5} = \binom{11}{5}.$$

**4. (a)** The binomial expansion of $(2x-y)^5$ is

$$(2x)^5 + \binom{5}{1}(2x)^4(-y)^1 + \binom{5}{2}(2x)^3(-y)^2 + \binom{5}{3}(2x)^2(-y)^3 + \binom{5}{4}(2x)^1(-y)^4 + (-y)^5$$

$$= \quad 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$$

**(b)** In the expansion of $(p-q)^8$ we have

$$(p-q)^8 = p^8 + \binom{8}{1}p^7(-q)^1 + \ldots + \binom{8}{5}p^3(-q)^5 + \ldots + (-q)^8.$$

So the coefficient of $p^3q^5$ will be $-\binom{8}{5} = -56$.

**5. Bonus Question**

**(a)** You are choosing 5 pasta dishes from 10, but it is possible to choose more than one of the same dish. The number of possible selections is

$$\binom{5+10-1}{5} = 2002.$$

**(b)** Two people choose ravioli with pesto sauce, so now three people have to choose dishes from a choice of 9 (since we cannot have any more ravioli and pesto sauce). The number of ways to do this is

$$\binom{3+9-1}{3} = 165.$$

**(c)** Two people choose ravioli and pesto sauce and now the remaining three people can choose any of the 10 dishes. The number of ways to do this is

$$\binom{3+10-1}{3} = 220.$$