6. (Hint: Note that all integers are also real numbers. To investigate floors and ceilings, first consider the case where x is an integer, and then consider the case where x is not an integer, so $x = k + d$ where k is an integer and $0 < d < 1$.)

(a) Either prove that the following statement is true, or else give a counterexample to show that it is false.

$$\forall x \in \mathbb{R}, \quad \lfloor x \rfloor + \lceil x \rceil = \lceil x - 1 \rceil + \lfloor x + 1 \rfloor$$

(6 marks)

This statement is true.

<u>Proof</u>

Suppose x is a real number. Then x is either an integer or not an integer.

If x is an integer, then

$\lfloor x \rfloor + \lceil x \rceil = x + x = 2x,$

and $\lceil x-1 \rceil + \lfloor x+1 \rfloor = x - 1 + x + 1 = 2x.$

$\therefore \lfloor x \rfloor + \lceil x \rceil = \lceil x-1 \rceil + \lfloor x+1 \rfloor$ in this case.

If x is not an integer, then $x = k + d$ where k is an integer and $0 < d < 1$.

Then

$\lfloor x \rfloor + \lceil x \rceil = \lfloor k+d \rfloor + \lceil k+d \rceil = k + k+1 = 2k+1,$

and

$\lceil x-1 \rceil + \lfloor x+1 \rfloor = \lceil k+d-1 \rceil + \lfloor k+d+1 \rfloor = k + k+1 = 2k+1.$

$\therefore \lfloor x \rfloor + \lceil x \rceil = \lceil x-1 \rceil + \lfloor x+1 \rfloor$ in this case.

All cases have been considered, so

$\forall x \in \mathbb{R}, \lfloor x \rfloor + \lceil x \rceil = \lceil x-1 \rceil + \lfloor x+1 \rfloor.$