1. (a)(i) \((e^x \tan x)' = (e^x)' \tan x + e^x (\tan x)' = e^x \tan x + e^x \sec^2 x\),
(ii) \((x^5 + \sin (x^2))' = (x^5)' + (\sin (x^2))' = 5x^4 + \frac{d}{dx}(x^2) = 5x^4 + \cos x^2 \cdot 2x\).

(b) (i) \(\lim_{x \to 1} \frac{x^2 + 2x + 1}{x + 1} = \lim_{x \to 1} \frac{(x+1)(x+1)}{x+1} = 2\),
(ii) \(\lim_{x \to 0} \frac{1}{x^2} - \frac{2}{x+4} = \lim_{x \to 0} \frac{1}{x} \frac{\frac{(x+4)-2(x+2)}{(x+2)(x+4)}}{\frac{1}{x}} = \lim_{x \to 0} \frac{-1}{(x+2)(x+4)} = \frac{-1}{(0+2)(0+4)} = -1/8\).

(c) (i) Since \(\lim_{x \to \infty} x = \infty = \lim_{x \to \infty} e^x\) we use L’Hopital’s rule so
\[\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{\frac{dx}{dx}}{\frac{d}{dx}e^x} = \lim_{x \to \infty} e^{-x} = 0,\]
(ii) Since \(\lim_{x \to 0} (1 - \cos x) = 0 = \lim_{x \to 0} x\) use L’Hopital’s rule so \(\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}x} = \lim_{x \to 0} \frac{\sin x}{1} = \sin 0 = 0\).

(d) Use Mathematical Induction to prove that \(1 + 2 + \cdots + n = \frac{n(n+1)}{2}\).
See lecture notes for the solutions. Clearly, in an exam, you will receive no marks if you say in answer to a question: ”See lecture notes or see tutorial sheets for the solution”.

2. (a) Show that \(A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\) satisfies \(A^2 - 4A + 3I = 0\). Hence deduce that \(A\) is invertible, with inverse \(A^{-1} = \frac{1}{3}(4I - A)\). See solutions to Problem Sheet 3 Question 2

2. (b) Let \(C = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & -1 \end{pmatrix}\). Show by direct multiplication that \(C^{-1} = \begin{pmatrix} -7 & 5 & 3 \\ 3 & -2 & -1 \\ 2 & -1 & -1 \end{pmatrix}\). Using (i) above, solve the system of equations
\[\begin{align*}
x + 2y + z &= -1, \\
x + y + 2z &= -1, \\
x + 3y - z &= 2.
\end{align*}\]
See solutions to Problem Sheet 3 Question 1
3. (a) Determine for which values of $x$ 
\[
\begin{vmatrix}
1 & 1 & 4 \\
2x & 1 & 1 \\
x^2 & x & 1
\end{vmatrix}
= 0.
\]
\[
\begin{vmatrix}
1 & 1 & 4 \\
2 & 1 & 1 \\
x & x & 1
\end{vmatrix}
= x\begin{vmatrix}
1 & 1 & 4 \\
2 & 1 & 1 \\
1 & 1 & 1
\end{vmatrix}
= 0 \text{ for all } x \text{ since row } 2 = \text{row } 3.
\]

3. (b) In 3-dimensional space, let $A, B, C$ be the points $A = (0, 1, 2), B = (2, 3, 2), C = (0, 5, 1)$. Show that $ABC$ is a right angled triangle. Find the area of triangle $ABC$ and a vector perpendicular to the plane containing it.

Now $\vec{AB} = \vec{OB} - \vec{OA} = (2, 3, 2) - (0, 1, 2) = (2, 2, 0)$ and $\vec{BC} = \vec{OC} - \vec{OB} = (0, 5, 1) - (2, 3, 2) = (-2, 2, -1)$. Thus $\vec{AB} \neq \vec{0} \neq \vec{BC}$ and $\vec{AB} \cdot \vec{BC} = (2, 2, 0) \cdot (-2, 2, -1) = 2(-2) + 2(2) + 0(-1) = 0$. Thus $\vec{AB}$ perpendicular to $\vec{BC}$ and $ABC$ is a right angled triangle.

Now $\vec{AB} \times \vec{BC} = (2, 2, 0) \times (-2, 2, -1) = \begin{vmatrix}
i & j & k \\
2 & 2 & 0 \\
-2 & 2 & -1
\end{vmatrix} = -2i + 2j + 8k$ is perpendicular to the plane of vectors $\vec{AB}$ and $\vec{BC}$ and hence to the plane of triangle $ABC$. Moreover, area of triangle $ABC = \frac{1}{2} |\vec{AB} \times \vec{BC}| = \sqrt{18} = 3\sqrt{2}$.

3. (c) An aeroplane points N45°W travelling at 300 km per hour. If the wind is travelling E45°N at 300 km per hour what is the velocity of the aeroplane relative to the ground?

The unit vector in the direction N45°W is $\frac{1}{\sqrt{2}}j - \frac{1}{\sqrt{2}}i$, while the unit vector in the direction E45°N is $\frac{1}{\sqrt{2}}j + \frac{1}{\sqrt{2}}i$. Thus the velocity of the plane relative to the wind,

\[
V_p \text{ rel } w = \frac{300}{\sqrt{2}}j - \frac{300}{\sqrt{2}}i,
\]

while the velocity of the wind relative to the ground

\[
V_w \text{ rel } G = \frac{300}{\sqrt{2}}j + \frac{300}{\sqrt{2}}i.
\]

Now the velocity of the plane relative to the ground,

\[
V_p \text{ rel } G = V_p \text{ rel } w + V_w \text{ rel } G = \frac{300}{\sqrt{2}}j - \frac{300}{\sqrt{2}}i + \frac{300}{\sqrt{2}}j + \frac{300}{\sqrt{2}}i = 600\sqrt{2}j.
\]

4. (a) A box with a square base and open top must have a volume of 32,000 cm$^3$. Find the dimensions of the box that minimise the amount of material used.

See the solutions of Problem Sheet 7, Question 6.

4. (b) Evaluate

\[
\sum_{n=1}^{\infty} \left[ \left( \frac{x}{3} \right)^n - 4 \left( \frac{2}{x} \right)^n \right],
\]

for those $x$ for which the sum converges.

\[
\sum_{n=1}^{\infty} at^{n-1} = \frac{a}{1 - t} \text{ for } -1 < t < 1
\]
and does not exist for other values of \( t \).

\[
\sum_{n=1}^{\infty} \left( \frac{x}{3} \right)^n = \sum_{n=1}^{\infty} at^n \quad \text{where} \quad t = \frac{x}{3} = a
\]

\[
\sum_{n=1}^{\infty} 4 \left( \frac{2}{x} \right)^n = \sum_{n=1}^{\infty} at^n \quad \text{where} \quad t = \frac{2}{x} \quad \text{and} \quad a = \frac{2}{x}.
\]

Thus

\[
\sum_{n=1}^{\infty} \left[ \left( \frac{x}{3} \right)^n - 4 \left( \frac{2}{x} \right)^n \right] = \sum_{n=1}^{\infty} \left( \frac{x}{3} \right)^n - \sum_{n=1}^{\infty} 4 \left( \frac{2}{x} \right)^n = \frac{x}{3} - \frac{4^2}{x}.
\]

when both \(-1 < \frac{x}{3} < 1\) and \(-1 < \frac{2}{x} < 1\). These both hold when \(-3 < x < -2\) and when \(2 < x < 3\).

4. (c) Suppose that

\[
f(x) = \begin{cases} 
  x^2 + 4cx + c & \text{if } x \leq 2 \\
  3cx + 3 & \text{if } x > 2.
\end{cases}
\]

Find the values of the constant \( c \) such that \( f \) is continuous at \( x = 2 \).

Need

\[
\lim_{x \to 2} f(x) = f(2) = 2^2 + 4c2 + c
\]

\[
eq \lim_{x \to 2^-} f(x) = 2^2 + 4c2 + c = \lim_{x \to 2^+} f(x)
\]

\[
eq \lim_{x \to 2^-} (x^2 + 4cx + c) = 2^2 + 4c2 + c = \lim_{x \to 2^+} (3cx + 3)
\]

\[
eq 2^2 + 4c2 + c = 2^2 + 4c2 + c = 3c2 + 3
\]

\[
\equiv 4 + 9c = 6c + 3
\]

\[
\equiv c = -1/3.
\]

5. (a) A street light is at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path.

(a) How fast is the tip of his shadow moving when he is 40 ft from the pole?

(b) How fast is his shadowing lengthening at that point?

See the solutions of Problem Sheet 7, Question 2.

5. (b) Let \( c \) be a constant. For the function

\[
y = f(x) = x^3 + \frac{9x^2}{2} + 6x + c,
\]

find:

(1) the intervals on which the function is increasing;

(2) the intervals on which the function is decreasing;

(3) the critical points.

\[
\frac{d}{dx}y = \frac{d}{dx} \left( x^3 + \frac{9x^2}{2} + 6x + c \right)
\]

\[
= 3x^2 + 9x + 6 = 3(x + 2)(x + 1)
\]

\[
eq 0 \text{ when } x = -2 \text{ or } x = -1
\]
Since \( f(x) \) is differentiable for all \( x \), the critical points are \( x = -2 \) and \( x = -1 \). Also \( \frac{d}{dx} y = 3(x + 2)(x + 1) > 0 \) when either \( x < -2 \) or \( x > -1 \). Thus \( f(x) \) is increasing when \( x \leq -2 \) and when \( x \geq -1 \). Moreover \( \frac{d}{dx} y = 3(x + 2)(x + 1) < 0 \) when \( -2 < x < -1 \) so that \( f(x) \) is decreasing when \( -2 \leq x \leq -1 \).

5. (c) Show that \( x \geq \sin x \), for all \( x \geq 0 \).

See lecture notes.

6. (a) If \( G(x) = \int_x^{e^x} \frac{1}{1+t^4} dt \), find \( G'(x) \).

**Hint:** \( G(x) = F(e^x) - F(x) \) where \( F(x) = \int_0^x \frac{1}{1+t^4} dt \).

Now \( G(x) = F(e^x) - F(x) \) where \( F(x) = \int_0^x \frac{1}{1+t^4} dt \) so that \( \frac{d}{dx} F(x) = \frac{1}{1+x^4} \), by the Fundamental Theorem. Thus

\[
\frac{d}{dx} G(x) = \frac{d}{dx} (F(e^x) - F(x))
\]

\[
= \frac{d}{dx} F(e^x) - \frac{d}{dx} (F(x))
\]

\[
= \frac{d}{du} F(u) \frac{du}{dx} - \frac{1}{1+u^4} \text{ by the Chain Rule with } u = e^x
\]

\[
= \frac{1}{1+u^4} e^x - \frac{1}{1+u^4} = \frac{1}{1+e^4x} e^x - \frac{1}{1+e^4x}.
\]

6. (b) Evaluate the following integrals:

   (i) \( \int 3x^2 \sqrt{2+x^3} dx \); (ii) \( \int xe^x dx \).

(i) The substitution rule is

\[
\int f(u) \frac{du}{dx} dx = \int f(u) du
\]

where \( u = g(x) \) has a continuous derivative and \( f(u) \) is continuous. Here we subst. \( u = 2 + x^3 \)

\[
\Rightarrow du = \frac{du}{dx} dx = 3x^2dx.
\]

Now

\[
\Rightarrow \int 3x^2 \sqrt{2+x^3} dx = \int \sqrt{u} du
\]

\[
= \frac{2}{3} u^{3/2} + c = \frac{2}{3} (2 + x^3)^{3/2} + c.
\]

(ii) The integration by parts formula is

\[
\int uv' dx = uv - \int vu' dx
\]

Consider \( \int xe^x dx \) Put \( u = x, v' = e^x \Rightarrow u' = 1, v = e^x \)

\[
\Rightarrow \int xe^x dx = uv - \int vu' dx
\]

\[
= xe^x - \int e^x dx
\]

\[
= xe^x - e^x + C.
\]