1. Contraction Mapping Principle

Let \((X, \|\cdot\|)\) be a Banach Space.

**Definitions:**

1. Let \(A : X \to X\). If there exists \(k > 0\) such that
   \[
   \|Ax - Ay\| \leq k\|x - y\| \quad \forall x, y \in X
   \]
   then \(A\) is called a **Lipschitz** mapping.\(^1\)

2. The smallest \(k\) for which (1) holds is called the **Lipschitz constant** of \(A\).

**Lemma 1.1.** A Lipschitz mapping is uniformly continuous.

**Proof:** Let \(\delta(\epsilon) = \epsilon/(k + 1)\) so
\[
\|Ax - Ay\| \leq k\|x - y\| < k\frac{\epsilon}{k + 1} \leq \epsilon \quad \text{when} \quad \|x - y\| < \delta.
\]

**Definitions:**

1. \(A\) is called a **contraction** if it is Lipschitz with constant \(\lambda < 1\).

2. \(\bar{x} \in X\) is called a fixed point of a map \(A : X \to X\) if \(A\bar{x} = \bar{x}\).

**Remark:** A mapping \(A : X \to X\) need not have a fixed point.

**Example:** If \(X = (0, 1)\) and \(Ax = x/2\),
\[
Ax = x \implies \frac{x}{2} = x
\]
which has no solution in \((0, 1)\).

**Note:**
\[
|Ax - Ay| = \left| \frac{1}{2}x - \frac{1}{2}y \right| = \frac{1}{2}|x - y|
\]
so \(A\) is a contraction with \(\lambda = \frac{1}{2}\).

**Theorem 1.2** (The Contraction Mapping Principle). In a Banach space \((X, \|\cdot\|)\) a contraction mapping \(A : X \to X\) has a unique fixed point \(\bar{x}\). Moreover if \(x_0 \in X\) and \(x_{n+1} = Ax_n \ \forall n = 0, 1, \ldots\) then \(\lim x_n = \bar{x}\) and
\[
\|x_n - \bar{x}\| \leq \frac{\lambda^n}{1 - \lambda} \|x_1 - x_0\|
\]
where \(\lambda\) is the Lipschitz constant of \(A\).

**Proof.** We claim that
\[
\|x_{n+1} - x_n\| \leq \lambda^n\|x_1 - x_0\|
\]
—obviously true for \(n = 0\). We use induction. Assume (2) is true for \(n = k\) and prove it for \(n = k + 1\). Now
\[
\|x_{(k+1)+1} - x_{(k+1)}\| = \|Ax_{k+1} - Ax_k\| \\
\leq \lambda\lambda^k\|x_1 - x_0\| \\
= \lambda^{k+1}\|x_1 - x_0\|
\]

\(^1\)Here it is convenient to write \(Ax\) for \(A(x)\), etc.
as required.

Now claim that \( \{x_t\} \) is a Cauchy sequence so \( \lim_{t \to \infty} x_t = \bar{x} \), say. In fact, using the triangle inequality

\[
\|x_{n+m} - x_n\| \leq \left\| x_{n} - x_{n+1} \right\| + \left\| x_{n+1} - x_{n+2} \right\| + \cdots + \left\| x_{n+m-1} - x_{n+m} \right\|
= \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\|
\leq \sum_{i=n}^{\infty} \lambda^i \|x_1 - x_0\|
= \frac{\lambda^n}{1 - \lambda} \|x_1 - x_0\|.
\] (3)

As \( 0 \leq \lambda < 1 \), \( \lambda \) constant, \( \lambda^n \to 0 \) as \( n \to \infty \) so \( \{x_t\} \) is Cauchy.

We now claim that \( A\bar{x} = \bar{x} \); that is, \( \|A\bar{x} - \bar{x}\| = 0 \). To see this,

\[
\|A\bar{x} - \bar{x}\| \leq \left\| \bar{x} - x_{n+1} \right\| + \|x_{n+1} - A\bar{x}\|
= \left\| \bar{x} - x_{n+1} \right\| + \|Ax_n - A\bar{x}\|
\leq \left\| \bar{x} - x_{n+1} \right\| + \lambda \|x_n - \bar{x}\|
\to 0 \quad \text{as} \ n \to \infty
\]

but \( \|A\bar{x} - \bar{x}\| \) is constant so \( \|A\bar{x} - \bar{x}\| = 0 \).

We now show that \( \bar{x} \) is unique. So let \( Ay = y \):

\[
\|\bar{x} - y\| = \|A\bar{x} - Ay\| \leq \lambda \|\bar{x} - y\|
\implies 0 \leq (1 - \lambda) \|\bar{x} - y\| \leq 0 \quad \text{as} \ 1 - \lambda > 0 \ \text{and} \ \|\bar{x} - y\| \geq 0
\implies \|\bar{x} - y\| = 0 \quad \text{or} \ \bar{x} = y.
\]

Finally, letting \( m \to \infty \) in (3), as \( \|x - x_n\| \) is a continuous function of \( x \) for fixed \( x_n \), we have

\[
\|\bar{x} - x_n\| \leq \frac{\lambda^n}{1 - \lambda} \|x_1 - x_0\|
\]
as required. \( \square \)

Remarks: An examination of the proof reveals that we do not need \( A \) defined on the whole of \( X \). If \( C \subseteq X \) is a closed subset and \( A : C \to C \) is a contraction mapping on \( C \) then the proof above still applies and we get a unique fixed point in \( C \) provided we start with \( x_0 \in C \). To see this we need only observe that Cauchy sequences \( x_n \) in \( C \) converge to some \( \bar{x} \in C \). Moreover to see this it suffices to notice that a Cauchy sequence \( x_n \) in \( C \) is a Cauchy sequence in \( B \) and hence converges to \( \bar{x} \in B \) and since \( C \) is closed and \( x_n \in C \) it follows that \( \bar{x} \in C \).

If we relax the condition \( \|Ax - Ay\| \leq \lambda \|x - y\| \) to

\[
\|Ax - Ay\| < \|x - y\| \quad \forall x, y \in X \text{ where } x \neq y
\]
and \( X, \|\cdot\| \) is a Banach space, then there need not exist a fixed point.

Example: \( X = [2, \infty) \) with norm \(|\cdot|\) the absolute value and \( Ax = x + 1/x \). Then
Ax = x \implies 1/x = 0 which is impossible. Let x > y, so

\[ |Ax - Ay| = (x - y) + \frac{y - x}{xy}, \quad \text{but } xy \geq 4 \]

\[ \implies |Ax - Ay| < x - y = |x - y|. \]

If there is a fixed point in this case, it is unique since otherwise

\[ \|\bar{x} - y\| < \|\bar{x} - y\| \] by the uniqueness argument in the theorem.

**Application 1:** Consider

(4) \[ y' = f(x, y) \quad \text{and} \quad y(0) = B \in \mathbb{R} \]

where \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous and

\[ |f(x, y) - f(x, z)| \leq l|y - z| \quad \forall y, z \in \mathbb{R}, \text{ some constant } l > 0. \]

Then \( \exists \epsilon > 0 \) such that (4) has a unique solution for \( |x| \leq \epsilon \).

(By a solution to (4) on \([-\epsilon, \epsilon]\) is meant a continuously differentiable function \( y : [-\epsilon, \epsilon] \to \mathbb{R} \) satisfying (4) on \([-\epsilon, \epsilon]\).)

**Proof:** Choose \( 2\epsilon l = \lambda < 1 \) and consider \( C[-\epsilon, \epsilon] \) with

\[ \|h - g\| = \sup_{|x| \leq \epsilon} \{|h(x) - g(x)|\}. \]

Finding a solution of (4) is equivalent to finding a solution of

(5) \[ y(x) = B + \int_0^x f(t, y(t)) \, dt. \]

By a solution of (5) on \([-\epsilon, \epsilon]\) we mean a continuous function \( y \) satisfying (5) on \([-\epsilon, \epsilon]\).

This can be seen as follows: if \( y' \) is continuous, and (4) holds, then

\[ y(x) - B = \int_0^x y'(t) \, dt = \int_0^x f(t, y(t)) \, dt. \]

Conversely, suppose \( y \) is continuous and a solution of (5). Then \( f(t, y(t)) \) is a continuous function of \( t \) and by the fundamental theorem,

\[ \frac{d}{dx} \left[ B + \int_0^x f(t, y(t)) \, dt \right] = f(x, y(x)) = y' \]

and

\[ y(0) = B + \int_0^0 f(t, y(t)) \, dt = B \]

so \( y \) is a solution of (4).

Let

\[ (Ah)(x) = B + \int_0^x f(t, h(t)) \, dt \]
for $h \in C[-\epsilon, \epsilon]$, and from the argument above $A : C[-\epsilon, \epsilon] \to C[-\epsilon, \epsilon]$. Now $h = Ah$ iff $h$ is a solution of (??) and

$$(Ah)(x) - (Ag)(x) = \left\{ B + \int_0^x f(t, h(t)) \, dt \right\} - \left\{ B + \int_0^x f(t, g(t)) \, dt \right\}$$

$$= \int_0^x \left\{ f(t, h(t)) - f(t, g(t)) \right\} \, dt$$

so

$$\|Ah - Ag\| = \sup_{|x| \leq \epsilon} \left| (Ah)(x) - (Ah)(x) \right|$$

$$= \sup_{|x| \leq \epsilon} \left\{ \int_0^x |f(t, h(t)) - f(t, g(t))| \, dt \right\}$$

$$\leq \sup_{|x| \leq \epsilon} \left\{ \int_0^x |f(t, h(t)) - f(t, g(t))| \, dt \right\}$$

$$\leq \sup_{|x| \leq \epsilon} \left\{ \int_0^x |h(t) - g(t)| \, dt \right\}$$

$$\leq l \int_{-\epsilon}^{\epsilon} \sup_{|s| \leq \epsilon} |h(s) - g(s)| \, dt$$

( as for $|t| \leq \epsilon$, $|h(t) - g(t)| \leq \sup_{|s| \leq \epsilon} |h(s) - g(s)|$)

$$= l \int_{-\epsilon}^{\epsilon} \|h - g\| \, dt$$

$$= 2\epsilon l \|h - g\|$$

$$= \lambda \|h - g\|$$

and $A$ is a contraction. We have seen previously that banach space $(C[-\epsilon, \epsilon], \|\|)$ is a Banach space and so there exists a unique solution $y$, say.

\[ \square \]

Moreover if $y_0(x)$ is a continuous function and

$$y_{n+1}(x) = (Ay_n)(x) = B + \int_0^x f(t, y_n(t)) \, dt,$$

then $\|y_n - y\| \to 0$ as $n \to \infty$; that is $y_n$ converges uniformly to the solution.

**APPLICATION 2:** Let $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ have continuous partial derivatives and assume

(6) \[ 1 - \frac{\partial}{\partial y} g(x, y) \leq \lambda < 1 \]

and

(7) \[ g(a, y_a) = 0 \text{ for some } y_a \in \mathbb{R} \]

Then there is a unique continuously differentiable function $y$ in $C[a, b]$ satisfying

(8) \[ g(x, y(x)) = 0 \]
and
\begin{equation}
y(a) = y_a.
\end{equation}

**Proof**: First we use the contraction mapping principle to show that there is a unique $y \in C[a, b]$ satisfying (8) and (9).

Let
\[ X = \{ y \in C[a, b] : y(a) = y_a \} \]
and
\[ \|h - k\| = \sup_{x \in [a, b]} \{ |h(x) - k(x)| \}. \]

Thus $X$ is a closed subspace of Banach space $(C[a, b], \|\|)$ which is a Banach space, so $X$ is complete. To see that $X$ is closed let $\phi : C[a, b] \to \mathbb{R}$ where $\phi(y) = y(a)$. Thus $\phi$ is continuous and $X = \phi^{-1}\{y_a\}$, the continuous preimage of a closed set, so $X$ is closed.

Define $A : X \to X$ by
\[ (Ah)(x) = h(x) - g(x, h(x)) \quad \forall x \in [a, b]. \]

It is easily seen that this is well-defined; i.e. that
\[ h \in C[a, b] \implies Ah \in C[a, b] \]
and \[ h \in X \implies Ah \in X \]
by (7) and (9).

To see that $A$ is a contraction mapping, fix $x \in [a, b]$ and let $h, k \in X$. Set $L(t) = t - g(x, t)$ so
\begin{align*}
(Ah)(x) - (Ak)(x) & = L(h(x)) - L(k(x)) \\
& = L'(\xi)(h(x) - k(x))
\end{align*}
for some $\xi$ between $h(x)$ and $k(x)$. Now
\[ L'(\xi) = 1 - \frac{\partial}{\partial y}g(x, \xi) \implies |L'(\xi)| \leq \lambda < 1. \]

Thus
\[ \|Ah - Ak\| = \sup_{x \in [a, b]} \{ |(Ah)(x) - (Ak)(x)| \} \]
\[ \leq \lambda \sup_{x \in [a, b]} \{ |h(x) - k(x)| \} \]
\[ = \lambda\|h - k\|, \]
as required, and by the contraction mapping principle $A$ has a unique fixed point.

To see that this unique $h \in C[a, b]$ is continuously differentiable requires a separate “regularity” argument.

□