(b) Either prove that the following statement is true, or else give a counter-example to show that it is false.

$$\forall x \in \mathbb{R}, \quad \lfloor x \rfloor \cdot \lceil x \rceil = \lceil x-1 \rceil \cdot \lfloor x+1 \rfloor$$

This statement is false. (3 marks)

Consider $x = 1$.

Then $\lfloor x \rfloor \cdot \lceil x \rceil = 1 \cdot 1 = 1$

and $\lceil x-1 \rceil \cdot \lfloor x+1 \rfloor = 0 \cdot 2 = 0$.

So, for $x = 1$,

$\lfloor x \rfloor \cdot \lceil x \rceil \neq \lceil x-1 \rceil \cdot \lfloor x+1 \rfloor$.

7. Use a proof by contradiction to prove the following statement.

The sum of any rational number and any irrational number is irrational.

Original statement $\forall r, s \in \mathbb{R}$, if r is rational and s is irrational, then $r+s$ is irrational. (7 marks)

Proof

Suppose not.

Suppose there exist real numbers r and s such that r is rational, s is irrational and $r+s$ is rational.

Since $r \in \mathbb{Q}$, $r = \frac{a}{b}$ for some integers a, b with $b \neq 0$.

Since $r+s \in \mathbb{Q}$, $r+s = \frac{c}{d}$ for some integers c, d with $d \neq 0$.

Now $r+s = \frac{c}{d}$, so $\frac{a}{b} + s = \frac{c}{d}$.

Thus $s = \frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{bd}$.

Since a, b, c, d are integers and $b \neq 0$, $d \neq 0$, we know that $cb-ad$ is an integer and bd is an integer and $bd \neq 0$. Thus s is rational.

This is a contradiction, so the negation cannot be true.

$\therefore$ The original statement is true.