11. (6 marks) Use Mathematical Induction to prove that, for all integers $n \geq 4$,

$$3n + 1 < 2^n.$$

Let $P(n)$ be the statement $3n+1<2^n$ for $n \geqslant 4$.

$P(4)$ is the statement $3(4)+1 < 2^4$.

$P(k)$ is the statement $3k+1 < 2^k$.

$P(k+1)$ is the statement $3(k+1)+1 < 2^{k+1}$.

$P(4)$ is true since $3(4)+1 = 13$ and $2^4=16$ and $13<16$.

Assume $P(k)$ is true and show that $P(k+1)$ is true,

R.H.S of $P(k+1) = 2^{k+1}$

$= 2^k \cdot 2$

$> (3k+1)\cdot 2$ (since $P(k)$ is assumed true)

$= 6k+2$

$= 3k+3k+2$

$> 3k+4$ (since $3k+2>4$ for $k \geqslant 4$)

$=$ L.H.S of $P(k+1)$

Note that it is easier to start with the R.H.S. of $P(k+1)$ here.

Hence $3(k+1)+1 < 2^{k+1}$, so by mathematical induction $3n+1<2^n$ for all integers $n \geqslant 4$.

Question 12 see next page.

12. (9 marks)
Binary relations α and β are defined on the set $\mathbf{Z}^+$ of positive integers by:

$m\,\alpha\,n$ if and only if $m+n$ is even;

$m\,\beta\,n$ if and only if $m \mid n$, that is, m divides n.

Insert ticks (for yes) or crosses (for no) into the following table, to show which properties these relations on $\mathbf{Z}^+$ have.

	α	β
Reflexive	✓	✓
Symmetric	✓	✗
Antisymmetric	✗	✓
Transitive	* ✓	✓
Equivalence relation	✓	✗

Explain your answer to the box marked *.

Suppose that $m,n,p \in \mathbf{Z}^+$ and $m\alpha n$ and $n\alpha p$.

Since $m\alpha n$, $m+n=2k$ for some integer k.

Since $n\alpha p$, $n+p=2\ell$ for some integer ℓ.

Then $m+p = 2k-n+2\ell-n = 2k+2\ell-2n = 2(k+\ell-n)$

So $m+p$ is even. $\therefore m\alpha p$

$\therefore \alpha$ is transitive.

Precisely ONE of these relations is an equivalence relation.
State which one, and give its equivalence classes.

α is an equivalence relation.

The equivalence classes are

$[1] = \{m \in \mathbf{Z}^+ \mid m \text{ is odd}\}$.

$[2] = \{m \in \mathbf{Z}^+ \mid m \text{ is even}\}$.

Question 13 see next page.