STABILITY AND APPROXIMATION OF RANDOM INVARIANT DENSITIES FOR LASOTA-YORKE MAP COCYCLES

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Abstract. We establish stability of random absolutely continuous invariant measures (acims) for cocycles of random Lasota-Yorke maps under a variety of perturbations. Our family of random maps need not be close to a fixed map; thus, our results can handle very general driving mechanisms. We consider (i) perturbations via convolutions, (ii) perturbations arising from finite-rank transfer operator approximation schemes and (iii) static perturbations, perturbing to a nearby cocycle of Lasota-Yorke maps. The former two results provide a rigorous framework for the numerical approximation of random acims using a Fourier-based approach and Ulam's method, respectively; we also demonstrate the efficacy of these schemes.

1. Introduction

Dynamical systems that are governed by laws that vary over time naturally arise in a variety of situations, including models of physical processes where the time variation is due to an external forcing. Random (or forced) dynamical systems are an extremely broad class of systems that exhibit time dependence, requiring only a stationarity assumption on the forcing process, and no assumptions on periodicity of forcing. The external forcing can also arise via a noise process, possibly modelling uncertainties in the dynamical system.

Although the word *random* appears throughout this work, the forcing may be deterministic. For example, in the context of multiscale systems of skew-product type (see eg. [PS08]), where aperiodic fast dynamics is driving slow dynamics, the "random" invariant measures are a family of probability measures on the slow space, indexed by the fast coordinates. This collection of probability measures on the slow space represent time-asymptotic distributions of slow orbits that have been driven by particular sample trajectories of the fast system. Such a situation occurs in many scenarios, including coupled ocean-atmosphere models; there is a large body of work in this area and we mention only Arnold *et al.*, [Arn98] and the references therein. Random invariant measures have also been studied in other geophysical contexts, including simplified El-Niño Southern Oscillation (ENSO) models [CSG11] and quasi-geostrophic models [DKS01] under random forcing.

The results of this paper deal with very general driving systems: the conditions on the base dynamics are that it should be stationary (i.e. have an invariant probability measure), ergodic and invertible; in particular, no mixing properties are needed. Furthermore, no uniform assumptions are made on the individual maps representing the evolution rules; rather, certain conditions are required to hold on average with respect to the stationary measure of the driving system.

Leaving technicalities for later, we denote our driving system by a measurable map $\sigma : \Omega \circlearrowleft$ and for each $\omega \in \Omega$, we have a dynamical law $T_\omega : X \circlearrowleft$ describing the evolution on our state space X. Putting these together we have a skew-product map $\Phi : \Omega \times X \circlearrowleft$, $\Phi(\omega, x) = (\sigma\omega, T_\omega x)$. We require σ to be invertible, but no hyperbolicity assumptions are imposed on it. The map σ could be, for example, an irrational torus rotation, or it could be discontinuous, or Ω could be a probability space without a topological structure. We assume that σ preserves a probability measure $\mathbb{P}$ on Ω and are concerned with measures μ preserved by Φ; that is, $\mu \circ \Phi^{-1}(A) = \mu(A)$ for each measurable $A \subset \Omega \times X$. By standard disintegration we can write, $\mu(A) = \int_\Omega \mu_\omega(A)\ d\mathbb{P}(\omega)$, where each μ_ω is a probability measure supported on on $\{\omega\} \times X$, satisfying $\mu_{\sigma\omega} = \mu_\omega \circ T_\omega^{-1}$ (thinking of $\mu_\omega, \mu_{\sigma\omega}$ as probability measures on X). Under certain conditions detailed shortly, there is a unique $\{\mu_\omega\}_{\omega\in\Omega}$, called a random absolutely continuous invariant measure (acim), such that each μ_ω has a density function with respect to Lebesgue, called f_ω, that satisfies

- $\lim_{n\to\infty} \mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n}\omega} \mathbf{1}_X = f_\omega$; thus each f_ω can be thought of as the asymptotic distribution arrived at by running the dynamics from the distant past, and
- $\mathcal{L}_\omega f_\omega = f_{\sigma\omega}$, where $\mathcal{L}_\omega$ is the Perron-Frobenius operator of T_ω (equivariance of the f_ω).

Our main goal in this paper is to demonstrate stability of random absolutely continuous invariant measures, f_ω, under a variety of perturbations. The type of stability we obtain is in a strong sense; the random acims converge *fibrewise* in L^1. We study the simple situation of piecewise smooth maps of the interval that expand on average; however, we see our results as a proof of concept, and expect results of this type to hold much more generally.

Numerical methods for approximating invariant densities rely on stability of the density under particular perturbations; those induced by the numerical method. A very common perturbation is Ulam's method, a relatively crude, but in practice extremely effective, approach. Positive stability results in a variety of settings include [BIS95, Fro95, DZ96, BK97, Mur97, KMY98, Fro99, Mur10]. A mechanism causing instability is described in [Kel82]. Ulam's method can also be used to estimate other non-essential spectral values [Fro97, BK98, BH99, Fro07]. Stability under convolution-type perturbations is treated in [BY93, AV13], and [BKL02, DL08] consider static perturbations, as well as of convolution-type. A seminal paper in this area is [KL99], which provides a rather general template for stability results for single maps.

Despite the considerable volume of results concerning stability of acims for single maps T, only a few results are known about stability of acims in the random or non-autonomous situation [BKS96, Bal97, Bog00]. Each of these results concerns stability of acims for small random perturbations of a *fixed* expanding map; thus these results concern stability of *non-random* objects associated with a fixed unperturbed transfer operator.

In contrast, we begin with a random Lasota-Yorke map that possesses a (random) invariant density, and demonstrate stability of this random invariant density under perturbations. In particular, our results answer a question raised by Buzzi in [Buz00, Buz99].

Our techniques can handle convolution-type perturbations (the random map experiences integrated noise), static perturbations (the random map is perturbed to another random map), and finite-rank perturbations (stability under numerical schemes, such as Ulam and Fourier-based schemes).

We remark that one may view random acims as the top element of an Oseledets splitting of the underlying Banach space, as considered in [FLQ10, FLQ13, GTQ13]. This viewpoint provides a broad framework for the study of random absolutely continuous invariant measures, exponential decay of correlations and coherent structures in random dynamical systems, by splitting Banach spaces into dynamically meaningful subspaces with specific growth rates. The references above provide explicit applications in the setting of random compositions of piecewise smooth expanding maps.

The approach we take here is motivated by the work of Keller and Liverani [KL99]; the latter considered random perturbations of a *single* (non-random) map T. There, for example, one replaces the Perron-Frobenius operator of T by the average of a family of Perron-Frobenius operators corresponding to small perturbations of T. This is known as the *annealed* Perron-Frobenius operator corresponding to the iid family. The evolution at the level of orbits is rather simple and is exactly described by a Markov chain; in fact one can recover the Markov chain from the averaged Perron-Frobenius operator.

If one has a random dynamical system $(\sigma, \{T_\omega\}_{\omega\in\Omega})$, where σ is Bernoulli but the T_ω are not necessarily close to some map T, it is dynamically meaningful to construct a single *annealed* Perron-Frobenius operator, which is the expectation of the individual Perron-Frobenius operators for each T_ω. With a little work, one can conclude that the annealed random invariant measure (a single probability measure) is stable under the types of perturbations considered in [KL99].

By contrast, our results are able to handle the *quenched* situation, in which we keep track of the current ω and assign a probability measure to *each* $\omega \in \Omega$, rather than a single probability measure for the entire expected action of the dynamics as in the annealed case. In our quenched setup, there is no requirement that the maps should be selected in an iid way – for example the situation in which there is a quasi-periodic base system, and the map to be applied is completely determined by the state of the base system falls within our framework, but cannot be treated within the KL framework. Even if the maps are selected in an iid way, the quenched result gives information about almost every composition of maps, rather than just the average over all possible compositions. In general, quenched results are much more delicate than annealed results.

1.1. **Statement of the main results.** A random dynamical system consists of a base dynamics (an invertible measure-preserving map σ of a probability space Ω) and a family of linear maps $\mathcal{L}_\omega$ from a Banach space to itself (in our applications these are the Perron-Frobenius operators of piecewise smooth maps, T_ω, of the circle or the interval). The results address stability of the random acim when the linear maps are perturbed (leaving the base dynamics unchanged). We consider three classes of perturbations:

(A) *Ulam-type perturbation.* For a fixed k, we define perturbed operators $\mathcal{L}_{k,\omega}$ to be $\mathbb{E}_k \circ \mathcal{L}_\omega$, where $\mathbb{E}_k$ is the conditional expectation operator with respect to the partition into intervals of length $1/k$.

(B) *Convolution-type perturbation.* Given a family of densities (Q_k) on the circle, we define perturbed operators $\mathcal{L}_{k,\omega}$ by $\mathcal{L}_{k,\omega} f = Q_k * \mathcal{L}_\omega f$. If one applies T_ω and then adds a noise term with distribution given by Q_k, then $\mathcal{L}_{k,\omega}$ is the random Perron-Frobenius operator. That is, the expectation of the Perron-Frobenius operators of $\tau_y \circ T_\omega$ where y has density Q_k and τ_y is translation by y. Notable examples of perturbations of this type are the cases where Q_k is uniformly distributed on an interval $[-\epsilon_k, \epsilon_k]$ or where Q_k is the kth Fejér kernel.

(C) *Static perturbation.* Here one replaces the entire family of transformations T_ω by nearby transformations $T_{k,\omega}$. These are much more delicate than the other two types of perturbation (composing with convolutions and conditional expectations generally make operators more benign, for example they reduce variation).

Notice that by enlarging the probability space, perturbations of this type can include transformations with (for example) independent identically distributed additive noise. To see this, let Ξ denote the space of sequences taking values in $[-1, 1]$, equipped with the product of uniform measures and let $\bar{\Omega} = \Omega \times \Xi$ and $\bar{\sigma}$ be the product of σ on the Ω coordinate and the shift on the Ξ coordinate. Then defining $T_{k,(\omega,\xi)}(x) = T_\omega(x) + \epsilon_k \xi$ gives a family of perturbed maps (with the common base dynamics being $\bar{\Omega}$). The unperturbed dynamics (T_ω) can, of course, also be seen as being driven by $\bar{\Omega}$. Notice that this is *not* the same thing as the perturbation obtained by convolving with a uniform Q_k as in (B). In the static case, the results obtained would give a result that holds for compositions of $\mathcal{L}_{\omega,\xi}$ for almost every ω and almost every sequence of perturbations ξ, whereas a result for the convolution perturbation would give a result that holds for the *expectation* of these operators obtained by integrating over the ξ variables. The convolution type perturbations are also known in the physics literature as *annealed* systems, while the static perturbations are *quenched* systems.

Below we outline the main application results of this paper. We refer the reader to §3 for definitions, and to Theorems 3.7, 3.9 and 3.11 for the precise statements.

Theorem A: *(Stability under Ulam discretization).* Let $\mathcal{L}$ be the Perron-Frobenius operator of a covering, good random Lasota-Yorke map acting on BV, the space of functions of bounded variation. Let $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ be the sequence of Ulam discretizations of $\mathcal{L}$, corresponding to uniform partitions of the domain into k bins. Then, for each sufficiently large k, $\mathcal{L}_k$ has a unique random acim. Let $\{F_k\}_{k\in\mathbb{N}}$ be the sequence of random acims for $\mathcal{L}_k$. Then, $\lim_{k\to\infty} F_k = F$ fibrewise in L^1.

Theorem B: *(Stability under convolutions).* Let $\mathcal{L}$ be the Perron-Frobenius operator of a covering, good random Lasota-Yorke map acting on BV. Let $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ be a family of perturbations, arising from convolution with positive kernels Q_k, such that $\lim_{k\to\infty} \int Q_k(x)|x|\,dx = 0$[1]. Then, for sufficiently large k, $\mathcal{L}_k$ has a unique random acim. Let us call it F_k. Then, $\lim_{k\to\infty} F_k = F$ fibrewise in L^1.

Theorem C: *(Stability under static perturbations).* Let $\mathcal{L}$ be the Perron-Frobenius operator of a covering, good random Lasota-Yorke map acting on BV. Let $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ be the Perron-Frobenius operators of a family of good random Lasota-Yorke maps over the same base as $\mathcal{L}$, satisfying the conditions of Definition 3.1, with the same bounds as $\mathcal{L}$. Assume that $d_{LY}(T_{k,\omega}, T_\omega)$ converges to 0 in $\mathbb{P}$-measure, where d_{LY} is a metric on the space of Lasota-Yorke maps. Then, for every sufficiently large k, $\mathcal{L}_k$ has a unique random acim. Let $\{F_k\}_{k\in\mathbb{N}}$ be the sequence of random acims for $\mathcal{L}_k$. Then, $\lim_{k\to\infty} F_k = F$ fibrewise in L^1.

1.2. **Structure of the paper.** The paper is organized as follows. An abstract stability result, Theorem 2.4, is presented in §2, after introducing the underlying setup. Examples are provided in §3. They include perturbations arising from Ulam's discretization scheme in §3.2, perturbations by convolution in §3.3 and static perturbations of random Lasota-Yorke maps in §3.4. The theoretical results are illustrated with a numerical example in §3.5.

2. A stability Result

We start by fixing some terminology. Let m denote Lebesgue measure on the interval $I := [0,1]$ (or on the circle). Let $(BV, \|\cdot\|_{\mathrm{BV}})$ be the Banach space of functions of bounded variation on I. That is, $f \in BV$ if

$$\mathrm{var}(f) := \inf_{g=f \pmod{m}} \sup_{0=x_0<x_1<\cdots<x_n=1, n\in\mathbb{N}} \sum_{j=1}^{n} |g(x_k) - g(x_{k-1})| < \infty,$$

and for every $f \in BV$, $\|f\|_{\mathrm{BV}} := \mathrm{var}(f) + |f|_1$.

Definition 2.1. *A* ***random linear system with ergodic and invertible base****, or for short a* ***random dynamical system****, is a tuple* $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ *such that*

[1]This condition is equivalent to weak convergence of Q_k to δ_0.

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\sigma : (\Omega, \mathcal{F}) \circlearrowleft$ is an invertible and ergodic $\mathbb{P}$-preserving transformation, X is a Banach space, $L(X)$ denotes the set of bounded linear maps of X, and $\mathcal{L} : \Omega \to L(X)$.

Remark 2.2. *Some measurability conditions on $\mathcal{L}$ will be required in the sequel. We use the notation $\mathcal{L}_\omega^{(n)} = \mathcal{L}(\sigma^{n-1}\omega) \circ \cdots \circ \mathcal{L}(\omega)$.*

2.1. **Setting.** Let us consider random dynamical systems of functions of bounded variation

$$\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, BV, \mathcal{L}) \text{ and } \mathcal{R}_k = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, BV, \mathcal{L}_k), k \geq 1,$$

with a common ergodic invertible base $\sigma : \Omega \circlearrowleft$ and such that the maps $(\omega, x) \mapsto \mathcal{L}_{k,\omega} G(\omega, x)$ is $\mathbb{P} \times m$ measurable for every $\mathbb{P} \times m$ measurable function G. Furthermore, suppose the following conditions hold:

- **(H0)** $\int \log^+ \|\mathcal{L}_\omega\|_{\mathrm{BV}} \, d\mathbb{P}(\omega) < \infty$ and for every $k \in \mathbb{N}$, $\int \log^+ \|\mathcal{L}_{k,\omega}\|_{\mathrm{BV}} \, d\mathbb{P}(\omega) < \infty$. Furthermore, for every $k \in \mathbb{N}$, $f \in X$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mathcal{L}_{k,\omega}$ and $\mathcal{L}_\omega$ preserve the cone of non-negative functions, and satisfy $\int \mathcal{L}_{k,\omega} f \, dm = \int f \, dm = \int \mathcal{L}_\omega f \, dm$.

Remark 2.3. *In all the examples of this paper, condition (H0) is clearly satisfied. Thus, we will henceforth assume it holds and use it wherever needed.*

The following condition generalizes the widely used Lasota-Yorke inequality to the context of random maps.

- **(H1)** There exist a constant $B > 0$ and a measurable $\alpha : \Omega \to \mathbb{R}_+$ with $\kappa := \int \log \alpha(\omega) \, d\mathbb{P}(\omega) < 0$, such that for every $f \in X$, $k \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\max\left(\|\mathcal{L}_\omega f\|_{\mathrm{BV}}, \sup_{k \in \mathbb{N}} \|\mathcal{L}_{k,\omega} f\|_{\mathrm{BV}} \right) \leq \alpha(\omega) \|f\|_{\mathrm{BV}} + B|f|_1. \tag{1}$$

A version of the following statement was used by Buzzi [Buz00]: Condition (H1) is implied by the following more practical condition.

- **(H1')** $\log \max(\|\mathcal{L}_\omega\|_{\mathrm{BV}}, \sup_{k \in \mathbb{N}} \|\mathcal{L}_{k,\omega}\|_{\mathrm{BV}})$ is $\mathbb{P}$-integrable, and there exist measurable functions $\tilde{\alpha}, \tilde{B} : \Omega \to \mathbb{R}_+$ with $\int \log \tilde{\alpha}(\omega) \, d\mathbb{P}(\omega) < 0$, such that for every $f \in X$, $k \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\max\left(\|\mathcal{L}_\omega f\|_{\mathrm{BV}}, \sup_{k \in \mathbb{N}} \|\mathcal{L}_{k,\omega} f\|_{\mathrm{BV}} \right) \leq \tilde{\alpha}(\omega) \|f\|_{\mathrm{BV}} + \tilde{B}(\omega)|f|_1. \tag{2}$$

Furthermore, κ in (1) may be chosen arbitrarily close to $\int \log \tilde{\alpha}(\omega) \, d\mathbb{P}(\omega)$.

Using the relative compactness of the unit ball of BV in L^1, one can verify that $\Phi_k(\omega) := \sup_{||g||_{BV}=1} |\mathcal{L}_{k,\omega}(g) - \mathcal{L}_\omega(g)|_1$ is a measurable function of ω. The following condition regarding smallness of the perturbations is required:

- **(H2)** $\Phi_k(\omega)$ converges in measure to 0. That is, for every $\delta > 0$,

$$\lim_{k \to \infty} \mathbb{P}\left(\{\omega : \sup_{\|g\|_{\mathrm{BV}}=1} |(\mathcal{L}_\omega - \mathcal{L}_{k,\omega})g|_1 > \delta\} \right) = 0.$$

The previous condition allows us to pursue a probabilistic version of the *triple norm* approach used by Keller and Liverani in the autonomous context [Kel82, KL99], where the triple norm of an operator $\mathcal{L}$ is defined by $|||\mathcal{L}||| := \sup_{\|g\|_{\mathrm{BV}}=1} |\mathcal{L}g|_1$.

A further assumption is made on $\mathcal{R}$, related to uniqueness of the random acim. We remark that this condition concerns the unperturbed system only, and not the perturbations.

(**U1**) For every $\epsilon, \delta > 0$, there exists $n_{\epsilon,\delta} \in \mathbb{N}$ such that

$$\mathbb{P}\left(\{\omega : \sup_{f \in V_0, \|f\|_{BV} \le 1} |\mathcal{L}^{(n)}_{\sigma^{-n}\omega} f|_1 > \delta\}\right) \le \epsilon \text{ for all } n \ge n_{\epsilon,\delta}, \tag{3}$$

where $V_0 := \{f \in BV : \int f dm = 0\}$.

2.2. **Stability of random acims.** For each $n \in \mathbb{N}$ and $G : \Omega \times I \to \mathbb{R}$, with $g_\omega := G(\omega, \cdot) \in X$, we let $\mathcal{L}^n G : \Omega \times I \to \mathbb{R}$ be the function defined fibrewise by $\mathcal{L}^n G(\omega, \cdot) := \mathcal{L}^{(n)}_{\sigma^{-n}\omega} g_{\sigma^{-n}\omega}$. Let $\mathcal{L}^n_k G$ be defined analogously.

Condition (H0) shows that if $\mathcal{L}$ (or $\mathcal{L}_k$) has a non-negative fixed point, then one can in fact choose it to be fibrewise normalized in $L^1(\text{Leb})$. We call any such fixed point F a *random acim* for $\mathcal{L}$ (or $\mathcal{L}_k$).

The main result of this section is the following.

Theorem 2.4. *Let $\mathcal{R}$ and $\mathcal{R}_k$, $k \ge 1$ be random dynamical systems of functions of bounded variation (BV). Suppose $\mathcal{R}$ and $\mathcal{R}_k$, $k \ge 1$ share a common ergodic invertible base and satisfy conditions* (H0)*–*(H2)*. Assume $\mathcal{R}$ satisfies* (U1)*.*

Then, $\mathcal{R}$ has a unique random acim, F, and for sufficiently large k, there is a unique random acim for $\mathcal{R}_k$, which is denoted by F_k. Furthermore, $\lim_{k\to\infty} F_k = F$ fibrewise in L^1. That is, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\lim_{k\to\infty} |f_\omega - f_{k,\omega}|_1 = 0$.

Proof of Theorem 2.4. The existence of random acims for $\mathcal{R}$ and $\mathcal{R}_k$ for each k follows from [Buz00, Proposition 2.1].

Let $0 < \epsilon < \frac{1}{2}$. We now assemble the ingredients that we use in the proof.

By [Buz00, Proposition 3.2], there exists an $A > 0$ and a set Ω_1 with $\mathbb{P}(\Omega_1) > 1 - \epsilon$ such that for every $\omega \in \Omega_1$, for every $k \in \mathbb{N} \cup \{0\}$ and every random acim, F'_k for $\mathcal{R}_k$, one has

$$\|F'_{k,\omega}\|_{\text{BV}} \le A. \tag{4}$$

By (U1), there exists n (depending on ϵ, but now fixed for the remainder of the proof) and a set Ω_2 with $\mathbb{P}(\Omega_2) > 1 - \epsilon$ such that for $\omega \in \Omega_2$, one has

$$\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} f\|_1 \le \epsilon \text{ for any } f \in V_0 \text{ with } \|f\|_{\text{BV}} \le 2A. \tag{5}$$

By (H1), there exists $B > 0$ and a set Ω_3 with $\mathbb{P}(\Omega_3) > 1 - \epsilon$ such that

$$\sum_{j=0}^{n-1} \|\mathcal{L}^{(j)}_{\sigma^{-n}\omega}\|_{\text{BV}} < B \text{ for all } \omega \in \Omega_3. \tag{6}$$

Finally, by (H2), there exists $k_0 \in \mathbb{N}$ such that for each $k \ge k_0$, there exists a set G_k of measure at least $1 - \frac{\epsilon}{n}$ such that

$$\|\mathcal{L}_\omega f - \mathcal{L}_{k,\omega} f\|_1 \le \epsilon/(AB)\|f\|_{\text{BV}} \text{ if } k \ge k_0 \text{ and } \omega \in G_k. \tag{7}$$

We now combine the ingredients. Let $\Omega_{4,k} = \bigcap_{j=1}^n \sigma^j(G_k)$, so that $\mathbb{P}(\Omega_{4,k}) > 1 - \epsilon$. Let $\tilde{\Omega}_k := \sigma^n \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_{4,k}$. Then $\mathbb{P}(\tilde{\Omega}_k) \ge 1 - 4\epsilon$. Let $H = \{h \in \text{BV}\colon \int h(x)\, dx = 1 \text{ and } \|h\|_{\text{BV}} \le A\}$.

Let $k \ge k_0$, $\omega \in \tilde{\Omega}_k$ and $h, h' \in H$. We then have

$$\begin{aligned} &\|\mathcal{L}^{(n)}_{k,\sigma^{-n}\omega} h - \mathcal{L}^{(n)}_{\sigma^{-n}\omega} h'\|_1 \\ &\le \|\mathcal{L}^{(n)}_{k,\sigma^{-n}\omega} h - \mathcal{L}^{(n)}_{\sigma^{-n}\omega} h\|_1 + \|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} h - \mathcal{L}^{(n)}_{\sigma^{-n}\omega} h'\|_1. \end{aligned} \tag{8}$$

We apply (5) to deduce that $\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} h - \mathcal{L}^{(n)}_{\sigma^{-n}\omega} h'\|_1 < \epsilon$.

For the first term of (8), we write

$$
\begin{aligned}
&\|\mathcal{L}^{(n)}_{k,\sigma^{-n}\omega}h - \mathcal{L}^{(n)}_{\sigma^{-n}\omega}h\|_1 \\
&\le \sum_{j=0}^{n-1} \|\mathcal{L}^{(n-j-1)}_{k,\sigma^{-(n-j-1)}\omega}\|_1 \,\| \left(\mathcal{L}_{k,\sigma^{-(n-j)}\omega} - \mathcal{L}_{\sigma^{-(n-j)}\omega}\right) \mathcal{L}^{(j)}_{\sigma^{-n}\omega}h\|_1.
\end{aligned} \tag{9}
$$

The L^1 norms of the Perron-Frobenius operators are 1. Furthermore, we have $\sigma^{-(n-j)}\omega \in G_k$ for each $0 \le j < n$, so that by (7), we obtain

$$\|\mathcal{L}^{(n)}_{k,\sigma^{-n}\omega}h - \mathcal{L}^{(n)}_{\sigma^{-n}\omega}h\|_1 \le \frac{\epsilon}{AB}\sum_{j=0}^{n-1} \|\mathcal{L}^{(j)}_{\sigma^{-n}\omega}h\|_{\mathrm{BV}}.$$

Using (4) and (6), we see $\|\mathcal{L}^{(n)}_{k,\sigma^{-n}\omega}h - \mathcal{L}^{(n)}_{\sigma^{-n}\omega}h\|_1 \le \epsilon$.

Combining this bound with (5) via (8), we now obtain

$$\|\mathcal{L}^{(n)}_{k,\sigma^{-n}\omega}h - \mathcal{L}^{(n)}_{\sigma^{-n}\omega}h'\|_1 \le 2\epsilon \text{ for } \omega \in \tilde{\Omega}_k \text{ if } h, h' \in H \tag{10}$$

We now demonstrate that $\mathcal{R}_k$ has a unique acim for each $k \ge k_0$. Indeed if $\mathcal{R}_k$ had two random acims, g and g', then the normalized positive and negative parts of $g - g'$, h_1 and h_2, would also be random acims and would satisfy $\|h_{1,\omega} - h_{2,\omega}\|_1 = 2$ for each ω. This contradicts (10) (taking $h' = 1$ and h to be h_1 and h_2 in turn) and establishes uniqueness.

Let f_k denote the (unique) random acim for $\mathcal{R}_k$ (where $k \ge k_0$) and f the random acim for $\mathcal{R}$. Then (10) shows that $\|f_{k,\omega} - f_\omega\|_1 \le 2\epsilon$ for $\omega \in \tilde{\Omega}_k$ for all $k \ge k_0$. Since $\mathbb{P}(\tilde{\Omega}_k) > 1 - 4\epsilon$ and ϵ is arbitrary, we see that the conclusion holds.

□

3. Examples

We present three applications of the stability theorem in the context of random Lasota-Yorke maps in sections §3.2–§3.4. These correspond to Ulam approximations, perturbations with additional randomness that arise by taking convolutions with non-negative kernels, and static perturbations, respectively. In §3.5, we illustrate the results with a numerical example.

3.1. Setting: Random (non-autonomous) Lasota-Yorke maps.

Our setup of unperturbed random Lasota-Yorke maps can handle maps as in [Buz99]. In particular, neither uniform expansion nor uniform bounds on number of branches are imposed on the individual maps, but rather some conditions are required to hold on average with respect to the ergodic invariant measure of the driving system.

Let LY be the space of non-singular, finite-branched piecewise monotonic, piecewise C^2 maps of the interval. For each $T \in LY$, let $\mu(T) := \operatorname{ess\,inf}_{x\in I} |T'(x)|$ and $N(T)$ the number of branches of T, and let $\{0 = a_0(T), a_1(T), \ldots, a_{N(T)}(T) = 1\}$ be the endpoints of the branches.

Definition 3.1. *Let $(\Omega, \mathcal{F})$ be a measurable space and let $\sigma : \Omega \circlearrowleft$ be an ergodic, invertible transformation preserving a probability measure $\mathbb{P}$. A* ***good random Lasota-Yorke map*** $\mathcal{T}$ *is a function $\mathcal{T} : \Omega \to LY$ given by $\omega \mapsto T_\omega$, such that*

- *$(\omega, x) \mapsto T_\omega(x)$ is measurable.*
- *$\lim_{K\to\infty} \int_\Omega \log\min(\mu(T_\omega), K)\,d\mathbb{P} > 0$.*
- *$\log^+(N(T_\omega)/\mu(T_\omega)) \in L^1(\mathbb{P})$.*

- $\log^+(\mathrm{var}(1/|T'_\omega|)) \in L^1(\mathbb{P})$.

A random Lasota-Yorke map is called ***covering*** *if for every non-trivial interval $J \subset I$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists some $n \in \mathbb{N}$ such that $T^{(n)}_\omega(J) = I \pmod 0$.*

Remark 3.2. *A good random Lasota-Yorke map can be made into a random dynamical system $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, BV, \mathcal{L})$, where $\mathcal{L}_\omega$ is the transfer operator associated to $\mathcal{T}(\omega) =: T_\omega$ acting on BV. For each $n \in \mathbb{N}$, $\mathcal{R}^n$ denotes the random dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \sigma^n, BV, \mathcal{L}^n)$. When the underlying $\mathcal{T}$ is clear from the context, we also refer to $\mathcal{R}$ and $\mathcal{L}$ as good random Lasota-Yorke maps, in a slight abuse of notation.*

Remark 3.3. *The measurability condition of Definition 3.1 is implicit in* [Buz99]*, and it implies measurability of $\omega \mapsto (\mu(T_\omega), \mathrm{var}(1/|T'_\omega|), N(T_\omega), a_0(T_\omega), \dots, a_{N(T_\omega)}(T_\omega), 0, 0, \dots)$, which is explicitly required in* [Buz99].

The following theorem of Buzzi will allow us to show the conditions of Theorem 2.4 are satisfied for the unperturbed map $\mathcal{R}$.

Theorem 3.4. [Buz99] *Let $\mathcal{R}$ be a covering good random Lasota-Yorke map. Then,*

(0) There exists $N \in \mathbb{N}$ such that $\mathcal{R}^N$ satisfies (H0) and (H1').
(1) There exists a unique random acim F for $\mathcal{R}$.
(2) There exist $\rho > 0$ and a function $n_0(\omega, M)$ such that if $h \geq 0$, $\|h\|_1 = 1$ and $\|h\|_{BV} \leq M$, then

$$\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} h - f_\omega\|_\infty \leq \rho^n \text{ for all } n \geq n_0(\omega, M).$$

Remark 3.5. *The choice of N in Theorem 3.4(0) is given by the requirement that the average expansion of the of the N-fold composition has to be sufficiently large. More precisely, it is necessary to have $\int \log \mu(T^{(N)}_\omega) d\mathbb{P}(\omega) > \log 3$.*

Proposition 3.6. *Let $\mathcal{R}$ be a covering good random Lasota-Yorke map. Then, $\mathcal{R}$ satisfies condition (U1).*

Proof. Let $\epsilon, \delta > 0$. We have to show that there exist $\Omega_{\epsilon,\delta} \subset \Omega$ with $\mathbb{P}(\Omega_{\epsilon,\delta}) \geq 1 - \epsilon$ and $n_{\epsilon,\delta} \in \mathbb{N}$ such that

$$\sup_{f \in V_0, \|f\| \leq 1} |\mathcal{L}^{(n)}_{\sigma^{-n}\omega} f|_1 \leq \delta \text{ for every } \omega \in \Omega_{\epsilon,\delta} \text{ and for all } n \geq n_{\epsilon,\delta}.$$

Let n_1 be such that $\rho^{n_1} < \delta/2$, where ρ comes from Theorem 3.4. Let $n_{\epsilon,\delta} > n_1$ be chosen so that $n_0(\omega, 2) < n_{\epsilon,\delta}$ for every ω in a set $\Omega_{\epsilon,\delta}$ with $\mathbb{P}(\Omega_{\epsilon,\delta}) \geq 1 - \epsilon$.

Now let $\omega \in \Omega_{\epsilon,\delta}$, and let $f \in V_0$ satisfy $\|f\|_{\mathrm{BV}} \leq 1$. Write $f = f_+ - f_-$ where f_+ and f_- are non-negative. Since $f \in V_0$, we have $\|f_+\|_{\mathrm{BV}} < 1$ and $\|f_-\|_{\mathrm{BV}} < 1$ and $\|f_+\|_1 = \|f_-\|_1 = c < 1$.

Let $h_+ = f_+ + (1 - c)$; and $h_- = f_- + (1 - c)$ so that $\|h_+\|_{\mathrm{BV}} < 2$, $\|h_-\|_{\mathrm{BV}} < 2$, $\|h_+\|_1 = 1$ and $\|h_-\|_1 = 1$. Note that $h_+ - h_- = f$.

Now we apply Theorem 3.4 to h_+ and h_-. We have $\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} h_i - f_\omega\|_\infty < \rho^{n_{\epsilon,\delta}}$ for $i \in \{+, -\}$ and $n \geq n_\epsilon$. It follows that $\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} f\|_\infty < 2\rho^{n_{\epsilon,\delta}} < \delta$, which yields the claim. □

3.2. **The Ulam scheme.** For each $k \in \mathbb{N}$, let $\mathcal{P}_k = \{B_1, \dots, B_k\}$ be the partition of I into k subintervals of uniform length, called bins. Let $\mathbb{E}_k$ be given by the formula

$$\mathbb{E}_k(f) = \sum_{j=1}^{k} \frac{1}{m(B_j)} \Big(\int 1_{B_j} \, f \, dm \Big) 1_{B_j},$$

where m denotes normalized Lebesgue measure on I.

Let $\mathcal{L}_k$ be defined as follows. For each $\omega \in \Omega$, $\mathcal{L}_{k,\omega} := \mathbb{E}_k \mathcal{L}_\omega$. This is the well-known Ulam discretization [Ula60], in the context of non-autonomous systems. It provides a way of approximating the transfer operator $\mathcal{L}$ with a sequence of (fibrewise) finite-rank operators $\mathcal{L}_k$, whose range consists of functions that are constant on each bin.

Theorem 3.7. *Let $\mathcal{L}$ be a covering good random Lasota-Yorke map*[2]*. For each $k \in \mathbb{N}$, let $\mathcal{L}_k$ be the sequence of Ulam discretizations, corresponding to the partition $\mathcal{P}_k = \{B_1, \dots, B_k\}$ introduced above.*

Then, for each sufficiently large k, $\mathcal{L}_k$ has a unique random acim. Let $\{F_k\}_{k\in\mathbb{N}}$ be the sequence of random acims for $\mathcal{L}_k$. Then, $\lim_{k\to\infty} F_k = F$ fibrewise in $|\cdot|_1$.

Proof. We will verify the assumptions of Theorem 2.4. (H0) is immediate. The assumptions combined with the fact that $\mathbb{E}_k$ reduces variation ensure that (H1') holds as well. The last condition to check in order for Theorem 2.4 to apply is (H2), which follows from e.g. [Kel82, §16-§18]. $\square$

3.3. **Convolution-type perturbations.**

In this section we consider perturbations of non-autonomous maps that arise from convolution with non-negative kernels $Q_k \in L^1(m)$, with $\int Q_k \, dm = 1$. They give rise to transfer operators as follows.

$$\mathcal{L}_{k,\omega} f(x) := \int \mathcal{L}_\omega f(y) Q_k(x-y) dy. \tag{11}$$

They model at least two interesting types of perturbations:

(1) Small iid noise. In this case, Q_k, supported on $[-\frac{1}{k}, \frac{1}{k}]$, represents the distribution of the noise, which is added after applying the corresponding map T_ω. See e.g. [Bal00, §3.3] for details.
(2) Cesàro averages of Fourier series. In this case, Q_k is the Fejér kernel $Q_k(x) = \frac{\sin(\pi k x)^2}{k \sin(\pi x)^2}$, and $Q_k * f = \frac{1}{k}\sum_{j=0}^{k-1} S_j(f)$, where $S_k(f)(x) = \sum_{j=-k}^{k} \hat{f}(j) e^{2\pi i j x}$ is the truncated Fourier series of f.

Remark 3.8. *We point out that the Galerkin projection on Fourier modes, corresponding to truncation of Fourier series, is obtained from convolution with Dirichlet kernels, which are not positive. Although a convergence result in this case remains open, the numerical behaviour appears to be good as well. This is illustrated in §3.5.*

Theorem 3.9. *Let $\mathcal{L}$ be a covering good Lasota-Yorke map*[3]*. Let $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ be a family of random perturbations, as in (11), such that $\lim_{k\to\infty} \int Q_k(x)|x| \, dx = 0$.*

Then, for sufficiently large k, $\mathcal{L}_k$ has a unique random acim. Let us call it F_k. Then, $\lim_{k\to\infty} F_k = F$ fibrewise in $|\cdot|_1$.

Proof. We will show that conditions (H0)–(H2) of Theorem 2.4 are satisfied. (H0) is clear. (H1') holds because of the assumption on $\mathcal{L}$ and the straightforward fact that taking convolution reduces variation. (H2) follows from [Kel82, §16-§18]. $\square$

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[2]We assume $N = 1$ in Theorem 3.4(0). If $N > 1$, the conclusions remain valid provided the projection $\mathbb{E}_k$ is taken after N compositions.

[3]We assume $N = 1$ in Theorem 3.4(0). If $N > 1$, the conclusions remain valid provided the convolutions are taken after N compositions.

3.4. **Static perturbations.** We consider the notion of distance in the space of Lasota-Yorke maps introduced by Keller in [Kel82, §3].

Definition 3.10. *The distance d_{LY} is defined as follows. Let $S, T \in LY$. Then,*

$$d_{LY}(S,T) := \inf\{\delta > 0 : \exists J \subset I,\ \exists \phi : I \circlearrowleft \ s.t.\ m(J) > 1-\delta, \phi \text{ is a diffeomorphism}, \\ S|_J = T \circ \phi|_J \text{ and } \forall x \in J, |\phi(x) - x| < \delta,\ |1/\phi'(x) - 1| < \delta\}.$$

Theorem 3.11. *Let $\mathcal{L}$ be a covering good Lasota-Yorke map, as defined in §3.1. For each $k \in \mathbb{N}$, let $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ be a family of good random Lasota-Yorke maps with the same base as $\mathcal{L}$, satisfying the conditions of Definition 3.1, with the same bounds as $\mathcal{L}$*[4]*. Assume that $d_{LY}(T_{k,\omega}, T_\omega)$ converges to 0 in measure ($\mathbb{P}$).*

Then, for every sufficiently large k, $\mathcal{L}_k$ has a unique random acim. Let $\{F_k\}_{k\in\mathbb{N}}$ be the sequence of random acims for $\mathcal{L}_k$. Then, $\lim_{k\to\infty} F_k = F$ fibrewise in $|\cdot|_1$.

Proof. We will show that there exists $n \in \mathbb{N}$ such that $\mathcal{T}^{(n)}$ and $\mathcal{T}_k^{(n)}$ satisfy the hypotheses of Theorem 2.4.

(H0) is straightforward to verify for $\mathcal{T}^{(n)}$ and $\mathcal{T}_k^{(n)}$. Condition (H1') holds for some $n \in \mathbb{N}$, because of the assumptions on $\mathcal{T}$ and $\mathcal{T}_k$ and [Buz99, §1.2]; we remark that the covering condition is not necessary for this part (see also [Buz00]). Condition (H2) follows from the next proposition.

Proposition 3.12. *Assume that $d_{LY}(T_{k,\omega}, T_\omega)$ converges to 0 in measure.*

Then, for every $n \in \mathbb{N}$, $\sup_{\|g\|_{BV}=1} |(\mathcal{L}_\omega^{(n)} - \mathcal{L}_{k,\omega}^{(n)})g|_1$ also converges to 0 in measure.

Proof of Proposition 3.12. For $n = 1$, the claim follows from [Kel82, §3], which shows that

$$\sup_{\|g\|_{\mathrm{BV}}=1} |(\mathcal{L}_\omega - \mathcal{L}_{k,\omega})g|_1 \le 12 d_{LY}(T_\omega, T_{k,\omega}).$$

The general case of fixed $n > 1$ then follows immediately from the identity

$$\mathcal{L}_{k,\omega}^{(n)} - \mathcal{L}_\omega^{(n)} = \sum_{j=0}^{n-1} \mathcal{L}_{k,\sigma^{n-j}\omega}^{(j)} (\mathcal{L}_{k,\sigma^{n-j-1}\omega} - \mathcal{L}_{\sigma^{n-j-1}\omega}) \mathcal{L}_\omega^{(n-j-1)},$$

with a similar argument to the one following (9). □

□

3.5. **Numerical examples.** In this section we provide a brief demonstration that the stability results of §3.2 and §3.3 can be used to rigorously approximate random invariant densities.

Let Ω be a circle of unit circumference let the driving system $\sigma : \Omega \circlearrowleft$ be a rigid rotation by angle $\alpha = 1/\sqrt{2}$. For $x \in \Omega$ considered to be a point in $[0, 1)$, we define a random

[4]More precisely, we assume there exist:

(i) $a > 0$ such that $\liminf_{k\to\infty} \lim_{K\to\infty} \int_\Omega \log \min(\mu(f_{k,\omega}), K) d\mathbb{P} > a$;
(ii) a $\mathbb{P}$-integrable function A such that for every sufficiently large $k \in \mathbb{N}$, $\log^+(N(f_{k,\omega})/\mu(f_{k,\omega}))$ and $\log^+(N(f_\omega)/\mu(f_\omega))$ are dominated by A; and
(iii) a $\mathbb{P}$-integrable function B such that for every sufficiently large $k \in \mathbb{N}$, $\log^+(\mathrm{var}(1/|f'_{k,\omega}|))$ and $\log^+(\mathrm{var}(1/|f'_\omega|))$ are dominated by B.

map as:

(12)

$$T_\omega(x) = \begin{cases} 3(x-\omega) - 2.9(x-\omega)(x-\omega-1/3), & \omega \le x < \omega + 1/3; \\ -3(x-\omega) + 1 - 2.9(x-\omega-1/3)(x-\omega-2/3), & \omega + 1/3 \le x < \omega + 2/3; \\ 7/3(x-\omega-2/3) + 2\omega/9, & \omega + 2/3 \le x < \omega + 1. \end{cases}.$$

Graphs of T_ω for three different ω are shown in Figure 1.

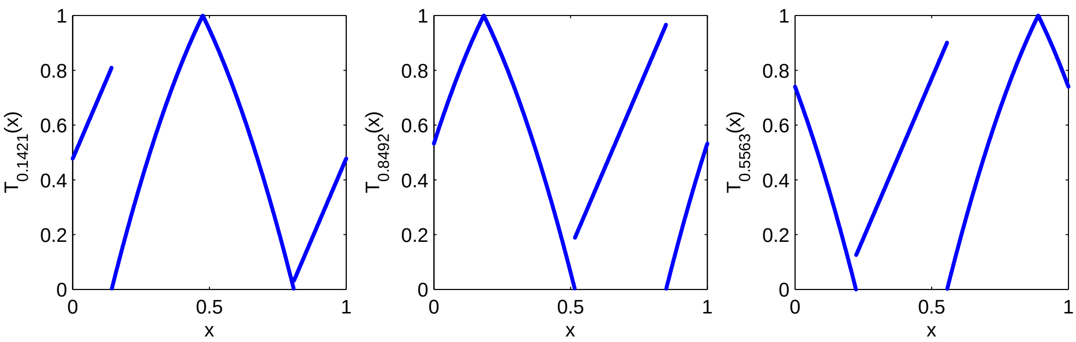

FIGURE 1. Graphs of the maps $T_{\sigma^{20}\omega}, T_{\sigma^{21}\omega}, T_{\sigma^{22}\omega}$, $\omega = 0$.

The graphs T_ω rotate with ω and one of three branches is also translated up/down with ω. The minimum slope of $\{T_\omega\}_{\omega\in\Omega}$ is bounded below by 2.

We employ the Ulam scheme with k equal subintervals, for $k = 100$ and $k = 1000$, and a Fejér kernel with $k = 100$ (100 Fourier modes). In the Ulam case, we use the well-known formula for the Ulam matrix [Ula60] to construct a matrix representation of $\mathcal{L}_{k,\omega}$: $[\mathcal{L}_{k,\omega}]_{ij} = m(B_i \cap T_\omega^{-1} B_j)/m(B_j)$, $i, j = 1, \ldots, k$, which is the result of Galerkin projection using the basis $\{\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_k}\}$. Lebesgue measure in the formula for $[\mathcal{L}_{k,\omega}]_{ij}$ is approximated by a uniform grid of 1000 test points per subinterval, and the estimate of $[\mathcal{L}_{k,\omega}]$ takes less than a second to compute in MATLAB.

In the Fourier case, we first use Galerkin projection onto the basis $\{1, \sin(2\pi x), \cos(2\pi x), \ldots, \sin(2k\pi x), \cos(2k\pi x)\}$. The relevant integrals are calculated using adaptive Gauss-Kronrod quadrature and we have limited the number of modes to $k = 100$ to place an upper limit of 10 minutes of CPU time (on a standard dual-core processor) to calculate the Galerkin projection matrix $[\mathcal{L}'_{k,\omega}]$, representing the projected action of $\mathcal{L}_\omega$ on the first k Fourier modes. We then take a Cesàro average to construct $[\mathcal{L}_{k,\omega}] = \frac{1}{k}\sum_{j=0}^{k-1}[\mathcal{L}'_{j,\omega}]$. Estimates of $f_{k,\sigma^j\omega}, \omega = 0, j = 21, 21, 22$ are shown in Figure 2.

The invariant density estimates $f_{k,\sigma^{20}\omega}$ were created by pushing forward Lebesgue measure (at "time" $\omega = 0$) by $[\mathcal{L}_{k,\sigma^{19}\omega}] \circ \cdots \circ [\mathcal{L}_{k,\omega}]$, and then pushing two more steps for the estimates $f_{k,\sigma^{21}\omega}$ and $f_{k,\sigma^{22}\omega}$. By inspecting Figures 1 and 2, one can see how $\mathcal{L}_{\sigma^j\omega}$ transforms the estimate of $f_{\sigma^j\omega}$ to $f_{\sigma^{j+1}\omega}$, $(j = 20, 21)$, coarse features such as a change in the number of inverse branches are particularly evident. Although the pure Galerkin estimates are more oscillatory, they appear to pick up more of the finer features than the smoother Fejér kernel estimates. The Ulam estimates are likely the most accurate, given the greater dimensionality of their approximation space. In fact, for the purpose of visualization, the Ulam approximation with $k = 1000$ is expected to be very close to the true invariant density. The Fourier-based estimates converge slower in this example (relative to computing time), but numerical tests on C^∞ random maps demonstrated rapid

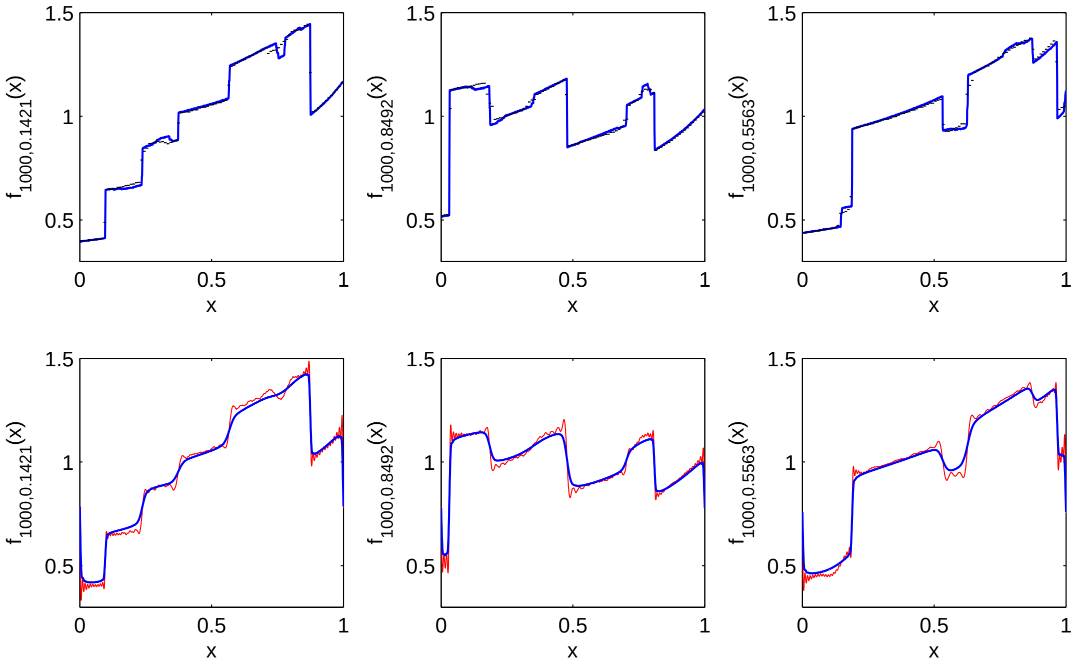

FIGURE 2. Estimates $f_{k,\sigma^j\omega}$, $\omega = 0, j = 20, 21, 22$ using Ulam's method (upper row, $k = 1000$ thick [blue, online] and $k = 100$ thin [black, online]) and Fejér kernels (lower row, thick [blue, online]). The pure Galerkin estimates using the Galerkin Fourier matrices $[\mathcal{L}'_{k,\omega}]$ are shown as thinner [red, online] curves in the lower row.

convergence, with the Fourier approach taking full advantage of the system's smoothness, to the extent that the influence of modes higher than $k = 20$ on the matrix $[\mathcal{L}'_{k,\omega}]$ was of the order of machine accuracy.

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