Solutions to the MATH1061 (Ipswich) Sample Exam

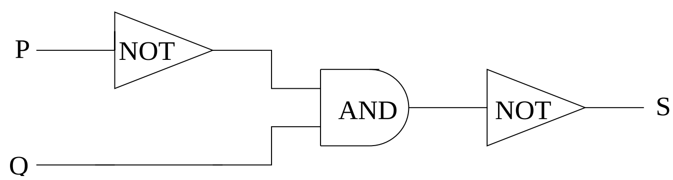
1. (a) Use a truth table to determine whether the following statement form is a tautology, a contradiction or neither.

$$((p \wedge q) \to (p \vee q)) \;\leftrightarrow\; (\sim p \vee \sim q). \qquad \text{(3 marks)}$$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$	$\sim p \vee \sim q$	given statement
T	T	T	T	T	F	F
T	F	F	T	T	T	T
F	T	F	T	T	T	T
F	F	F	F	T	T	T

This statement form is <u>neither a tautology nor a contradiction</u>.

(b) Consider the following digital logic circuit.



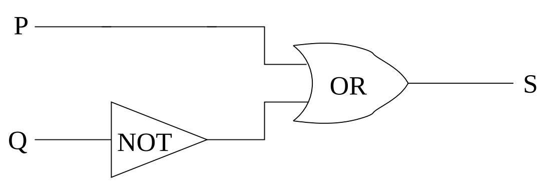
(i) Complete the input/output table for this circuit. (2 marks)

P	Q	S
1	1	1
1	0	1
0	1	0
0	0	1

(ii) Use your input/output table to draw a simpler circuit which is equivalent to the circuit given above.

(2 marks)

The input/output table corresponds to the truth table for $P \vee \sim Q$, so a simpler circuit is:



2. Consider the statement

$$\forall r \in \mathbb{R}, \text{if } r \text{ is irrational, then } r^2 \text{ is irrational.}$$

(a) Write the contrapositive of this statement. (2 marks)

$\forall r \in \mathbb{R}$, if r^2 is not irrational, then r is not irrational.
(or equivalently, $\forall r \in \mathbb{R}$, if r^2 is rational , then r is rational.)

(b) Write the negation of this statement. (2 marks)

$\exists r \in \mathbb{R}$ such that r is irrational and r^2 is not irrational.
(or equivalently, $\exists r \in \mathbb{R}$ such that r is irrational and r^2 is rational.)

(c) State which is true, the original statement or the negation. (1 mark)

The negation is true. (For example, let $r = \sqrt{2}$.)

3. Prove that the following statement is true.

$$\text{For all integers } a \text{ and } b, \text{ if } 4 \mid a \text{ and } 4 \mid b, \text{ then } 4 \mid (a + b).$$

(4 marks)

Proof: Suppose that a and b are integers and that $4 \mid a$ and $4 \mid b$.
Since $4 \mid a$, we know that $a = 4 \cdot m$ for some integer m.
Since $4 \mid b$, we know that $b = 4 \cdot n$ for some integer n.
Thus $a + b = 4m + 4n = 4(m + n)$.
Since $m + n$ is an integer, $a + b = 4 \cdot k$ for some integer k.
Therefore $4 \mid (a + b)$.

4. Use mathematical induction to prove that the following statement is true.

$$\text{For all integers } n \geq 3, \quad n^2 > 2n + 1.$$

(6 marks)

Let $P(n)$ be the statement $n^2 > 2n + 1$, for $n \geq 3$.
$P(3)$ is the statement $3^2 > 2 \cdot 3 + 1$.
$P(k)$ is the statement $k^2 > 2 \cdot k + 1$.
$P(k+1)$ is the statement $(k+1)^2 > 2 \cdot (k+1) + 1$.

$P(3)$ is true since $3^2 = 9$ and $2 \cdot 3 + 1 = 7$ and $9 > 7$.
Assume that $P(k)$ is true and use this to prove that $P(k+1)$ is true.

$$\begin{aligned}
\text{L.H.S. of } P(k+1) &= (k+1)^2 \\
&= k^2 + 2k + 1 \\
&> 2k + 1 + 2k + 1 \quad \text{since } P(k) \text{ is assumed true} \\
&= 2k + 2 + 2k \\
&> 2k + 3 \quad \text{since } k \geq 3 \\
&= \text{R.H.S of } P(k+1)
\end{aligned}$$

So $(k+1)^2 > 2 \cdot (k+1) + 1$.

Thus, by mathematical induction, $n^2 > 2n + 1$ for all integers $n \geq 3$.

5. **(a)** Use the Euclidean Algorithm to find $\gcd(423, 81)$. (2 marks)

$$\begin{aligned} 423 &= 5 \cdot 81 + 18 \\ 81 &= 4 \cdot 18 + 9 \\ 18 &= 2 \cdot 9 + 0 \end{aligned}$$

Thus $\gcd(423, 81) = 9$.

(b) Determine whether or not there exist integers x and y that satisfy the following linear diophantine equation. If so, then find one such pair of integers. If not, explain why not.

$$81x + 423y = 36$$

(2 marks)

From (a) we know that $\gcd(423, 81) = 9$ and since 9 divides 36, a solution exists.

$$\begin{aligned} 9 &= 81 - 4 \cdot 18 \\ &= 81 - 4(423 - 5 \cdot 81) \\ &= 81(21) + 423(-4) \end{aligned}$$

Therefore $81(84) + 423(-16) = 36$, so $x = 84$, $y = -16$ is one solution to this linear diophantine equation.

6. Let $U = \{10, 11, 12, 13, 14, 15\}$ be a universal set with subsets

$$A = \{x \in U \mid x \text{ is even}\} \text{ and } B = \{x \in U \mid x \text{ is divisible by } 3\}.$$

Determine the elements in each of the following sets and write each set in the space provided.

(5 marks)

$A \cap B \quad = \quad \{12\}$

$A \cup B \quad = \quad \{10, 12, 14, 15\}$

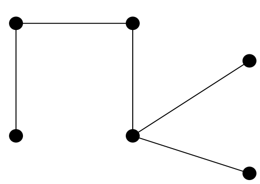
$A - B \quad = \quad \{10, 14\}$

$A^c \quad = \quad \{11, 13, 15\}$

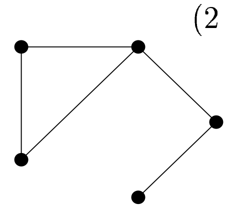
$\mathcal{P}(B) \quad = \quad \{\emptyset, \{12\}, \{15\}, \{12, 15\}\}$

7. Determine whether or not each of the following graphs is a tree. Give a brief justification for your answers.

(2 marks)



(1)

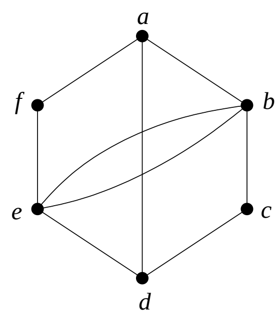


(2)

(1) is a tree. It is connected and contains no circuits. It has 6 vertices and 5 edges.

(2) is not a tree. It contains a circuit. It has 5 vertices and 5 edges (one edge too many to be a tree).

8. Let G be the following graph.



(a) Determine whether or not G has an Euler circuit. If it does, give one such circuit; if it does not, explain why not.

(2 marks)

G does not have an Euler circuit. It has two vertices of odd degree.

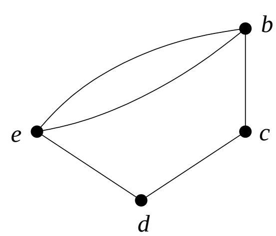
(b) Determine the total degree of G. (2 marks)

Total degree $= 3 + 4 + 2 + 3 + 4 + 2 = 18$.

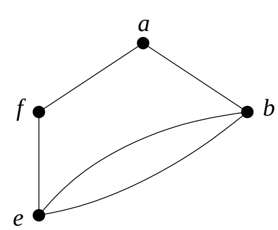
(c) Determine whether or not G has a subgraph that has four vertices and five edges. Justify your answer by drawing the subgraph or by explaining why such a subgraph does not exist.

(2 marks)

G does have a subgraph that has four vertices and five edges.



or



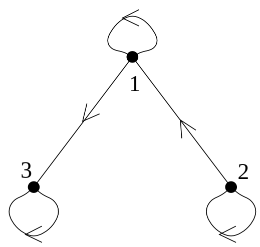
9. Let ρ and σ be relations on the set $A = \{1, 2, 3\}$ defined as follows:

$$\rho = \{(1,3),\ (1,1),\ (2,1),\ (2,2),\ (3,3)\}$$

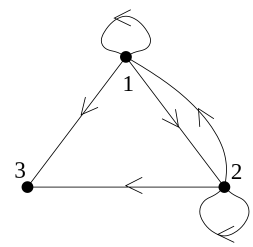
and

$$\sigma = \{(1,2),\ (1,3),\ (2,1),\ (1,1),\ (2,2),\ (2,3)\}.$$

(a) Draw the directed graphs that correspond to ρ and σ. (2 marks)



ρ



σ

(b) Write the elements of the relation σ^{-1} in the box below.

(2 marks)

σ^{-1}	$\{(2,1),\ (3,1),\ (1,2),\ (1,1),\ (2,2),\ (3,2)\}$

9. **(c)** Let ρ and σ be the relations on the set $A = \{1, 2, 3\}$ as defined on the previous page; that is,

$$\rho = \{(1, 3),\ (1, 1),\ (2, 1),\ (2, 2),\ (3, 3)\}$$

and

$$\sigma = \{(1, 2),\ (1, 3),\ (2, 1),\ (1, 1),\ (2, 2),\ (2, 3)\}.$$

For each of the relations ρ and σ, state whether or not it is (i) reflexive, (ii) symmetric, (iii) antisymmetric, (iv) transitive? Write your answer in the appropriate box below, with a brief justification.

(8 marks)

	ρ	σ
(i) reflexive?	Yes $(1, 1), (2, 2), (3, 3) \in \rho$. Loops on all three vertices of the graph.	No $(3, 3) \notin \sigma$ No loop at vertex 3 of the graph.
(ii) symmetric?	No $(2, 1) \in \rho$ but $(1, 2) \notin \rho$. Not all edges are double in the graph.	No $(1, 3) \in \sigma$ but $(3, 1) \notin \sigma$. Not all edges are double in the graph.
(iii) antisymmetric?	Yes For distinct $a, b \in A$, if $(a, b) \in \rho$ then $(b, a) \notin \rho$. No double edges in the graph.	No $(1, 2) \in \sigma$ and $(2, 1) \in \sigma$. There is a double edge in the graph.
(iv) transitive?	No $(2, 1) \in \rho$ and $(1, 3) \in \rho$, but $(2, 3) \notin \rho$.	Yes For all $a, b, c \in A$, if $(a, b) \in \sigma$ and $(b, c) \in \sigma$ then $(a, c) \in \sigma$.

10. Let S be a relation on the set $\mathbb{Z}$ defined as follows

$$a\,S\,b \quad \text{if and only if} \quad a \leq b.$$

(a) Prove that S is an partial order relation. (5 marks)

(1) Show that S is reflexive.
Let $a \in \mathbb{Z}$. Then $a \leq a$, so $a\ S\ a$.
Therefore S is reflexive.

(2) Show that S is antisymmetric.
Let a, b be distinct integers and suppose that $a\ S\ b$. Since $a\ S\ b$ and $a \neq b$, we have that $a < b$. Hence $b \not\leq a$, so b is not related to a.
Therefore S is antisymmetric.

(3) Show that S is transitive.
Let $a, b, c \in \mathbb{Z}$ and suppose that $a\ S\ b$ and $b\ S\ c$. Then $a \leq b$ and $b \leq c$. Thus $a \leq c$, so $a\ S\ c$.
Therefore S is transitive.

Since S is reflexive, antisymmetric and transitive, S is a partial order relation.

(b) Explain why S is a total order relation. (2 marks)

For every pair of integers a and b, either $a \leq b$ or $b \leq a$, so every integer is related to every other integer. Therefore S is a total order relation.

11. Let f and g be functions defined as

$$f : \mathbb{R} \to \mathbb{R} \text{ such that } f(x) = 3x + 6,$$
$$g : \mathbb{R} \to \mathbb{R} \text{ such that } g(x) = x^2 - 2.$$

(a) Is the function f one-to-one and/or onto? Justify your answer. (5 marks)

f is one-to-one.
Suppose that x_1 and x_2 are real numbers and that $f(x_1) = f(x_2)$.
Then $3x_1 + 6 = 3x_2 + 6$, so $3x_1 = 3x_2$ and $x_1 = x_2$.

f is onto.
Suppose that y is a real number.
There exists an $x \in \mathbb{R}$ such that $f(x) = y$, namely $x = \dfrac{y-6}{3}$.

(b) Is the function g one-to-one and/or onto? Justify your answer. (5 marks)

g is not one-to-one.
$g(-3) = 9 - 2 = 7$ and $g(3) = 9 - 2 = 7$ but $-3 \neq 3$.

g is not onto.
$-4 \in \mathbb{R}$ but there is no $x \in \mathbb{R}$ such that $g(x) = -4$.
(If $x^2 - 2 = -4$ then $x^2 = -2$ and x would not be a real number.)

(c) Calculate $(g \circ f)(x)$ and $(f \circ g)(x)$. (4 marks)

$(g \circ f)(x) = g(f(x)) = g(3x + 6) = (3x + 6)^2 - 2 = 9x^2 + 36x + 34.$

$(f \circ g)(x) = f(g(x)) = f(x^2 - 2) = 3(x^2 - 2) + 6 = 3x^2.$

12. **(a)** Consider the systems $(\mathbb{Z}_7 - \{0\}, \odot)$, $(\mathbb{Z}_6 - \{0\}, \otimes)$, $(\mathbb{Z}_n, \oplus)$, $(\mathbb{R}, \times)$. Complete the following table with ticks (for yes) and crosses (for no) to indicate whether the given system is a group. Note that $\odot$ denotes multiplication modulo 7, $\otimes$ denotes multiplication modulo 6, $\oplus$ denotes addition modulo n and $\times$ denotes multiplication.

(4 marks)

	$(\mathbb{Z}_7 - \{0\}, \odot)$	$(\mathbb{Z}_6 - \{0\}, \otimes)$	$(\mathbb{Z}_n, \oplus)$	$(\mathbb{R}, \times)$
Group?	$\surd$	$\times$	$\surd$	$\times$

(b) For each system in part (a) which is not a group give a brief explanation of why it is not a group.

(3 marks)

$(\mathbb{Z}_6 - \{0\}, \otimes)$ is not a group because it is not closed.
$2 \otimes 3 = 6 = 0$ which is not in $\mathbb{Z}_6 - \{0\}$.

$(\mathbb{R}, \times)$ is not a group because $0 \in \mathbb{R}$ and 0 does not have an inverse.
The element 1 would be the identity in this system and there does not exist an element $r \in \mathbb{R}$ such that $0 \times r = 1$.

13. **(a)** Construct the Cayley table for the group $(\mathbb{Z}_5 - \{0\}, \odot)$ where $\odot$ is the binary operation of multiplication modulo 5.

(3 marks)

$\odot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(b) Is $(\mathbb{Z}_5, \oplus, \odot)$ a field? Recall that $\oplus$ is the binary operation of addition modulo 5 and $\odot$ is the binary operation of multiplication modulo 5. Justify your answer.

(3 marks)

Yes $(\mathbb{Z}_5, \oplus, \odot)$ is a field.

$(\mathbb{Z}_5, \oplus)$ is an abelian group since
it is closed,
$\oplus$ is associative and commutative,
0 is the identity,
and each element has an inverse: $1^{-1} = 4$, $2^{-1} = 3$ and $0^{-1} = 0$.

$(\mathbb{Z}_5 - \{0\}, \odot)$ is an abelian group since
it is closed,
$\odot$ is associative and commutative,
1 is the identity,
and each element has an inverse: $1^{-1} = 1$, $2^{-1} = 3$ and $4^{-1} = 4$.

Modular multiplication distributes over modular addition.

14. (a) How many distinct arrangements can be made from the letters of the word MISSISSIPPI?

(2 marks)

There are 11 letters: 1M, 4I, 4S, 2P.

The number of distinct arrangements is

$$\frac{11!}{1!\ 4!\ 4!\ 2!} = 34\ 650.$$

(b) How many distinct arrangements can be made from the letters of the word MISSISSIPPI that begin with an S and end with an M?

(2 marks)

If we start with an S and end with an M, we are effectively arranging 9 letters: 4I, 3S, 2P.

The number of distinct arrangements is

$$\frac{9!}{4!\ 3!\ 2!} = 1260.$$

(c) How many distinct arrangements can be made from the letters of the word MISSISSIPPI which contain the two letters PM next to each other in the given order?

(2 marks)

We are effectively arranging 10 letters: 1PM, 1P, 4I, 4S.

The number of distinct arrangements is

$$\frac{10!}{1!\ 1!\ 4!\ 4!} = 6300.$$

15. **(a)** Evaluate $\frac{5 \times \binom{9}{4}}{\binom{8}{4}}$. Show your working. (2 marks)

$$5 \times \frac{9!}{4!\ 5!} \times \frac{4!\ 4!}{8!} = 5 \times \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} \times \frac{4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5} = \frac{5 \times 9}{5} = 9.$$

(b) A regular 6-sided die is rolled five times. In each case the result (1, 2, 3, 4, 5 or 6) is recorded. The order of the five results is also recorded. For example, one possible outcome is 3 4 6 1 3.

(i) What is the total number of possible outcomes of this die-rolling experiment?

(1 mark)

There are 6 possible results for each roll, so $6^5 = 7776$ possible outcomes.

(ii) In how many of the possible outcomes is the first result a three and exactly two threes are obtained in total?

(2 marks)

The possible outcomes (where $*$ is one of 1, 2, 4, 5 or 6) are:
$3\,3 * * *$, $3 * 3 * *$, $3 * * 3 *$, and $3 * * * 3$.

There are 5^3 possiblities for each of these four cases, so the total number of possible outcomes of this type is $4 \cdot 5^3 = 500$.

(iii) In how many of the possible outcomes is the first result a three and at least two threes are obtained in total?

(2 marks)

This is equal to the number of outcomes in which the first result is a 3 minus the number of outcomes in which the first 3 is the only 3. This is $6^4 - 5^4 = 671$.

This could also be calculated as those outcomes in which the first result is a three and exactly 2 3s are obtained plus exactly 3 3s are obtained plus exactly 4 3s are obtained plus exactly 5 3s are obtained. This is $4 \cdot 5^3 + 6 \cdot 5^2 + 4 \cdot 5 + 1 = 671$.

(iv) What is the probability of obtaining an outcome in which there are no fives or sixes?

(2 marks)

The number of outcomes with no fives or sixes is $4^5 = 1024$.

The probability of obtaining an outcome that has no fives or sixes is

$$\frac{1024}{7776} = \frac{32}{243}.$$