4. (a) (3 marks) Let a, b, c be any integers.
Are the following true or false?
(*State* true or false in each box; no need to give reasons.)

(i) If $a \mid b$ and $a \mid c$, then $a \mid (b-c)$. $\boxed{\text{True}}$

(ii) If $a \mid b$ and $b \mid c$, then $a \mid c$. $\boxed{\text{True}}$

(iii) If $a \mid b^2$ then $a \mid b$. $\boxed{\text{False}}$

(b) (5 marks) One (at least) of (i), (ii) or (iii) in part **(a)** is true! For whichever is true, give a proof of that true statement. For whichever is false, give a counterexample.

(i) Suppose $a,b,c \in \mathbb{Z}$ and $a \mid b$ and $a \mid c$.
Then $b = a \cdot m$ for some integer m and
$c = a \cdot n$ for some integer n.
So $b - c = a \cdot m - a \cdot n = a(m-n)$.
$\therefore a \mid b-c$

(ii) Suppose $a,b,c \in \mathbb{Z}$ and $a \mid b$ and $b \mid c$.
Then $b = a \cdot m$ for some integer m and
$c = b \cdot n$ for some integer n.
So $c = b \cdot n = a \cdot mn$.
$\therefore a \mid c$.

(iii) Let $a = 8$ and $b = 4$.
Then $a \mid b^2$ but a does not divide b.
$(8 \mid 16$ but $8 \nmid 4)$.

Question 5 see next page.

5. (12 marks)

(a) Find all solutions to the equation $m^2 - n^2 = 63$ for which both m and n are positive integers. (Hint: $m^2 - n^2 = (m+n)(m-n)$.)

$(m+n)(m-n) = 63$.

Since we require m and n to be positive integers, $m+n$ will be a positive integer and so $m-n$ must also be a positive integer.

So we want solutions with $m > n$.

The set of all pairs of positive integers whose product is 63 is:
$\{\{9,7\}, \{21,3\}, \{63,1\}\}$.

If $m+n = 9$, $m-n = 7$ then $\boxed{m = 8 \text{ and } n = 1}$.

If $m+n = 21$, $m-n = 3$ then $\boxed{m = 12 \text{ and } n = 9}$.

If $m+n = 63$, $m-n = 1$ then $\boxed{m = 32 \text{ and } n = 31}$.

These are all the solutions in which m and n are positive integers.

See question 5(b) next page