NOTATION: If $X$ is a Hilbert or a Banach space then $\Omega \subseteq X$ denotes a bounded open set and $B_r$ the sphere $\{x \in X: \|x\| < r\}$. If $S \subseteq X$ then $\partial S$ denotes its boundary and $\overline{S}$ denotes its closure.

If $X$ is a Hilbert space and $x, y \in X$ let $\langle x, y \rangle$ denote the usual inner product. Thus $\|x\|^2 = \langle x, x \rangle$. Moreover, if $X = \mathbb{R}^n$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ where $x_i, y_i \in \mathbb{R}$, $i \leq i \leq n$, then $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$.

1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ and $\operatorname{det} f'(x) \neq 0$ for all $x$.

(a) Let $p \in \mathbb{R}^n$ satisfy $p \notin f(\partial \Omega)$. Give the weighted sum formula for the Brouwer degree $d(f, \Omega, p)$.

If $d(f, \Omega, p) = m$ show that $f(x) = p$ has at least $m$ solutions $x \in \Omega$.

Give an example of a function $g$ satisfying $g(0) = 0$, $\operatorname{det} g'(0) = 0$, $g(x) \neq 0$ for all $x \neq 0$ and $d(g, \Omega, 0) = 2$. [9]

(b) If $\Omega$ is symmetric, $0 \in \Omega$ and $f$ is an odd function (that is, $f(x) = -f(-x)$ for all $x$) satisfying $0 \notin f(\partial \Omega)$, show that $d(f, \Omega, 0)$ is an odd number. [7]

(c) Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by

$$F(x, y) = (x^2 - y^2 - x, -y + 2yx).$$

If $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$ find $d(F, \Omega, (0, 1))$. [5]

2. (a) List the basic axioms of Brouwer degree. If $f \in C(\overline{B}_1; B_1)$ use these properties to show $f$ has a fixed point. [10]
2. (b) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, $x_0 \in \Omega$, and $f \in C(\overline{\Omega}; \mathbb{R}^n)$. If $f(x) \neq \lambda x + (1 - \lambda)x_0$ for all $x \in \partial \Omega$ and $\lambda > 1$, show that $f$ has a fixed point in $\overline{\Omega}$. [7]

(Hint: Consider the homotopy $H(t, x) = (t - 1)x_0 + x - tf(x).$)

(c) Let $f \in C(\mathbb{R}^n; \mathbb{R}^n)$. If $\langle x, f(x) \rangle \geq \|x\|^2$ for all $x \in \mathbb{R}^n$ show that $f(\mathbb{R}^n) = \mathbb{R}^n$. [10]

3. (a) Define what is meant by the following:
   A normed space $(N, \|\cdot\|)$;
   A Banach space $(B, \|\cdot\|)$;
   A continuous function $f : B \rightarrow B$;
   A bounded linear mapping $L : B \rightarrow B$;
   An open covering and a compact set.
   [10]

(b) Prove that a linear mapping $L : B \rightarrow B$ is continuous if and only if it is bounded. [7]

(c) Prove that a compact set is totally bounded. [5]

(d) Let $(B, \|\cdot\|)$ be a Banach space and $K \subseteq B$ be a compact subset of $B$. Given $\epsilon > 0$ prove that there is a mapping $S_\epsilon : K \rightarrow B$ such that $\|x - S_\epsilon(x)\| < \epsilon$ for all $x \in K$. [10]

4. (a) Let $B$ be a Banach space, $\Omega \subseteq B$ be a bounded open subset, $0 \in \Omega$, and $f \in C(\overline{\Omega}; B)$ satisfy $f(\overline{\Omega}) \subseteq K$ for some compact set $K$. If $f(x) \neq \lambda x$ for all $x \in \partial \Omega$ and $\lambda < 1$, show that $f$ has a fixed point in $\overline{\Omega}$. [9]

(b) Let $H$ be a Hilbert space, $\Omega \subseteq H$ be a bounded open subset, $0 \in \Omega$ and $f \in C(\overline{\Omega}; H)$ satisfy $f(\overline{\Omega}) \subseteq K$ for some compact set $K$. If

$$ (f(x), x) \leq \|x\|^2 $$

for all $x \in \partial \Omega$, show that $f$ has a fixed point in $\overline{\Omega}$. [9]

(Hint: You may assume without proof that $d(I, \Omega, 0) = 1$, where $I(x) = x$ for all $x$ is the identity mapping.)

Questions 5–6 see next page

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5. (a) State Schauder’s Fixed Point Theorem. \[3\]

(b) Let \( f: [0, 1] \times \mathbb{R} \to \mathbb{R} \) be continuous and assume that \( |f(x, y)| \leq k \) for all \((x, y) \in [0, 1] \times \mathbb{R}\) and some constant \(k\). Show that there is a solution \(y \in C([0, 1])\) of
\[
y(x) = \int_0^x f(t, y(t)) \, dt + \sin x
\]

(c) Let \( B \) be a Banach Space and \( F: B \to B \) be completely continuous. Show that either there is \(x \in B\) such that \(x = F(x)\) or the set \( \{x \in B: x = \lambda F(x) \text{ some } 0 < \lambda < 1\} \) is unbounded. \[9\]

6. (a) Let \( \Omega \subset \mathbb{R}^n \) be bounded open, \( f, g: \overline{\Omega} \to \mathbb{R}^n \) be continuous, and \( d(f, \Omega, 0) = 0 \neq d(g, \Omega, 0) \).

Show that \( f(x) = \lambda g(x) \), has solutions \((\lambda_1, x_1)\) and \((\lambda_2, x_2)\) with \( \lambda_1 < 0 < \lambda_2 \) and \( x_1, x_2 \in \partial \Omega \). \[8\]

(b) Let \( K \subset \mathbb{R}^n \) be a cone and \( \Omega \subset \mathbb{R}^n \) be bounded open with \( 0 \in \Omega \). Let \( f: \mathbb{R}^n \to K \setminus \{0\} \) be continuous. Show there is \( \lambda > 0 \) and \( x \in \partial \Omega \cap K \) such that \( f(x) = \lambda x \). \[8\]

(Hint: Compute \( d(f, \Omega, 0) \) and \( d(g, \Omega, 0) \), where \( g(x) = x \) for all \( x \).)

(c) Let \( \Omega \subset \mathbb{R}^n \) be bounded open, \( 0 \in \Omega \) and \( V \subset \mathbb{R}^n \) be a proper subspace. If \( f: \mathbb{R}^n \to V \) is continuous and \( 0 \notin f(\partial \Omega) \), show that there is \( x \in \partial \Omega \cap V \) and \( \lambda \geq 0 \) such that \( f(x) = \lambda x \). \[9\]
7. (a) State and prove the Banach Contraction Mapping Principle. 

(b) Show that there is a unique function \( y \in C[0,1] \) such that

\[
y(x) = 1 + \int_0^1 \frac{|x - t|}{3 + |x - t|} \frac{y(t)}{2 + y^2(t)} \, dt
\]

for all \( x \in [0,1] \).

You may use without proof the fact that if \( u \in C[0,1] \) then

\[
\int_0^1 \frac{|x - t|}{3 + |x - t|} \frac{u(t)}{2 + u^2(t)} \, dt
\]

is a continuous function of \( x \).