13. (8 marks)

(a) What is the minimum number of people in a group, to guarantee that at least 3 people in the group all have their birthday falling in the same month?

There are 12 months. The way to guarantee that at least 3 people have their birthdays in the same moth is to have 2 in each month and 3 in one month.

So we need 25 people.

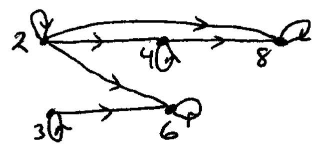
(b) Let the binary relation τ on $S = \{2,3,4,6,8\}$ be given by

$$a \tau b \quad \text{if and only if} \quad a \mid b.$$

• List the ordered pairs in the relation τ.

$\tau = \{(2,4),(2,6),(2,8),(3,6),(4,8),$
$(2,2),(3,3),(4,4),(6,6),(8,8)\}$

• Draw the directed graph which corresponds to the relation τ.



• Is τ a partial order relation on S? (Explain briefly.)

τ is a partial order relation
It is reflexive, antisymmetric and transitive.

• Is τ a total order relation on S? (Explain briefly.)

No ~~$2~~ 2 and 3 are not related in any way.
$(2,3) \notin \tau$ and $(3,2) \notin \tau$.

Question 14 see next page.

14. (10 marks)

(a) Let the binary operation $\bullet$ on the integers $\mathbf{Z}$ be defined by $a \bullet b = a + b + 2ab$.

Is $\bullet$ commutative? (Explain.) Yes.

$a \bullet b = a + b + 2ab$ and $b \bullet a = b + a + 2ab$

So $a \bullet b = b \bullet a$.

Is $\bullet$ associative? (Explain.) Yes

$(a \bullet b) \bullet c = a + b + 2ab + c + 2(a+b+2ab)c$
$= a + b + c + 2ab + 2ac + 2bc + 4abc$
$a \bullet (b \bullet c) = a + b + c + 2bc + 2a(b+c+2bc)$
$= a + b + c + 2bc + 2ab + 2ac + 4abc$

equal

Is there an identity in $\mathbf{Z}$ for $\bullet$? (Explain.)

Yes. 0 is the identity since

$a \bullet 0 = a + 0 + 2(a)(0) = a + 0 + 0 = a$

and $0 \bullet a = a \bullet 0$.

Are there inverses in $\mathbf{Z}$ for $\bullet$? (Explain.) No.

Solving $a \bullet a^{-1} = 0$ for a^{-1} gives

$a^{-1} = \dfrac{-a}{1+2a}$ which is not always an integer.

$a \bullet a^{-1} = 0$
$a + a^{-1} + 2aa^{-1} = 0$
$a^{-1} + 2aa^{-1} = -a$
$a^{-1} = \dfrac{-a}{1+2a}$

(b) Consider the systems $(\mathbf{Z}, \times)$, $(\mathbf{Q} - \{0\}, \times)$, $(\mathbf{Z}_p, \oplus)$, $(\mathbf{Z}_p - \{0\}, \otimes)$, where p is any integer and $\times$ denotes multiplication, $\oplus$ denotes addition modulo p and $\otimes$ denotes multiplication modulo p. Complete the following table with ticks (for yes) and crosses (for no) to indicate whether the given system is an abelian group.

	$(\mathbf{Z}, \times)$	$(\mathbf{Q} - \{0\}, \times)$	$(\mathbf{Z}_p, \oplus)$	$(\mathbf{Z}_p - \{0\}, \otimes)$
Abelian Group	✗	✓	✓	✗