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A CONCISE PROOF OF THE MULTIPLICATIVE ERGODIC THEOREM ON BANACH SPACES

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ABSTRACT. We give a new proof of a multiplicative ergodic theorem for quasi-compact operators on Banach spaces with a separable dual. Our proof works by constructing the finite-codimensional 'slow' subspaces (those where the growth rate is dominated by some λ_i), in contrast with earlier infinite-dimensional multiplicative ergodic theorems which work by constructing the finite-dimensional fast subspaces. As an important consequence for applications, we are able to get rid of the injectivity requirements that appear in earlier works.

1. INTRODUCTION

The multiplicative ergodic theorem (MET) is a very powerful result in ergodic theory establishing the existence of *generalized eigenspaces* for stationary compositions of linear operators. It is of great interest in many areas of mathematics, including analysis, geometry and applications. The MET was first established by Oseledets [8] in the context of matrix cocycles. The decomposition into generalized eigenspaces is called the *Oseledets splitting*.

After the original version, the MET was proved by a different method by Raghunathan [10]. The result was subsequently generalized to compact operators on Hilbert spaces by Ruelle [11]. Mañé [7] proved a version for compact operators on Banach spaces under some continuity assumptions on the base dynamics and the dependence of the operator on the base point. Thieullen [12] extended this to quasi-compact operators. Recently, Lian and Lu [6] proved a version in the context of linear operators on separable Banach spaces, in which the continuity assumption was relaxed to a measurability condition.

We prove a non-invertible Oseledets theorem (i.e. we obtain a filtration) for a random dynamical system (the full definition is below) acting on a Banach space with separable dual. We do not make any assumption about injectivity of

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the operators, unlike most previous Banach-space valued versions of the Multiplicative Ergodic Theorem. We also prove a semi-invertible Oseledets theorem (i.e. we obtain a splitting) under the assumption that the underlying Banach space is separable and reflexive.

An important feature of the present approach is its constructive nature. Indeed, it provides a robust way of approximating the Oseledets splitting, following what could be considered a *power method* type strategy. This makes the work also relevant from an applications perspective.

The approach of this work is similar in spirit to that of Raghunathan, in that we primarily work with the 'slow Oseledets spaces'. Mañé's proof works hard to build the fast space, as do the subsequent works based on Mañé's template. These proofs rely on injectivity of the operators; some of them make use of natural extensions to extend the result to non-invertible operators – this was the strategy in [12], and it was also used by Doan in [2] to extend [6] to the non-invertible context. In contrast, we establish the non-invertible version first and recover the (semi-)invertible one, including the 'fast spaces', straightforwardly using duality. Another key simplifying feature of our method is that we prove measurability at the end of the proof, rather than working to ensure that all intermediate constructions are measurable.

While Raghunathan's proof uses singular value decomposition and hence relies on the notion of orthogonality, we study instead collections of vectors with maximal volume growth. Another important difference with Raghunathan's approach is that instead of dealing with the exterior algebra, we work with the Grassmannian. We claim this is more natural since subspaces correspond to *rank one* elements of the exterior algebra (those that can be expressed as $v_1 \wedge \ldots \wedge v_k$). In the Euclidean setting, rank one elements naturally appear as eigenvectors of $\Lambda^k(A^* A)$, but this does not seem to generalize to the Banach space case.

Section 2 analyses notions of volume growth for bounded linear maps T on a Banach space X. We establish an asymptotic equivalence between k-dimensional volume growth under T and T^*, as well as other measures of volume growth and Section 3 uses these results to obtain the multiplicative ergodic theorems. The main results in this article are Theorem 16 and Corollary 17. After submitting the current article, we learned of an independent proof of essentially the same result via closely related methods due to Blumenthal [1].

2. VOLUME CALCULATIONS IN BANACH SPACES

Let X be a Banach space with norm $\|\cdot\|$. As usual, given a non-empty subset A of X and a point $x \in X$, we define $d(x, A) = \inf_{y\in A} d(x, y)$. We denote by B_X and S_X the unit ball and unit sphere in X, respectively. The linear span of a finite collection C of vectors in X will be denoted by $\mathrm{lin}(C)$ with the convention that $\mathrm{lin}(\emptyset) = \{0\}$. The dual of X will be denoted by X^*. In this section, we study the relationships between various notions of volume and singular value for maps of Banach spaces. Other closely related notions are due to Gelfand

and Kolmogorov and are described in Pisier's book [9]. For the purposes of later sections, it will suffice to show that two quantities agree up to a bounded multiplicative factor. We make no attempt to optimize the bounds. We use the notation $Q \asymp Q'$ if the ratio of the quantities Q and Q' is bounded above and below by constants independent of the Banach space(s).

We define the *k-dimensional volume* of a collection, $(v_1,\dots,v_k)$, of vectors in a Banach space by

$$\mathrm{vol}_k(v_1,\dots,v_k) = \prod_{i=1}^{k} d(v_i, \mathrm{lin}(\{v_j : j < i\})).$$

It is easy to see that $\mathrm{vol}_k(\alpha_1 v_1,\dots \alpha_k v_k) = |\alpha_1|\dots|\alpha_k|\mathrm{vol}_k(v_1,\dots,v_k)$. In the case where the normed space is Euclidean this notion corresponds with the standard notion of k-dimensional volume. Notice that $\mathrm{vol}_k(v_1,\dots,v_k)$ is not generally invariant under permutation of the vectors.

Given a bounded linear map T from X to Y, we define $d_kT(v_1,\dots,v_k)$ to be $\mathrm{vol}_k(Tv_1,\dots,Tv_k)$ and $D_kT = \sup_{\|v_1\|=1,\dots,\|v_k\|=1} d_kT(v_1,\dots v_k)$.

LEMMA 1 (Submultiplicativity)**.** *Let $T\colon X \to Y$ and $S\colon Y \to Z$ be linear maps. Then $D_k(S\circ T) \le D_k(S)D_k(T)$.*

Proof. Let $T(v_1),\dots,T(v_k) \in X$ be linearly independent. Then one checks from the definition that for any collection of coefficients $(\alpha_{ij})_{j<i}$, the following holds

$$\mathrm{vol}_k(v_1,\dots,v_k) = \mathrm{vol}_k\left(v_1, v_2-\alpha_{21}v_1,\dots,v_k-\sum_{j<k}\alpha_{kj}v_j\right). \tag{1}$$

Since the linear spans in the definition of volume are finite-dimensional spaces, the minima are attained so that

$$\begin{aligned} &d_kT(v_1,\dots,v_k)\\ &\quad= \|T(v_1)\|\|T(v_2)-\alpha_{21}T(v_1)\|\dots\|T(v_k)-\alpha_{k1}T(v_1)-\dots-\alpha_{k,k-1}T(v_{k-1})\| \end{aligned}$$

for appropriate choices of $(\alpha_{ij})_{j<i}$.

Let $w_j = v_j - \sum_{i<j}\alpha_{ji}v_i$ so that $d_kT(v_1,\dots,v_k) = \|T(w_1)\|\dots\|T(w_k)\|$ and set $u_j = T(w_j)/\|T(w_j)\|$. Using (1), we have

$$\begin{aligned} d_k(S\circ T)(v_1,\dots,v_k) &= \mathrm{vol}_k(ST(v_1),\dots,ST(v_k))\\ &= \mathrm{vol}_k(ST(w_1),\dots,ST(w_k))\\ &= \|T(w_1)\|\dots\|T(w_k)\|\,\mathrm{vol}_k(S(u_1),\dots,S(u_k))\\ &\le d_kT(v_1,\dots,v_k)D_kS. \end{aligned}$$

Taking a supremum over $v_1,\dots,v_k$ in the unit ball of X, one obtains the bound $D_k(S\circ T) \le D_k(S)D_k(T)$ as required. □

LEMMA 2. *Let $T\colon X \to Y$ be linear. Suppose that V is a k-dimensional subspace and $\|Tx\| \ge M\|x\|$ for all $x \in V$. Then $D_kT \ge M^k$.*

Proof. Let $v_1,\dots,v_k$ belong to $V\cap S_X$ and satisfy $d(v_j, \mathrm{lin}(\{v_i : i<j\})) = 1$. Then $d_kT(v_1,\dots,v_k) \ge M^k$. □

We now proceed to compare volume estimates for a linear operator $T\colon X \to Y$ and its dual $T^*\colon Y^* \to X^*$. We introduce a third quantity to which we compare both $D_k(T)$ and $D_k(T^*)$. Given linear functionals $\theta_1,\dots,\theta_k \in Y^*$ and points $x_1,\dots,x_k \in X$, we let $U((\theta_i),(x_j))$ be the matrix with entries $U_{ij} = \theta_i(T(x_j))$ and define

$$E_k(T) = \sup\left\{\det U\big((\theta_i),(x_j)\big)\colon \|\theta_i\| = 1 \text{ and } \|x_j\| = 1 \text{ for all } i,j\right\}.$$

LEMMA 3 (Relationship between volumes for T and T^*). *For all $k > 0$, there exist positive constants c_k and C_k with the following property: for every bounded linear map T between Banach spaces X and Y,*

$$c_k D_k(T) \le D_k(T^*) \le C_k D_k(T).$$

Proof. The statement will follow from the following inequalities:

$$D_k(T) \le E_k(T) \le k! D_k(T) \tag{2}$$

$$D_k(T^*) \le E_k(T) \le k! D_k(T^*). \tag{3}$$

The second inequality of (2) is proved as follows. Let $x_1,\dots,x_k$ and $\theta_1,\dots,\theta_k$ all be of norm 1 in X and Y^* respectively. Let $\alpha_j = d(Tx_j, \mathrm{lin}(Tx_1,\dots,Tx_{j-1}))$. Let $c_1^j,\dots,c_{j-1}^j$ be chosen so that $\|Tz_j\| = \alpha_j$, where z_j is defined by $z_j = x_j - (c_1^j x_1 + \dots + c_{j-1}^j x_{j-1})$. Note that $U' = U((\theta_i),(z_j))$ may be obtained from $U = U((\theta_i),(x_j))$ by column operations that leave the determinant unchanged. Notice also that $|U'_{ij}| = |\theta_i(Tz_j)| \le \alpha_j$. From the definition of a determinant, we see that $\det U = \det U' \le k!\alpha_1 \dots \alpha_k$. This inequality holds for all choices of θ_i in the unit sphere of Y^*. Now, maximizing over choices of x_j in the unit sphere of X, we obtain the desired result.

The second inequality of (3) may be obtained analogously. We let

$$\beta_i = d(T^*\theta_i, \mathrm{lin}(T^*\theta_1,\dots,T^*\theta_{i-1}))$$

and choose linear combinations ϕ_i of the θ_i for which the minimum is obtained. The matrix $U'' = U((\phi_i),(x_j))$ is obtained by row operations from U and the $|U''_{i,j}| = |\phi_i(Tx_j)| = |(T^*\phi_i)(x_j)| \le \beta_i$.

To show the first inequality of (2), fix $x_1,\dots,x_k$ of norm 1. As before, let $\alpha_j = d(Tx_j, \mathrm{lin}(Tx_1,\dots,Tx_{j-1}))$. By the Hahn-Banach theorem, there exist linear functionals $(\theta_i)_{i=1}^k$ in S_{Y^*} such that $\theta_i(Tx_i) = \alpha_i$ and $\theta_i(x_k) = 0$ for all $k < i$. Now

$$\det U((\theta_i),(x_j)) = \prod \alpha_i.$$

Maximizing over the choice of (x_j), we obtain $E_k(T) \ge D_k(T)$ as required.

Finally, for the first inequality of (3), we argue as follows. Let $\epsilon > 0$ be arbitrary and let $\theta_1,\dots,\theta_k$ belong to the unit sphere of Y^*. We may assume that $T^*\theta_1,\dots,T^*\theta_k$ are linearly independent – otherwise the inequality is trivial. Let $\phi_i = T^*\theta_i - \sum_{k<i} a_{ik} T^*\theta_k$ be such that $\|\phi_i\| = d(T^*\theta_i, \mathrm{lin}(\{T^*\theta_k\colon k < i\}))$. We shall pick $x_1,\dots,x_k$ inductively in such a way that $|\det((\phi_i(x_j))_{i,j\le l})|$ is at least $\prod_{i=1}^l (\|\phi_i\| - \epsilon)$ for each $1 \le l \le k$. Suppose $x_1,\dots,x_{l-1}$ have been chosen. Then since $\det((\phi_i(x_j))_{i,j<l})$ is non-zero, the rows span $\mathbb{R}^{l-1}$. Hence there exist $(b_i)_{i<l}$

such that $\psi_l := \phi_l + \sum_{i<l} b_i\phi_i$ satisfies $\psi_l(x_j) = 0$ for all $j < l$. By assumption, $\|\psi_l\| \geq \|\phi_l\|$. Pick $x_l \in S_X$ such that $\psi_l(x_l) > \|\psi_l\| - \epsilon$. Then the matrix with a row for ψ_l and a column for x_l adjoined has determinant of absolute value at least $\prod_{i=1}^{l}(\|\phi_i\| - \epsilon)$. The matrix with ϕ_l replacing ψ_l has the same determinant, completing the induction. Maximizing over the choice of $(\theta_j)_{j\leq k}$, letting ϵ shrink to 0, and observing that $\det((\phi_i(x_j))_{i,j\leq k}) = \det((T^*\theta_i(x_j))_{i,j\leq k})$ completes the proof. □

A fourth quantity that will play a crucial role in what follows is $F_k(T)$, defined as

$$F_k(T) = \sup_{\dim(V)=k} \inf_{v\in V\cap S_X} \|Tv\|.$$

We make use of the following lemma due to Gohberg and Krein whose proof may be found in Kato's book [5] (Chapter 4, Lemma 2.3).

LEMMA 4 (Gohberg and Krein)**.** *Let V_1 be a proper finite-dimensional subspace of a subspace V_2 of a Banach space, X. Then there exists $v \in V_2 \setminus \{0\}$ such that $d(v, V_1) = \|v\|$.*

LEMMA 5 (Relation between determinants and F_k)**.** *Let T be a bounded linear map from a Banach space X to a Banach space Y. Then*

$$E_{k-1}(T)F_k(T) \leq E_k(T) \leq k2^{k-1}E_{k-1}(T)F_k(T).$$

Proof. We first show $E_k(T) \leq k2^{k-1}E_{k-1}(T)F_k(T)$. We may assume $E_k(T) > 0$ as otherwise the inequality is trivial. Let $\theta_1, \ldots, \theta_k$ be elements of the unit sphere of X^* and $x_1, \ldots, x_k$ be elements of the unit sphere of X. Let U be the matrix with entries $\theta_i(Tx_j)$. Assume that $\det U \neq 0$. Since $x_1, \ldots, x_k$ span a k-dimensional space, there exists a $v = a_1x_1 + \ldots + a_kx_k$ of norm 1 such that $\|Tv\| \leq F_k(T)$. By the triangle inequality, one of the $|a|$'s, say $|a_{j_0}|$, must be at least $\frac{1}{k}$. Let $\tilde{x}_j = x_j$ for $j \neq j_0$ and $\tilde{x}_{j_0} = v$ and set $\tilde{U}$ to be the matrix with entries $\theta_i(T\tilde{x}_j)$. By properties of determinants, we see $|\det\tilde{U}| = |a_{j_0}||\det U| \geq \frac{1}{k}|\det U|$. Next, there exists i_0 for which $|\theta_{i_0}(Tv)|$ is maximal, this maximum not being 0 since $|\det\tilde{U}|$ is positive. Let $\bar{\theta}_i = \theta_i - (\theta_i(Tv)/\theta_{i_0}(Tv))\theta_{i_0}$ for $i \neq i_0$ and $\bar{\theta}_{i_0} = \theta_{i_0}$, so that $\|\bar{\theta}_i\| \leq 2$ and $\bar{\theta}_i(Tv) = 0$ for $i \neq i_0$.

Now let $\bar{U}_{ij} = \bar{\theta}_i(T\tilde{x}_j)$, so that $|\det U| \leq k|\det\tilde{U}| = k|\det\bar{U}|$. Finally, the j_0th column of $\bar{U}$ has a single non-zero entry that is at most $\|Tv\| \leq F_k(T)$ in absolute value. The absolute value of the cofactor is $\left|\det\left(\bar{\theta}_i(T(\tilde{x}_j))\right)_{i\neq i_0,\ j\neq j_0}\right| \leq 2^{k-1}E_{k-1}(T)$. Taking a supremum over choices of (θ_i) and (x_j), we have shown $E_k(T) \leq k2^{k-1}E_{k-1}(T)F_k(T)$.

For the other inequality, we may suppose that T has kernel of codimension at least k, otherwise $F_k(T) = 0$ and there is nothing to prove. Let $\theta_1, \ldots, \theta_{k-1}$ and $x_1, \ldots, x_{k-1}$ be arbitrary. Let Δ be the determinant of the matrix with entries $\theta_i(Tx_j)$. Let V be a k-dimensional subspace such that $V \cap \ker T = \{0\}$. Let $W = \mathrm{lin}(Tx_1, \ldots, Tx_{k-1})$. Using Lemma 4, let z be a point in the unit sphere of $T(V)$ such that $d(z, W) = 1$. Let $v \in V \cap S_X$ be such that $T(v)$ is a multiple of z. Let θ_k be a linear functional of norm 1 such that $\theta_k|_W = 0$ and $\theta_k(z) = 1$ and let $x_k = v$.

Now forming the $k \times k$ matrix $\big(\theta_i(x_j)\big)_{1\le i,j\le k}$, we see the absolute value of the determinant is $\Delta\cdot\theta_k(Tv) = \Delta\cdot\|Tv\| \ge \Delta\cdot\inf_{x\in V\cap S_X}\|Tx\|$. Taking suprema over choices of x's, θ's and k-dimensional V's, we see that $E_k(T) \ge F_k(T)E_{k-1}(T)$ as required. □

COROLLARY 6 (of Lemmas 3 and 5)**.** *For each $k>0$, the quantities $D_k(T)$, $D_k(T^*)$, $E_k(T)$ and $\prod_{i\le k} F_i(T)$ agree up to multiplicative factors that may be bounded by constants independent of the bounded linear map T and the Banach spaces X and Y. Further, $F_i(T)$ and $F_i(T^*)$ agree up to a uniformly bounded multiplicative factor.*

We comment that besides these approximate Banach space versions of singular values, additional related quantities are given by Gelfand numbers and Kolmogorov numbers (see the book of Pisier [9] for more information). It can be checked that these quantities also agree with the sequence of F_i's up to bounded multiplicative factors (dependent on i, but independent of X and T).

By definition, for each natural number k, one can find sequences $(\theta_i)_{i\le k}$ and $(x_j)_{j\le k}$ such that $\det(U((\theta_i),(x_j))) \asymp E_k(T)$. We now show that we can find infinite sequences (θ_i) and (x_j) so that, for each k, $\det(U((\theta_i)_{i\le k},(x_j)_{i\le k})) \asymp E_k(T)$.

LEMMA 7 (Existence of consistent sequences)**.** *Let X and Y be infinite-dimensional Banach spaces. For any linear map $T\colon X\to Y$, there exist $(\theta_i)_{i\ge1}$ in S_{Y^*} and $(x_j)_{j\ge1}$ in S_X such that for all k,*

$$\det\big((\theta_i(Tx_j))_{1\le i,j\le k}\big) \ge \frac{1}{2^k}\prod_{i\le k} F_i(T) \quad \textit{and}$$
$$\|Tx\| \ge 4^{-k}F_k(T)\|x\| \quad \textit{for all } x\in \operatorname{lin}(x_1,\dots,x_k).$$

Proof. The proof is by induction: suppose $(\theta_i)_{i<k}$ and $(x_j)_{j<k}$ have been chosen and satisfy the desired inequalities at stage $k-1$. Then pick an arbitrary k-dimensional space V such that $\|Tv\| \ge \frac12 F_k(T)\|v\|$ for all $v\in V$. Using Lemma 4, let $x_k\in V\cap S_X$ be such that $d(Tx_k,\operatorname{lin}(Tx_1,\dots,Tx_{k-1})) = \|Tx_k\|$. Finally choose θ_k of norm 1 such that $\theta_k(Tx_i)=0$ for $i<k$ and $\theta_k(Tx_k)=\|Tx_k\|$. The determinant inequality at stage k follows.

Let $x = a_1x_1+\ldots+a_kx_k$ be of norm 1. Then

$$\|Tx\| \ge |a_k|d(Tx_k,\operatorname{lin}(Tx_1,\dots,Tx_{k-1})) = |a_k|\|Tx_k\| \ge |a_k|F_k(T)/2. \tag{4}$$

Also,

$$\|Tx\| \ge \Big\|T(\textstyle\sum_{j<k} a_jx_j)\Big\| - |a_k|\|Tx_k\|.$$

Averaging the inequalities, we get

$$\|Tx\| \ge \tfrac12\Big\|T(\textstyle\sum_{j<k} a_jx_j)\Big\|. \tag{5}$$

If $|a_k| > \frac12$, (4) yields $\|Tx\| \ge \frac14 F_k(T)$. If $|a_k|\le\frac12$, then $\|\sum_{j<k}a_jx_j\| \ge \frac12$, and (5) combined with the inductive hypothesis gives $\|Tx\| \ge \frac14 4^{-(k-1)}F_{k-1}(T)$. □

LEMMA 8 (Lower bound on volume growth in a subspace of finite codimension)**.** *For any natural numbers $k > m$, there exists C_k such that if X, Y are Banach spaces, $T: X \to Y$ is a linear map and V is a closed subspace of X of codimension m, then $D_k(T) \le C_k D_m(T) D_{k-m}(T|_V)$.*

Proof. Let $\epsilon > 0$. Let P be a projection from X to V of norm at most $\sqrt{m}+\epsilon$ (such a projection exists by Corollary III.B.11 in the book of Wojtaszczyk [13]). Then, $\|I - P\| \le \sqrt{m}+\epsilon+1$. Let $x_1,\dots,x_k$ be a sequence of vectors in X of norm 1. The proof of Lemma 3 shows that there exist $\psi_1,\dots,\psi_k$ in S_{Y^*} such that $\det(\psi_i(x_j)) \ge d_k T(x_1,\dots,x_k)$. Write P_1 for P and P_0 for $I-P$, which has m-dimensional range. There exists a choice $\epsilon_1,\dots\epsilon_k \in \{0,1\}^k$ such that

$$|\det(\psi_i(Tx_j))| > 2^{-k} d_k T(x_1,\dots,x_k),$$

by multilinearity of the determinant. At most m of the ϵ_j can be 0, as otherwise more than m vectors lie in a common m-dimensional space, so that at least $k-m$ of them lie in V. Hence, there exist vectors $z_1,\dots,z_m$ in S_X and $z_{m+1},\dots,z_k$ in $S_X \cap V$ such that

$$|\det(\psi_i(Tz_j))| \ge (2(\sqrt{m}+\epsilon+1))^{-k} d_k T(x_1,\dots,x_k).$$

Using the proof of Lemma 3 again, we deduce that

$$\begin{aligned} d_m T(z_1,\dots,z_m) d_{k-m} T(z_{m+1},\dots,z_k) &\ge d_k T(z_1,\dots,z_k) \\ &\ge (2(\sqrt{m}+\epsilon+1))^{-k}/(k!)\, d_k T(x_1,\dots,x_k). \end{aligned}$$

This completes the proof. □

3. RANDOM DYNAMICAL SYSTEMS

A closed subspace Y of X is called *complemented* if there exists a closed subspace Z such that X is the direct sum of Y and Z, written $X = Y \oplus Z$. That is, for every $x \in X$, there exist $y \in Y$ and $z \in Z$ such that $x = y + z$, and this decomposition is unique. The Grassmannian $\mathcal{G}(X)$ is the set of closed complemented subspaces of X. We equip $\mathcal{G}(X)$ with the metric $d(Y,Y') = d_H(Y \cap S_X, Y' \cap S_X)$ where d_H denotes the Hausdorff distance. We denote by $\mathcal{G}^k(X)$ the collection of closed k-codimensional subspaces of X (these are automatically complemented), by $\mathcal{G}_k(X)$ the k-dimensional subspaces of X. If U and V are closed subspaces of X such that $U \oplus V = X$, then $\mathrm{Proj}_{U\|V}$ is the projection onto U parallel to V (that is $\mathrm{Proj}_{U\|V}(x) \in U$ and $x - \mathrm{Proj}_{U\|V}(x) \in V$). We record some facts about Grassmannians in the following lemma.

LEMMA 9. *Let X be a Banach space with separable dual. Let $k \in \mathbb{N}$. The following facts hold:*

1. *$\mathcal{G}^k(X)$ is complete and separable.*
2. *If $V \in \mathcal{G}(X)$, $W \in \mathcal{G}(X)$ and $V \oplus W = X$, then $\frac{1}{\delta} \le \|\mathrm{Proj}_{V\|W}\| \le \frac{2}{\delta}$, where $\delta = \inf_{x\in V\cap S_X, y\in W\cap S_X} \|x - y\|$.*
3. *There exists $K > 0$ (independent of X) such that if $V \in G^k(X)$, there exists a subspace $W \in \mathcal{G}_k(X)$ such that $\|\mathrm{Proj}_{W\|V}\| \le K$ and $\|\mathrm{Proj}_{V\|W}\| \le K$.*

4. *(Symmetry of closeness) There exists* $K > 0$ *such that if* $V, V' \in \mathscr{G}^k(X)$*, then*

$$\sup_{v' \in V' \cap S_X} \inf_{v \in V \cap S_X} \|v - v'\| \le K \sup_{v \in V \cap S_X} \inf_{v' \in V' \cap S_X} \|v - v'\|.$$

Proof. The map $\perp : \mathscr{G}^k(X) \to \mathscr{G}_k(X^*)$ defined by $V^\perp = \{\theta \in X^* : \theta|_V = 0\}$ is a bi-Lipschitz bijection [5]. Separability of $\mathscr{G}^k(X)$ and symmetry of closeness are proved in [4]. The completeness is stated but not proved in Kato's book. We sketch a proof using results from the appendix of [4].

Let V be a k-dimensional subspace of a Banach space Z and let $v_1 \dots, v_k$ be an Auerbach basis. By the Hahn-Banach theorem, there exist $\theta_1, \dots, \theta_k \in Z^*$ of norm 1 such that $\theta_i(v_j) = \delta_{ij}$. Now if $\tilde v_1, \dots, \tilde v_k$ in Z satisfy $\|\tilde v_i - v_i\| < \epsilon/k$ for each k, then one has $\|\sum a_i \tilde v_i\| \ge (1-\epsilon)\max|a_i|$ (to see this, apply θ_{i_0} where $|a_{i_0}| = \max|a_i|$). From this, we see that $\tilde v_1, \dots, \tilde v_k$ is an ϵ-nice basis (as defined in [4]). Now let (V_n) be a Cauchy sequence in $\mathscr{G}_k(Z)$. By refining the sequence, one may assume $d(V_n, V_{n+1}) < (3k)^{-n}$. Choosing an Auerbach basis $v_1^1, \dots, v_k^1$ for V_1, one may then obtain elements $v_1^n, \dots, v_k^n$ of V_n satisfying $\|v_i^{n+1} - v_i^n\| < (2k+1)^{-n}$. This is a convergent sequence of $\frac{1}{2}$-nice bases. Letting v_i^* be the limit of v_i^n, Corollary B6 of [4] shows that $d(V_n, V_*) \to 0$, where V_* is the subspace spanned by the v_i^*. This establishes completeness of $\mathscr{G}_k(X^*)$ and hence completeness of $\mathscr{G}^k(X)$. To see (2), if $v_n \in V \cap S_X$ and $w_n \in W \cap S_X$, satisfy $\|v_n - w_n\| \to \delta$ then $\|\mathrm{Proj}_{V\|W}(v_n - w_n)\| = 1$ shows the first inequality. For the second inequality, let $v \in V \cap S_X$. If $1 - \frac{\delta}{2} < \|w\| < 1 + \frac{\delta}{2}$, then $\|v + w\| \ge \|v + \frac{\|v\|}{\|w\|} w\| - |\|v\| - \|w\|| \ge \frac{\delta}{2}$. If $\|w\|$ lies outside this range, then the same conclusion follows from the triangle inequality, so that $\|\mathrm{Proj}_{V\|W}(v + w)\| = \|v\| \le \frac{2}{\delta}\|v + w\|$. (3) can be found in [13], Corollary III.B.11. $\square$

For a Banach space X, the bounded linear maps from X to itself will be written $B(X, X)$ and $\mathscr{B}_X$ will be the Borel σ-algebra on X. In this section, we consider *random dynamical systems*. These consist of a tuple $\mathscr{R} = (\Omega, \mathscr{F}, \mathbb{P}, \sigma, X, \mathscr{L})$, where $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space; σ is a measure preserving transformation of Ω; X is a separable Banach space; the *generator* $\mathscr{L} : \Omega \to B(X, X)$ is strongly measurable (that is for fixed $x \in X$, $\omega \mapsto \mathscr{L}_\omega x$ is $(\mathscr{F}, \mathscr{B}_X)$-measurable); and $\log\|\mathscr{L}_\omega\|$ is integrable. An alternative description of strong measurability is that the map $\omega \mapsto \mathscr{L}_\omega$ is $(\mathscr{F}, \mathscr{S})$-measurable, where $\mathscr{S}$ is the Borel σ-algebra of the strong operator topology on $B(X, X)$ (see Appendix A of [4] for details). In the context where X is separable and the operators are bounded, strong measurability is equivalent to $(\mathscr{F} \otimes \mathscr{B}_X, \mathscr{B}_X)$-measurability of the map $(\omega, x) \mapsto \mathscr{L}_\omega x$ ([4]).

A random dynamical system gives rise to a cocycle of bounded linear operators $\mathscr{L}_\omega^{(n)}$ on X, defined by $\mathscr{L}_\omega^{(n)}(x) = \mathscr{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathscr{L}_\omega x$. We will consider $\mathscr{F}$ and $\mathbb{P}$ to be fixed, and thus refer to a random dynamical system as $\mathscr{R} = (\Omega, \sigma, X, \mathscr{L})$. We say $\mathscr{R}$ is ergodic if σ is ergodic.

When the base σ is invertible, we can also define the dual random dynamical system $\mathscr{R}^* = (\Omega, \mathscr{F}, \mathbb{P}, \sigma^{-1}, X^*, \mathscr{L}^*)$, where X^* is the dual of X and $\mathscr{L}_\omega^*(\theta) := (\mathscr{L}_{\sigma^{-1}\omega})^*\theta$. Notice that $\mathscr{L}_\omega^*$ is *not* $(\mathscr{L}_\omega)^*$. The rationale for this is that $\mathscr{L}_\omega$ maps the X-fibre over ω to the X-fibre over $\sigma(\omega)$ and similarly $\mathscr{L}_\omega^*$ maps the X^*-fibre over ω to the X^*-fibre over $\sigma^{-1}\omega$. In this way, $\theta(\mathscr{L}_\omega x) = \mathscr{L}_{\sigma\omega}^*\theta(x)$ and, more generally, $\mathscr{L}_{\sigma^n\omega}^{*(n)}\theta(x) = \theta(\mathscr{L}_\omega^{(n)}x)$ for every $x \in X, \theta \in X^*$. Thus, $\mathscr{L}_{\sigma^n\omega}^{*(n)} = (\mathscr{L}_\omega^{(n)})^*$.

LEMMA 10 (Measurable dense subset of a family of subspaces)**.** *Let X be a separable Banach space. Let $V: \Omega \to \mathscr{G}^k(X)$ be measurable. Then there exist sequences of measurable functions $u_n: \Omega \to S_X$ and $u'_n: \Omega \to B_X$ such that $\{u_n(\omega): n \in \mathbb{N}\}$ is a dense subset of $V(\omega) \cap S_X$ and $\{u'_n(\omega): n \in \mathbb{N}\}$ is a dense subset of $V(\omega) \cap B_X$.*

Proof. First, for fixed $v \in X, \omega \mapsto d(v, V(\omega))$ is a measurable function, as it is the composition of continuous and measurable functions. Fix a dense sequence $v_1, v_2, \ldots \in S_X$. Now for each j, set $u_j^0(\omega) = v_j$, and let $u_j^{k+1}(\omega) = v_l$, where

$$\begin{aligned} l = \min\big\{m : d(v_m, V(\omega) \cap S_X) \le \tfrac{1}{4} d(u_j^k(\omega), V(\omega) \cap S_X) \text{ and} \\ d(v_m, u_j^k(\omega)) \le 2d(u_j^k(\omega), V(\omega) \cap S_X)\big\}. \end{aligned}$$

For each j, this is a measurable convergent sequence and hence the limit point $u_j^\infty(\omega)$ is measurable, and belongs to $V(\omega) \cap S_X$. The sequence $(u_j^\infty(\omega))$ is dense in $V(\omega) \cap S_X$ because there are v_j arbitrarily close to all points of $V(\omega) \cap S_X$. The functions u'_n are produced exactly analogously. □

LEMMA 11 (Measurability of growth measurements)**.** *Let $\mathscr{R}$ be a random dynamical system $\mathscr{R} = (\Omega, \sigma, X, \mathscr{L})$ acting on a separable Banach space. The following functions are measurable:*

- $\omega \mapsto D_k(\mathscr{L}_\omega)$;
- $\omega \mapsto \|\mathscr{L}_\omega\|$;
- $\omega \mapsto \alpha(\mathscr{L}_\omega) := \inf\big\{r > 0 : \mathscr{L}_\omega(B_X)$ *can be covered by finitely many balls of radius* $r\big\}$.

Further, if $V: \Omega \to \mathscr{G}^k(X)$ is measurable, then $\omega \mapsto \|\mathscr{L}_\omega|_{V(\omega)}\|$ is measurable.

Proof. Let (x_n) be a dense subsequence of B_X. By strong measurability, for each fixed n, $\omega \mapsto \|\mathscr{L}_\omega x_n\|$ is measurable. Then for each $j_1, \ldots, j_i$, we have that $f_{j_i|j_1,\ldots,j_{i-1}}(\omega) := \inf_{q_1,\ldots,q_{i-1}\in\mathbb{Q}} \|\mathscr{L}_\omega x_{j_i} - \sum_{1\le l<i} q_l \mathscr{L}_\omega x_{j_l}\|$ is measurable, so

$$D_k(\mathscr{L}_\omega) = \sup_{j_1,\ldots,j_k} \prod_{i\le k} f_{j_i|j_1,\ldots,j_{i-1}}(\omega)$$

is measurable. In particular, $\omega \mapsto \|\mathscr{L}_\omega\| = D_1(\mathscr{L}_\omega)$ is measurable. We claim that

$$\alpha(\mathscr{L}) = \lim_{n\to\infty} \sup_j \inf_{k\le n} \big\|\mathscr{L} x_j - 2\|\mathscr{L}\| x_k\big\|. \tag{6}$$

If this limit is r, then there exists n such that $\sup_j \inf_{k\le n} \big\|\mathscr{L} x_j - 2\|\mathscr{L}_\omega\| x_k\big\| < r + \epsilon$. This gives a covering of $\{\mathscr{L} x_j : j \in \mathbb{N}\}$ by n balls of radius $r + \epsilon$, so that the left side of (6) is dominated by the right side. Conversely, if $\alpha(\mathscr{L}) = r$, let $\mathscr{L}(B_X)$ be covered by finitely many balls of radius $r + \epsilon$. These must have centres with norm at most $2\|\mathscr{L}\|$ otherwise they do not intersect $\mathscr{L}(B_X)$. The centres must therefore be ϵ-approximable by points of the form $2\|\mathscr{L}\| x_k$, so that the right side of (6) is at most $r + 2\epsilon$. We deduce (6) holds and $\omega \mapsto \alpha(\mathscr{L}_\omega)$ is measurable.

Finally, if $V:\Omega\to\mathscr{G}^k(X)$ is measurable, let $(u_n(\omega))_{n\in\mathbb{N}}$ be a sequence of measurable functions such that $\{u_n(\omega)\colon n\in\mathbb{N}\}$ is dense in $S_{V(\omega)}$. Then $\|\mathscr{L}_\omega|_{V(\omega)}\| = \sup_n \|\mathscr{L}_\omega u_n(\omega)\|$, which is therefore measurable. $\square$

When $\mathscr{R}$ is ergodic, Lemma 11 combined with Kingman's sub-additive ergodic theorem ensures the existence of the *maximal Lyapunov exponent* of $\mathscr{R}$, defined by

$$\lambda(\mathscr{R}) := \lim_{n\to\infty}\tfrac{1}{n}\log\|\mathscr{L}_\omega^{(n)}\|,$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$. Similarly, using the fact that the Kuratowski index of compactness, $\alpha(\mathscr{L})$, is also sub-multiplicative and bounded above by the norm, we have existence of the *index of compactness* of $\mathscr{R}$, defined by

$$\kappa(\mathscr{R}) := \lim_{n\to\infty}\tfrac{1}{n}\log\alpha(\mathscr{L}_\omega^{(n)}),$$

with the property that $\kappa(\mathscr{R})\le\lambda(\mathscr{R})$.

In the case where $\mathscr{L}_\omega$ is independent of ω, $\lambda(\mathscr{R})$ and $\kappa(\mathscr{R})$ are the spectral radius and essential spectral radius respectively, so that $\kappa(\mathscr{R})<\lambda(\mathscr{R})$ is the quasi-compact case. If the operator is compact, then $\kappa(\mathscr{R})$ is 0.

Our previous paper [4] studies the case in which $\mathscr{R}$ is a random dynamical system where the operators $\mathscr{L}_\omega$ are Perron-Frobenius operators of a family of expanding maps and gives sufficient conditions for $\kappa(\mathscr{R})<\lambda(\mathscr{R})$.

LEMMA 12. *Given an ergodic random dynamical system $\mathscr{R}$, there exist constants $\Delta_k=\Delta_k(\mathscr{R})$ such that for almost every $\omega\in\Omega$,*

$$\lim_{n\to\infty}\tfrac{1}{n}\log D_k(\mathscr{L}_\omega^{(n)}) = \Delta_k.$$

Furthermore, $\frac{1}{n}\log E_k(\mathscr{L}_\omega^{(n)})\to\Delta_k$. Define $\Delta_0=0$ and let $\mu_k=\Delta_k-\Delta_{k-1}$ for each $k\ge 1$. Then, $\frac{1}{n}\log F_k(\mathscr{L}_\omega^{(n)})\to\mu_k$.

Proof. The first claim follows from Kingman's sub-additive ergodic theorem, via Lemma 11 and Lemma 1. The remaining two claims are consequences of Corollary 6. $\square$

The μ_k's of the previous lemma are called the *Lyapunov exponents* of $\mathscr{R}$. When $\mu_k>\kappa(\mathscr{R})$, μ_k is called an *exceptional Lyapunov exponent*.

THEOREM 13 (Lyapunov exponents and index of compactness)**.** *Let $\mathscr{R}$ be a random dynamical system with ergodic base acting on a separable Banach space X. Then*

- *$\mu_1\ge\mu_2\ge\dots$;*
- *For any $\rho>\kappa(\mathscr{R})$, there are only finitely many exponents that exceed ρ;*
- *If σ is invertible, then $\mathscr{R}$ and $\mathscr{R}^*$ have the same Lyapunov exponents.*

Proof. That the μ_i are decreasing follows from Lemma 12 and the observation that $F_k(T)\le F_{k-1}(T)$. That the system and its dual have the same exponents follows from Lemma 3 together with the simple result (in [3, Lemma 8.2]) that if (f_n) is sub-additive and satisfies $f_n(\omega)/n\to A$ almost everywhere, then one has $f_n(\sigma^{-n}\omega)/n\to A$ also.

It remains to show that for $\rho > \kappa$, the system has at most finitely many exponents that exceed ρ. Let $\kappa < \alpha < \beta < \rho$. Since $\log\|\mathscr{L}_\omega\|$ is integrable, there exists a $0 < \delta < (\beta - \alpha)/2|\alpha|$ such that if $\mathbb{P}(E) < \delta$, then $\int_E \log^+ \|\mathscr{L}_\omega\|\, d\mathbb{P}(\omega) < (\beta - \alpha)/2$. By the sub-additive ergodic theorem, there exists $L > 0$ such that $\mathbb{P}(\alpha(\mathscr{L}_\omega^{(L)}) \geq e^{\alpha L}) < \delta/2$. If $\alpha(\mathscr{L}_\omega^{(L)}) < e^{\alpha L}$, then by definition, $\mathscr{L}_\omega^{(L)} B_X$ may be covered by finitely many balls of size $e^{\alpha L}$. By linearity, if $\alpha(A) = \zeta$, one sees that if B is a ball with arbitrary centre and radius ρ, then $A(B)$ may be covered by finitely many balls of size $\zeta\rho$.

Let r be chosen large enough so that $\mathbb{P}(G) > 1 - \delta$, where G (the good set) is defined by

$$G = \left\{\omega : \mathscr{L}_\omega^{(L)} B_X \text{ may be covered with } e^{rL} \text{ balls of size } e^{\alpha L}\right\}.$$

We split the orbit of ω into blocks: if $\sigma^i\omega \in G$, then the block length is L; otherwise, if $\sigma^i\omega$ is bad, we take a block of length 1. Consider the following iterative process: start with a ball of radius $\rho_0 = 1$. Then look at the current iterate of ω, $\sigma^i\omega$, and suppose that $\mathscr{L}_\omega^{(i)} B_X$ is covered by N_i balls of radius ρ_i. If $\sigma^i\omega \in G$, then $\mathscr{L}_\omega^{(i+L)} B_X$ is covered by at most $N_{i+L} = N_i e^{rL}$ balls of radius $\rho_{i+L} = e^{\alpha L}\rho_i$ and the new iterate is $\sigma^{i+L}\omega$. If $\sigma^i\omega \notin G$, then $\mathscr{L}_\omega^{(i+1)} B_X$ is covered by at most $N_{i+1} = N_i$ balls of radius $\rho_{i+1} = \|\mathscr{L}_{\sigma^i\omega}\|\rho_i$ and the new iterate is $\sigma^{i+1}\omega$.

We claim that for almost all ω, for sufficiently large N, $\mathscr{L}_\omega^{(N)}(B_X)$ is covered by at most e^{rN} balls of size $e^{\beta N}$. Indeed, given ω, let n_0 be chosen such that for all $N \geq n_0$, one has $\sum_{i=0}^{N-1} \mathbf{1}_{G^c}(\sigma^i\omega)\log^+ \|\mathscr{L}_{\sigma^i\omega}\| < (\beta - \alpha)N/2$. If $\alpha \geq 0$, then for large N, through the good steps, the balls are inflated by a factor at most $e^{\alpha N}$. If $\alpha < 0$, then combining the good blocks, the balls are scaled by a factor of $e^{\alpha(1-\delta)N} < e^{(\alpha+\beta)N/2}$ or smaller. In both cases, we see that overall, balls are scaled by at most $e^{\beta N}$. The splitting only takes place in the good blocks, and yields at most e^{rN} balls.

Now suppose that $\mu_k > \rho$. For almost all ω, we have that for all large N, $D_k(\mathscr{L}_\omega^{(N)}) > e^{kN\rho}$. Fix such an N, and suppose that $x_1, \ldots, x_k$ belong to S_X and have the property $\prod_{i \leq k} D_i > e^{kN\rho}$, where

$$D_i = d\big(\mathscr{L}_\omega^{(N)} x_i, \operatorname{lin}(\{\mathscr{L}_\omega^{(N)} x_j :\ j < i\})\big).$$

Let $T_i = \{0, 1, \ldots, \lfloor D_i/(2ke^{\beta N})\rfloor\}$ and notice that $|T_1 \times \cdots \times T_k| \geq e^{kN(\rho-\beta)}/(2k)^k$. For $(j_1, \ldots, j_k) \in T_1 \times \cdots \times T_k$, define

$$y_{j_1,\ldots,j_k} = \sum_{i=1}^{k} \frac{2j_i e^{\beta N}}{D_i} \mathscr{L}_\omega^{(N)} x_i.$$

It is not hard to see that all of these points belong to the image of the unit ball of X under $\mathscr{L}_\omega^{(N)}$. Further, from the definition of D_i, one can check that these points are mutually separated by at least $2e^{\beta N}$, so that one requires at least $e^{kN(\rho-\beta)}/(2k)^k$ balls of radius $e^{\beta N}$ to cover $\mathscr{L}_\omega^{(N)}(B_X)$.

Hence we obtain

$$\frac{e^{kN(\rho-\beta)}}{(2k)^k} \leq e^{rN}.$$

Since this holds for all large N, we deduce $k \le r/(\rho-\beta)$ as required. □

LEMMA 14 (Measurability II). *Suppose that X is a Banach space with separable dual. Suppose further that $\mathcal{R}$ is an ergodic random dynamical system acting on X. Assume there exist $\lambda' > \lambda \in \mathbb{R}$ and $d \in \mathbb{N}$ such that for $\mathbb{P}$-almost every ω, there is a closed d-codimensional subspace $V(\omega)$ of X such that:*

1. *For all $v \in V(\omega)$, $\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} v\| \le \lambda$;*
2. *For each $a > 0$ and $\epsilon > 0$, there is an n_0 such that for $v \in S_X$ satisfying $d(v, V(\omega)) > a$, one has $\|\mathcal{L}_\omega^{(n)} v\| \ge e^{n(\lambda'-\epsilon)}$ for all $n \ge n_0$.*

Then $\omega \mapsto V(\omega)$ is measurable.

Proof. Given $V \in \mathcal{G}^d(X)$, fix $w_1, \dots, w_d$ such that $V \oplus \operatorname{lin}(w_1, \dots, w_d) = X$ and define a neighbourhood of V by

$$N_{V,k} = \big\{U \in \mathcal{G}^d(X) \colon U \cap \operatorname{lin}(w_1, \dots, w_d) = \{0\} \text{ and } \\ \|\operatorname{Proj}_{\operatorname{lin}(w_i)\|U\oplus\operatorname{lin}(\{w_j\colon j\ne i\})}|_V\| \le \tfrac{1}{k} \text{ for } 1 \le i \le d\big\}.$$

Since $\mathcal{G}^d(X)$ is separable, fix a countable sequence (V_n) of subspaces, dense in $\mathcal{G}^d(X)$. Using Lemma 9 items (3) and (4), there exists $K > 0$ such that for each $V \in \mathcal{G}^d(X)$, there exists $W \in \mathcal{G}_d(X)$ such that $\|\operatorname{Proj}_{W\|V}\| \le K$, $\|\operatorname{Proj}_{V\|W}\| \le K$. If $w_1, \dots, w_d$ is an Auerbach basis for W, then $\|\operatorname{Proj}_{\operatorname{lin}(w_i)\|\operatorname{lin}(\{w_j\colon j\ne i\})}|_W\| = 1$. For each V_n, let W_n be a subspace satisfying the above inequalities and let $w_{n,1}, \dots, w_{n,d}$ be an Auerbach basis. Let $P_{n,i}$ denote $\operatorname{Proj}_{\operatorname{lin}(w_{n,i})\|V(\omega)\oplus\operatorname{lin}(\{w_{n,j}\colon j\ne i\})}$.

We obtain a countable collection of basic sets $(N_{V_n,k})$ which generate the Borel σ-algebra on $\mathcal{G}^d(X)$. To see this, we claim that for each $V \in \mathcal{G}^d(X)$ and each open set O containing V, there are n and k such that $V \in N_{V_n,k} \subset O$. Then each open set is the union of the basic sets that it contains.

Given $V \in \mathcal{G}^d(X)$ and an open set O containing it, let $B_r(V) \subset O$. Let $k > 4Kd/r$ and $\delta = \min(1/(2K), 1/(4kK), r/2)$. Let n be such that $d_H(V \cap S_X, V_n \cap S_X) < \delta$. Let $v \in V \cap S_X$ and $w \in W_n \cap S_X$. There exists $v' \in V_n \cap S_X$ such that $\|v - v'\| < \delta$. By Lemma 9(2), $\|w - v'\| \ge 1/K$, so that $\|w - v\| \ge 1/(2K)$. Hence $V \cap W_n = \{0\}$ and $\|\operatorname{Proj}_{W_n\|V}\| \le 4K$. Now given $v' \in V_n \cap S_X$, there exists $v \in V \cap S_X$ and $x \in X$ with $v' = v + x$ and $\|x\| < \delta$. We have

$$\|P_{n,i}(v')\| = \|P_{n,i}(x)\| = \|P_{n,i} \circ \operatorname{Proj}_{W_n\|V}(x)\| \le 4K\delta,$$

so that $V \in N_{V_n,k}$. Finally let $U \in N_{V_n,k}$ and let $v' \in V_n \cap S_X$. By definition, we have $\|P_{n,i}(v')\| \le \frac{1}{k}$ for each i, so that $\|\operatorname{Proj}_{W_n\|U}(v')\| \le \frac{d}{k}$. In particular, there exists $u \in U$ such that $\|u - v'\| \le \frac{d}{k}$ and hence there is $u \in U \cap S_X$ such that $\|u - v'\| \le \frac{2d}{k}$. Using Lemma 9(4), we deduce $d_H(U \cap S_X, V_n \cap S_X) \le \frac{2Kd}{k}$, so that $d_H(U \cap S_X, V \cap S_X) \le \delta + \frac{2Kd}{k} < r$, showing $N_{V_n,k} \subset O$.

Hence to show the desired measurability, it suffices to show that for each $N = N_{V_n,k}$, $\{\omega \colon V(\omega) \in N\}$ is measurable. First, $\{U \colon U \cap W_n = \{0\}\}$ is an open set, so that $\{\omega \colon V(\omega) \cap W_n = \{0\}\}$ is measurable.

Fix a dense set $v_1, v_2, \dots$ in the unit sphere of V_n. We claim that for those ω lying in the set G of full measure on which $\dim V(\omega) = d$ and hypotheses (1) and

(2) of the lemma hold, we have that $V(\omega)$ lies in N if and only if the following condition holds:

> For each rational $\epsilon > 0$ and each $j \in \mathbb{N}$, there is $m_0 > 0$ such that for each $m \geq m_0$, there are rationals $a_1^j, \ldots, a_d^j$ in $[-\frac{1}{k}, \frac{1}{k}]$ such that $\|\mathscr{L}_\omega^{(m)}(v_j - \sum_{i=1}^d a_i^j w_{n,i})\| \leq e^{(\lambda+\epsilon)m}$.

To see the 'only if' direction, suppose that $V(\omega) \in N$. Now given $v_j \in V_n \cap S_X$, by definition of N, there are $b_1, \ldots, b_d$ in $[-\frac{1}{k}, \frac{1}{k}]$ such that $v_j = v' + b_1 w_{n,1} + \ldots + b_d w_{n,d}$ with $v' \in V(\omega)$. Hence $v' = v_j - \sum_{i=1}^d b_i w_{n,i} \in V(\omega)$, and therefore we have $\|\mathscr{L}_\omega^{(m)} v'\| < e^{(\lambda+\epsilon)m}$ for all sufficiently large m. Now for any such m, one can take a_i's that are suitably close rational approximations to b_i so that $\|\mathscr{L}_\omega^{(m)}(v_j - \sum_{i=1}^d a_i w_{n,i})\| \leq e^{(\lambda+\epsilon)m}$.

Conversely, suppose that $V(\omega) \cap \operatorname{lin} W_n = \{0\}$, but $V(\omega) \notin N$. Then there exists a $v \in V_n \cap S_X$ and an i such that $\|P_{n,i}(v)\| > \frac{1}{k}$. By continuity, there exists a v_j satisfying the same property. Let $\delta = \|P_{n,i}(v_j)\| - \frac{1}{k}$. Then $\|P_{n,i}(v - \sum_{l=1}^d a_l w_l)\| \geq \delta$ for all $(a_l)_{l=1}^d \in [-\frac{1}{k}, \frac{1}{k}]^d$. By hypothesis, we now see that the condition is not satisfied.

Since this condition is obtained by taking countable unions and intersections of measurable sets, the measurability of $G \cap \{\omega\colon V(\omega) \in N\}$ is demonstrated. Using completeness of $(\Omega, \mathscr{F}, \mathbb{P})$, we deduce that $\{\omega\colon V(\omega) \in N\}$ is measurable, so that $\omega \mapsto V(\omega)$ is measurable as required. □

LEMMA 15. *Let σ be an ergodic measure-preserving transformation of a probability space $(\Omega, \mathbb{P})$. Let g be a non-negative measurable function and let $h \geq 0$ be integrable. Suppose further that $g(\omega) \leq h(\omega) + g(\sigma(\omega))$, $\mathbb{P}$-a.e. Then g is tempered, that is $\lim_{n\to\infty} g(\sigma^n\omega)/n = 0$, $\mathbb{P}$-a.e.*

A proof of this lemma appears in Mañé's paper [7].

Proof. Let $\epsilon > 0$ and let $K > \int h$. By the maximal ergodic theorem,

$$B_1 = \{\omega\colon h(\omega) + \ldots + h(\sigma^{n-1}\omega) < nK \text{ for all } n\}$$

has positive measure. Let M be such that $B_2 := \{\omega\colon g(\omega) < M\}$ has positive measure. As a consequence of the Birkhoff ergodic theorem, for any measurable set B with $\mathbb{P}(B) > 0$, for $\mathbb{P}$-a.e. ω, for all sufficiently large k, there exists $j \in [(1+\epsilon)^k, (1+\epsilon)^{k+1})$ such that $\sigma^j(\omega) \in B$. Now for $\omega \in \Omega$, let k_0 be such that for all $k \geq k_0$, there exist $j \in [(1+\epsilon)^k, (1+\epsilon)^{k+1})$ such that $\sigma^j\omega \in B_1$ and $j' \in ((1+\epsilon)^{k+2}, (1+\epsilon)^{k+3})$ such that $\sigma^{j'}\omega \in B_2$. If $n > (1+\epsilon)^{k_0+1}$, then $n \in [(1+\epsilon)^{k+1}, (1+\epsilon)^{k+2})$ for some $k \geq k_0$. Let $j \in [(1+\epsilon)^k, (1+\epsilon)^{k+1})$ and $j' \in [(1+\epsilon)^{k+2}, (1+\epsilon)^{k+3})$ be as above. Then

$$g(\sigma^n\omega) \leq \sum_{k=n}^{j'-1} h(\sigma^k\omega) + g(\sigma^{j'}\omega) \leq \sum_{k=j}^{j'-1} h(\sigma^k\omega) + g(\sigma^{j'}\omega) \leq K(j'-j) + M,$$

so that $\limsup g(\sigma^n\omega)/n \leq 4\epsilon$. Since ϵ is arbitrary, the conclusion follows. □

THEOREM 16 (Multiplicative ergodic theorem: the Oseledets filtration)**.** *Let $\mathscr{R}$ be an ergodic random dynamical system acting on a Banach space X with separable dual. Suppose that $\kappa(\mathscr{R}) < \lambda(\mathscr{R})$. Then there exist $1 \le r \le \infty$*[1] *and:*

- *a sequence of exceptional Lyapunov exponents $\lambda(\mathscr{R}) = \lambda_1 > \lambda_2 > \ldots > \lambda_r > \kappa(\mathscr{R})$,*
- *a sequence $m_1, m_2, \ldots, m_r$ of positive integers, and*
- *a measurable filtration of closed subspaces,*

$$X = V_1(\omega) \supset V_2(\omega) \supset \cdots \supset V_r(\omega) \supset V_\infty(\omega),$$

with the equivariance property $\mathscr{L}_\omega(V_i(\omega)) \subset V_i(\sigma(\omega))$ for each i,

such that for $\mathbb{P}$-a.e. ω, $\operatorname{codim} V_\ell(\omega) = m_1 + \cdots + m_{\ell-1}$; for all $v \in V_\ell(\omega) \setminus V_{\ell+1}(\omega)$, one has $\lim \frac{1}{n} \log \|\mathscr{L}_\omega^{(n)} v\| = \lambda_\ell$; and for $v \in V_\infty(\omega)$, $\limsup \frac{1}{n} \log \|\mathscr{L}_\omega^{(n)} v\| \le \kappa(\mathscr{R})$.

While the theorem is stated for ergodic random dynamical systems, a standard application of ergodic decomposition allows one to deduce a version for non-ergodic systems, in which constants are replaced by invariant functions.

Proof. Let $\mu_1 \ge \mu_2 \ge \ldots$ be as in Lemma 12. Let $\lambda_1 > \lambda_2 > \ldots$ be the decreasing enumeration of the distinct μ-values that exceed $\kappa(\mathscr{R})$ (if this an infinite sequence, then Theorem 13 establishes that $\lambda_i \to \kappa(\mathscr{R})$). The fact that $\lambda(\mathscr{R}) = \lambda_1$ is straightforward from the definitions. Let m_ℓ be the number of times that λ_ℓ occurs in the sequence (μ_i) and let $\mathscr{M}_\ell = m_1 + \ldots + m_\ell$, so that $\mu_{\mathscr{M}_{\ell-1}} = \lambda_{\ell-1}$ and $\mu_{\mathscr{M}_{\ell-1}+1} = \lambda_\ell$.

We now turn to the construction of $V_\ell(\omega)$. For a fixed ω, let the sequences $(\theta_i^{(n)})_{i\ge1}$ and $(x_j^{(n)})_{j\ge1}$ be as guaranteed by Lemma 7 for the operator $\mathscr{L}_\omega^{(n)}$. We let $V_\ell^{(n)}(\omega)$ be $\operatorname{lin}((\mathscr{L}_\omega^{(n)})^* \theta_1^{(n)}, \ldots, (\mathscr{L}_\omega^{(n)})^* \theta_{\mathscr{M}_{\ell-1}}^{(n)})^\perp$, $Y_\ell^{(n)}(\omega)$ be $\operatorname{lin}(\{x_j^{(n)} : j \le \mathscr{M}_\ell\})$. Thus, $X = V_\ell^{(n)}(\omega) \oplus Y_{\ell-1}^{(n)}(\omega)$. All of these depend on the choice of θ's and x's. No claim of uniqueness or measurability is made.

The space $V_\ell^{(n)}(\omega)$ is an approximate slow space. The proof will go by the following steps:

(a) For almost all ω, for arbitrary $\epsilon > 0$ and for sufficiently large n, $\|\mathscr{L}_\omega^{(n)} x\| \le e^{(\lambda_\ell+\epsilon)n}\|x\|$ for all $x \in V_\ell^{(n)}(\omega)$;
(b) $V_\ell^{(n)}(\omega)$ is a Cauchy sequence for almost all ω – we define the limit to be $V_\ell(\omega)$;
(c) The $V_\ell(\omega)$ are equivariant: $\mathscr{L}_\omega(V_\ell(\omega)) \subseteq V_\ell(\sigma(\omega))$;
(d) If $x \notin V_{\ell+1}(\omega)$, then $\|\mathscr{L}_\omega^{(n)} x\| > e^{(\lambda_\ell-\epsilon)n} d(x, V_{l+1}(\omega))$ for large n;
(e) For all $a > 0$ and $\epsilon > 0$, there exists n_0 so that for all $n \ge n_0$ and all $x \in S_X$ such that $d(x, V_{\ell+1}(\omega)) \ge a$, one has $\|\mathscr{L}_\omega^{(n)} x\| \ge e^{(\lambda_\ell-\epsilon)n}$.

The remaining steps are proved by induction on ℓ:

(f) If $x \in V_\ell(\omega)$, then $\limsup \frac{1}{n} \log \|\mathscr{L}_\omega^{(n)} x\| \le \lambda_\ell$;
(g) $\omega \mapsto V_\ell(\omega)$ is measurable;

[1] If $r = \infty$, the conclusions are replaced by: $\lambda(\mathscr{R}) = \lambda_1 > \lambda_2 > \ldots \to \kappa(\mathscr{R})$, $m_1, m_2, \ldots \in \mathbb{N}$ and $X = V_1(\omega) \supset V_2(\omega) \supset \ldots$; $V_\infty(\omega) = \bigcap V_i(\omega)$.

(h) The restriction $\mathscr{R}_\ell$ of $\mathscr{R}$ to $V_\ell(\omega)$ has the same exponents as $\mathscr{R}$ with the initial $\mathscr{M}_{\ell-1}$ exponents removed.

Proof of (a). Note that by construction

$$\det\Big(\big(\theta_i^{(n)}(\mathscr{L}_\omega^{(n)}x_j^{(n)})\big)_{1\le i,j\le \mathscr{M}_{\ell-1}}\Big) \ge KE_{\mathscr{M}_{\ell-1}}(\mathscr{L}_\omega^{(n)}),$$

where K is a constant depending only on $\mathscr{M}_{\ell-1}$ arising from Lemmas 5 and 7. For an arbitrary $x \in V_\ell^{(n)} \cap S_X$, let $\phi \in S_{X^*}$ be such that $\phi(\mathscr{L}_\omega^{(n)}x) = \|\mathscr{L}_\omega^{(n)}x\|$. Then, adding a column for x and a row for ϕ to the matrix $U((\theta_i^{(n)}),(x_j^{(n)}))_{1\le i,j\le\mathscr{M}_{\ell-1}}$, we see that the x column has all 0 entries except for the $1+\mathscr{M}_{\ell-1}$-st (by definition of $V_\ell^{(n)}(\omega)$), and so we arrive at the bound (uniform over $x\in S_X\cap V_\ell^{(n)}$),

$$KE_{\mathscr{M}_{\ell-1}}(\mathscr{L}_\omega^{(n)})\|\mathscr{L}_\omega^{(n)}x\| \le E_{1+\mathscr{M}_{\ell-1}}(\mathscr{L}_\omega^{(n)}). \tag{7}$$

The conclusion follows from Lemma 12.

Proof of (b). Let us assume that n_0 is chosen large enough that for all $n\ge n_0$, the following conditions are satisfied: $\|\mathscr{L}_\omega\|$, $\|\mathscr{L}_{\sigma^n\omega}\|$ are less than $e^{\epsilon n}$; $\|\mathscr{L}_\omega^{(n)}x\| \le e^{(\lambda_\ell+\epsilon)n}\|x\|$ for all $x\in V_\ell^{(n)}$; and $\|\mathscr{L}_\omega^{(n)}x\| \ge e^{(\lambda_{\ell-1}-\epsilon)n}\|x\|$ for all $x\in Y_{\ell-1}^{(n)}(\omega)$ (using integrability of $\log\|\mathscr{L}_\omega\|$; (a); and Lemma 7). Let $n\ge n_0$. Let $x\in V_\ell^{(n)}(\omega)\cap S_X$ and write $x=u+w$ where $u\in V_\ell^{(n+1)}(\omega)$ and $w\in Y_{\ell-1}^{(n+1)}(\omega)$. Now we have

$$\|\mathscr{L}_\omega^{(n+1)}x\| \le e^{(\lambda_\ell+\epsilon)n}\|\mathscr{L}_{\sigma^n\omega}\| \le e^{(\lambda_\ell+2\epsilon)n}.$$

We also have $\|u\|\le 1+\|w\|$, $\|\mathscr{L}_\omega^{(n+1)}w\| \ge e^{(\lambda_{\ell-1}-\epsilon)(n+1)}\|w\|$ and $\|\mathscr{L}_\omega^{(n+1)}u\| \le e^{(\lambda_\ell+\epsilon)(n+1)}(1+\|w\|)$. Manipulation with the triangle inequality yields

$$\|w\| \le e^{-n(\lambda_{\ell-1}-\lambda_\ell-4\epsilon)}. \tag{8}$$

Hence, we have shown that each point in the unit sphere of $V_\ell^{(n)}(\omega)$ is exponentially close to $V_\ell^{(n+1)}(\omega)$. Since the two spaces have the same codimension, one obtains a similar inequality in the opposite direction by Lemma 9(4). This establishes that $V_\ell^{(n)}(\omega)$ is a Cauchy sequence.

Proof of (c). We argue essentially as in (b). For large n, we take $v\in V_\ell^{(n+1)}(\omega)\cap S_X$. We write $\mathscr{L}_\omega(v)$ as $u+w$ with $u\in V_\ell^{(n)}(\sigma(\omega))$ and $w\in Y_{\ell-1}^{(n)}(\sigma(\omega))$. We have bounds of the form $\|\mathscr{L}_\omega^{(n+1)}v\| \lesssim e^{\lambda_\ell n}$; $\|u\|\lesssim 1+\|w\|$, $\|\mathscr{L}_{\sigma(\omega)}^{(n)}u\| \lesssim e^{\lambda_\ell n}(1+\|w\|)$ and $\|\mathscr{L}_{\sigma(\omega)}^{(n)}w\| \gtrsim e^{\lambda_{\ell-1}n}\|w\|$ (here $\lesssim$ means 'is smaller up to sub-exponential factors'). Combining the inequalities as before, one obtains a bound $\|w\| \lesssim e^{-(\lambda_{\ell-1}-\lambda_\ell)n}$. Taking a limit, we obtain $\mathscr{L}_\omega V_\ell(\omega)\subset V_\ell(\sigma(\omega))$ as required.

Proof of (d). Let $x\notin V_{\ell+1}(\omega)$, with $\|x\|=1$. For large n, if x is written as u_n+v_n with $u_n\in V_{\ell+1}^{(n)}(\omega)$ and $v_n\in Y_\ell^{(n)}(\omega)$, then $\|v_n\|\ge \frac{1}{2}d(x,V_{\ell+1}(\omega))$ and $\|u_n\|\le 1+\|v_n\|$. By (7), $\|\mathscr{L}_\omega^{(n)}u_n\| \le e^{(\lambda_{\ell+1}+\epsilon)n}(1+\|v_n\|)$ for large n, and Lemma 7 gives $\|\mathscr{L}_\omega^{(n)}v_n\| \ge 4^{-\mathscr{M}_\ell}F_{\mathscr{M}_\ell}(\mathscr{L}_\omega^{(n)})\|v_n\| \ge e^{(\lambda_\ell-\epsilon)n}\|v_n\|$ for large n. The conclusion follows. The proof of (e) is the same, using the uniformity in Lemma 7.

For the inductive part, notice that the case $\ell=1$ is trivial. Let $\ell\ge 2$ and suppose that the claims have been established for the case $\ell-1$. We know from (a) that elements of $V_\ell^{(n)}(\omega)$ expand at exponential rate approximately λ_ℓ

under $\mathscr{L}_\omega^{(n)}$. We need to show that the analogous statement holds for the limiting subspace $V_\ell(\omega)$. We mimic the start of the proof to control the slow $m_{\ell-1}$-codimensional subspace of $V_{\ell-1}(\omega)$. This will be exactly $V_\ell(\omega)$.

Proof of (f). By the inductive hypothesis, the top Lyapunov exponent of $\mathscr{L}_\omega^{(n)}$ applied to the bundle $\{V_{\ell-1}(\omega) : \omega \in \Omega\}$ is $\lambda_{\ell-1}$ with multiplicity $m_{\ell-1}$, with the following Lyapunov exponent being λ_ℓ. Let $(z_j^{(n)})_{j=1}^{m_{\ell-1}} \in S_X \cap V_{\ell-1}(\omega)$ and $(\psi_j^{(n)})_{j=1}^{m_{\ell-1}}$ be as guaranteed by Lemma 7 and let $V_\ell'^{(n)}(\omega) = V_{\ell-1}(\omega) \cap \operatorname{lin}(\psi_1^{(n)}, \dots, \psi_{m_{\ell-1}}^{(n)})^\perp$. The same argument as in (a) shows that for arbitrary $\epsilon > 0$ and sufficiently large n, $\|\mathscr{L}_\omega^{(n)} x\| \le e^{(\lambda_\ell+\epsilon)n}\|x\|$ for $x \in V_\ell'^{(n)}(\omega)$. The argument used in (b) also works, showing that $V_\ell'^{(n)}(\omega)$ converges to a space $V_\ell'(\omega) \subset V_{\ell-1}(\omega)$ and, crucially, we obtain an analogue of (8): for all sufficiently large n, if $x \in V_\ell'(\omega)$, then x may be expressed as $u + w$ with $u \in V_\ell'^{(n)}(\omega)$ and $w \in V_{\ell-1}(\omega)$ satisfying $\|w\| \le e^{-n(\lambda_{\ell-1}-\lambda_\ell-4\epsilon)}$. Then, $\|\mathscr{L}_\omega^{(n)} u\| \lesssim e^{\lambda_\ell n}$ by the above, and $\|\mathscr{L}_\omega^{(n)} w\| \lesssim e^{-(\lambda_{\ell-1}-\lambda_\ell)n} \cdot e^{\lambda_{\ell-1} n}$. So $\|\mathscr{L}_\omega^{(n)} x\| \lesssim e^{\lambda_\ell n}$ by the triangle inequality. From (d), we deduce $V_\ell'(\omega) \subseteq V_\ell(\omega)$. Since, by (h) (applied to $\mathscr{R}_{\ell-1}$), $V_\ell(\omega)$ and $V_\ell'(\omega)$ have the same finite co-dimension as subspaces of $V_{\ell-1}(\omega)$, $V_\ell(\omega) = V_\ell'(\omega)$ and (f) follows.

Proof of (g). From (f) and (d), we see that

$$V_\ell(\omega) = \left\{v\colon \limsup \tfrac{1}{n} \log \|\mathscr{L}_\omega^{(n)} v\| \le \lambda_\ell\right\}$$

and the assumptions of Lemma 14 hold. Measurability of $V_\ell(\omega)$ follows.

Proof of (h). Let $W(\omega) = \{w \in V_{\ell-1}(\omega) \colon d(w, V_\ell(\omega)) \ge \frac{1}{2}\|w\|\}$. Let $\epsilon > 0$. We claim that for sufficiently large n,

$$d(w', v') > e^{-\epsilon n} \text{ for all } w' \in S_X \cap \mathscr{L}_\omega^{(n)} W(\omega) \text{ and } v' \in V_\ell(\sigma^n \omega). \tag{9}$$

Let $\delta < \frac{\epsilon(\lambda_{\ell-1}-\lambda_\ell)}{4(\lambda_{\ell-1}-\lambda_\ell+\epsilon)}$. Let $g_\ell(\omega) = \sup_{p\in\mathbb{N}} e^{-p(\lambda_\ell+\delta)} \left\| \mathscr{L}_\omega^{(p)}|_{V_\ell(\omega)} \right\|$. This is measurable by Lemma 11. Notice that

$$\log^+ g_\ell(\omega) \le \log^+ \|\mathscr{L}_\omega\| + \max(-\lambda_\ell - \delta, 0) + \log^+ g_\ell(\sigma\omega).$$

By Lemma 15, $\lim_{n\to\infty} \frac{1}{n} \log^+ g_\ell(\sigma^n\omega) = 0$. Then, there exists $n_0(\omega)$ such that for $p, n \ge n_0$, one has

$$\begin{aligned} &\|\mathscr{L}_{\sigma^n\omega}^{(p)} z\| \le \tfrac{1}{3} \exp(n\tfrac{\epsilon}{2} + p(\lambda_{\ell-1}+\delta))\|z\| \text{ for all } z \in V_{\ell-1}(\sigma^n\omega); \\ &\|\mathscr{L}_{\sigma^n\omega}^{(p)} v\| \le \tfrac{1}{3} \exp(n\tfrac{\epsilon}{2} + p(\lambda_\ell+\delta))\|v\| \text{ for all } v \in V_\ell(\sigma^n\omega). \end{aligned} \tag{10}$$

Additionally by (e), n_0 may be chosen so that

$$e^{n(\lambda_{\ell-1}-\delta)} < \|\mathscr{L}_\omega^{(n)} w\| < e^{n(\lambda_{\ell-1}+\delta)} \text{ for all } w \in W(\omega) \cap S_X \text{ and } n \ge n_0. \tag{11}$$

We will show (9) by contradiction. Suppose $d(\mathscr{L}_\omega^{(n)} w, V_\ell(\sigma^n\omega)) < e^{-\epsilon n}\|\mathscr{L}_\omega^{(n)} w\|$ for some $n > n_0$, and such that $\epsilon n/(\lambda_{\ell-1} - \lambda_\ell) > n_0$. Write $\mathscr{L}_\omega^{(n)} w = v + z$, with $v \in V_\ell(\sigma^n\omega)$ and $\|z\| < e^{-\epsilon n}\|v\|$. Now $\mathscr{L}_\omega^{(p+n)} w = \mathscr{L}_{\sigma^n\omega}^{(p)} v + \mathscr{L}_{\sigma^n\omega}^{(p)} z$. Taking $p = \epsilon n/(\lambda_{\ell-1} - \lambda_\ell)$, the bounds on the two terms coming from (10) agree, giving

$$\|\mathscr{L}_\omega^{(n+p)} w\| \le e^{-\epsilon n/2} e^{(p+n)(\lambda_{\ell-1}+\delta)}.$$

One checks, however, that by the choice of δ, this is smaller than $e^{(p+n)(\lambda_{\ell-1}-\delta)}$, contradicting (11). This establishes claim (9). Notice that combining (9) and (11), we see that the restriction of the random dynamical system to the equivariant family $Q_{\ell-1}(\omega)=V_{\ell-1}(\omega)/V_\ell(\omega)$ satisfies for all sufficiently large n,

$$\|\bar{\mathscr{L}}_\omega^{(n)}\bar{w}\|_{Q_{\ell-1}(\sigma^n\omega)} \geq e^{(\lambda_{\ell-1}-\epsilon)n}\|\bar{w}\|_{Q_{\ell-1}(\omega)} \text{ for all } \bar{w}\in Q_{\ell-1}(\omega), \tag{12}$$

where $\bar{\mathscr{L}}_\omega$ denotes the induced action of $\mathscr{L}_\omega$ on $Q_{\ell-1}(\omega)$.

Set $m=m_{\ell-1}$. To complete the proof of (h), let $n>n_0$ be arbitrary; let $v_1,\dots,v_{k-m}$ be unit vectors in $V_\ell(\omega)$ and $w_1,\dots,w_m$ be unit vectors in $V_{\ell-1}(\omega)$. Then

$$\begin{aligned} &d_k\mathscr{L}_\omega^{(n)}(v_1,\dots,v_{k-m},w_1,\dots,w_m)\\ &\qquad\geq d_{k-m}\mathscr{L}_\omega^{(n)}(v_1,\dots,v_{k-m})\,d_m\bar{\mathscr{L}}_\omega^{(n)}(\bar{w}_1,\dots,\bar{w}_{k-m}),\end{aligned}$$

where $\bar{w}_i$ is $w_i+V_\ell(\omega)$. We therefore see

$$D_k\mathscr{L}_\omega^{(n)}|_{V_{\ell-1}(\omega)} \geq D_{k-m}\mathscr{L}_\omega^{(n)}|_{V_\ell(\omega)}\cdot D_m\bar{\mathscr{L}}_\omega^{(n)}.$$

By (12) and Lemma 2, $D_m\bar{\mathscr{L}}_\omega^{(n)} \gtrsim e^{\lambda_{\ell-1}mn}$.

This gives a matching upper bound for $D_{k-m}\mathscr{L}_\omega^{(n)}|_{V_\ell(\omega)}$ to the lower bound that we obtained in Lemma 8. Hence we deduce the first k exponents of $\mathscr{R}_{\ell-1}$ are $m=m_{\ell-1}$ repetitions of $\lambda_{\ell-1}$ followed by the first $k-m$ exponents of $\mathscr{R}_\ell$, establishing (h). □

The next corollary provides a *splitting* when the base σ is invertible and the Banach space is reflexive. Note that the methods of [4] obtain the same conclusion under the weaker assumption that X^* has separable dual. We include this new proof, as we find it to be illuminating.

COROLLARY 17 (Multiplicative Ergodic Theorem: The Oseledets splitting)**.** *Let $\mathscr{R}$ be a random dynamical system acting on a reflexive separable Banach space. Suppose that the base, σ, is invertible; and that $\kappa(\mathscr{R})<\lambda(\mathscr{R})$. Then there exist $1\leq r\leq\infty$ and exceptional Lyapunov exponents and multiplicities as in Theorem 16. Furthermore, there is a measurable direct sum decomposition*[2]

$$X=Z_1(\omega)\oplus\cdots\oplus Z_r(\omega)\oplus V_\infty(\omega),$$

such that for $\mathbb{P}$-a.e. ω, $\mathscr{L}_\omega(Z_i(\omega))=Z_i(\sigma(\omega))$ for each i, $\mathscr{L}_\omega(V_\infty(\omega))\subset V_\infty(\sigma(\omega))$, $\dim Z_i(\omega)=m_i$, and

$$\lim_{n\to\infty}\frac{1}{n}\log\|\mathscr{L}_\omega^{(n)}v\|=\lambda_i \textit{ for } v\in Z_i(\omega)\smallsetminus\{0\},$$

$$\limsup_{n\to\infty}\frac{1}{n}\log\|\mathscr{L}_\omega^{(n)}v\|\leq\kappa(\mathscr{R}) \textit{ for } v\in V_\infty(\omega).$$

We make use of the following facts valid for reflexive Banach spaces. If X is reflexive and Θ is a closed subspace of X^* of codimension k, then its annihilator, $\Theta^\perp$ is k-dimensional. Further if θ is a bounded functional such that $\theta|_{\Theta^\perp}=0$, then $\theta\in\Theta$.

[2] In the case $r=\infty$, the decomposition is $X=\bigoplus_{i=1}^\infty Z_i(\omega)\oplus V_\infty(\omega)$.

Proof. Let $\mathscr{R}^*$ be the dual random dynamical system to $\mathscr{R}$ as defined above. Applying Theorem 16 to $\mathscr{R}^*$, and recalling from Theorem 13 that the Lyapunov exponents and multiplicities of $\mathscr{R}$ and $\mathscr{R}^*$ coincide, yields a σ^{-1} equivariant measurable filtration $X^* = V_1^*(\omega) \supset \cdots \supset V_r^*(\omega) \supset V_\infty^*(\omega)$, with the same codimensions as those of $\mathscr{R}$.

Let $Y_\ell(\omega) = V_\ell^*(\omega)^\perp$. Notice that $\dim Y_\ell(\omega) = \mathscr{M}_{\ell-1}$. Since $V_\ell^*(\omega)$ is measurable and $(\cdot)^\perp : \mathscr{G}(X^*) \to \mathscr{G}(X)$ is continuous [5, IV §2], then $Y_\ell(\omega)$ is measurable. Also, for every $\psi \in V_\ell^*(\sigma\omega)$, we have $\mathscr{L}^*_{\sigma\omega}\psi \in V_\ell^*(\omega)$ by equivariance of $V_\ell^*(\cdot)$. Hence, for every $y \in Y_\ell(\omega)$, $0 = \mathscr{L}^*_{\sigma\omega}\psi(y) = \psi(\mathscr{L}_\omega y)$. Thus, $\mathscr{L}_\omega Y_\ell(\omega) \subset Y_\ell(\sigma\omega)$, yielding equivariance.

We define $Z_\ell(\omega) = Y_{\ell+1}(\omega) \cap V_\ell(\omega)$. It remains to show that $V_{\ell-1}(\omega) = V_\ell(\omega) \oplus Z_{\ell-1}(\omega)$. To prove this, it suffices to show that $V_\ell(\omega) \oplus Y_\ell(\omega) = X$. Suppose this is not the case. Then, there exists $v \in V_\ell(\omega) \cap Y_\ell(\omega) \cap S_X$. Let $\theta \in S_{X^*}$ be such that $\theta(v) = 1$. Let $\bar\theta$ be the equivalence class of θ in $Q^*(\omega) = X^*/V_\ell^*(\omega)$.

We record a corollary of (12). For almost every ω, one has for all large n

$$\|\bar{\mathscr{L}}_\omega^{*(n)}\bar\psi\|_{Q^*(\sigma^{-n}\omega)} \ge e^{-(\lambda_{\ell-1}-\epsilon)n}\|\bar\psi\|_{Q^*(\omega)} \text{ for all } \bar\psi \in Q^*(\omega).$$

Since $Q^*(\omega)$ is a finite-dimensional space whose dimension does not depend on ω, the above implies that $\bar{\mathscr{L}}_\omega^*$ is bijective. Furthermore, the quantity

$$C(\omega) = \inf_{n\in\mathbb{N};\bar\psi\in S_{X^*}\cap Q^*(\omega)} e^{-(\lambda_{\ell-1}-\epsilon)n}\|\bar{\mathscr{L}}_\omega^{*(n)}\bar\psi\|_{Q^*(\sigma^{-n}\omega)}$$

is positive. We claim that $C(\omega)$ is measurable. Let $(\zeta_n(\omega))$ be a measurable dense subsequence of $V_\ell^*(\omega)$. If $\bar\psi(\omega)$ is the equivalence class of ψ in $Q^*(\omega)$, then we have $\|\bar\psi(\omega)\|_{Q^*(\omega)} = \inf_k \|\psi(\omega) - \zeta_k(\omega)\|$, which depends measurably on ω. Proceeding as in Lemma 11, we see that $C(\omega)$ is measurable and by (12) is positive almost everywhere. Hence $C(\omega)$ exceeds some quantity c on a set of positive measure.

Let $\bar\phi_n \in Q^*(\sigma^n\omega)$ be such that $\bar{\mathscr{L}}^{*(n)}_{\sigma^n\omega}\bar\phi_n = \bar\theta$. Then

$$\|\bar{\mathscr{L}}^{*(n)}_{\sigma^n\omega}\bar\phi_n\|_{Q^*(\omega)} \ge C(\sigma^n\omega)e^{(\lambda_{\ell-1}-\epsilon)n}\|\bar\phi_n\|_{Q^*(\sigma^n\omega)}.$$

By ergodicity, there exist arbitrarily large values of n for which

$$\|\bar\phi_n\|_{Q^*(\sigma^n\omega)} \le c^{-1}e^{-(\lambda_{\ell-1}-\epsilon)n}\|\bar\theta\|_{Q^*(\omega)}. \tag{13}$$

On the other hand, one has $v \in Y_l(\omega)$, so that $\psi(v) = 0$ for every $\psi \in V_l^*(\omega)$. Thus, if we express $\mathscr{L}^{*(n)}_{\sigma^n\omega}\phi_n + \psi_n = \theta$, where $\phi_n \in X^*$ is a representative of $\bar\phi_n$ and $\psi_n \in V_l^*(\omega)$, the following holds

$$1 = \theta(v) = \mathscr{L}^{*(n)}_{\sigma^n\omega}\phi_n(v) + \psi_n(v) = \phi_n(\mathscr{L}^{(n)}_\omega v).$$

In addition, for every $\psi \in V_l^*(\sigma^{-n}\omega)$, $(\phi_n + \psi)(\mathscr{L}^{(n)}_\omega v) = \phi_n(\mathscr{L}^{(n)}_\omega v) = 1$. Thus, for sufficiently large n and every $\psi \in V_l^*(\sigma^{-n}\omega)$, $\|\phi_n + \psi\| e^{(\lambda_\ell+\epsilon)n} \ge 1$. Therefore, $\|\bar\phi_n\|_{Q^*(\sigma^n(\omega))} \ge e^{-(\lambda_\ell+\epsilon)n}$, giving a contradiction with (13). Hence, $V_{\ell-1}(\omega) = V_\ell(\omega) \oplus Z_{\ell-1}(\omega)$ as required. □

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