Invariant densities for metastable systems

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Introduction

- **Metastable systems**. Systems with two or more nearly invariant subsets. We consider one-dimensional piecewise expanding maps T_ϵ that are perturbations of a map T_0 with two invariant sets, I_l and I_r.
- **Invariant densities**. Functions that describe invariant measures of physical interest.
- **Question**. For small ϵ, what is the best approximation to the invariant density of T_ϵ by a linear combination of the ergodic invariant densities of T_0?

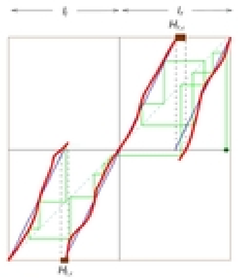


Figure: Metastable system: red. Initial system: blue. Holes: brown.

Call $H_{l,\epsilon} := I_l \cap T_\epsilon^{-1}(I_r)$ and $H_{r,\epsilon} := I_r \cap T_\epsilon^{-1}(I_l)$ the *holes*.

Initial system

$T_0 : I \circlearrowleft$ is a piecewise C^2 expanding map with critical set $\mathcal{C}_0 = \{-1 = c_0^0 < c_1^0 < \cdots < c_d^0 = 1\}$, satisfying:

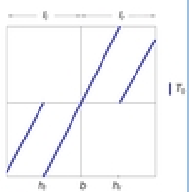
(I1) Two invariant sets with unique ACIMs.
- There is a boundary point $b \in (-1, 1)$ such that $I_l := [-1, b]$ and $I_r := [b, 1]$ are invariant under T_0.
- $T_0|_{I_*}$ has a unique ACIM μ_* with density ϕ_*.

(I2) Restriction on infinitesimal holes.
- Let the *infinitesimal holes* be $H_0 := T_0^{-1}\{b\} \setminus \{b\}$.
- $I_l \cap H_0 = \{h_l\}$, $I_r \cap H_0 = \{h_r\}$.

(I3) No-return of the critical set to the infinitesimal holes.

(I4) Positive densities at infinitesimal holes.

(I5) Restriction on periodic critical points.
- Either the expansion is greater than 2, or T_0 has no periodic critical points except possibly -1 or 1 fixed.



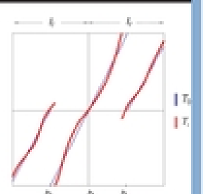
Perturbations

$T_\epsilon : I \circlearrowleft$ is a perturbation of T_0 such that for $\epsilon > 0$, the following hold:

(P1) Small C^2 perturbation.

(P2) Unique ACIM μ_ϵ, with density $\phi_\epsilon := d\mu_\epsilon/dx$.

(P3) Boundary condition.
- When $b \notin \mathcal{C}_0$, $T_\epsilon(b) = b$.
- When $b \in \mathcal{C}_0$, $b \in \mathcal{C}_\epsilon$, $T_\epsilon(b_-) \leq b$ and $T_\epsilon(b_+) \geq b$.



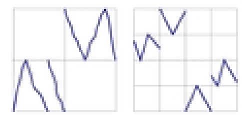
Main theorem

Consider T_0 and T_ϵ satisfying assumptions (I1)-(I5) and (P1)-(P3), respectively.
Suppose that $A := \lim_{\epsilon\to 0} \frac{\mu_r(H_{r,\epsilon})}{\mu_l(H_{l,\epsilon})}$ exists. Let $\alpha = \frac{A}{1+A}$.
Then as $\epsilon \to 0$,

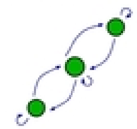
$$\phi_\epsilon \xrightarrow{L^1} \alpha\phi_l + (1-\alpha)\phi_r.$$

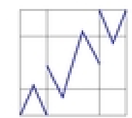
Generalizations

- Multiple *holes*.
- Disconnected invariant sets.
- Boundary point can depend on ϵ.

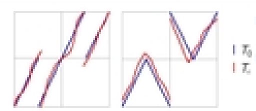


- Multiple invariant sets: The limiting invariant density is determined by the invariant densities of the initial system and the stationary measure of a (limiting) finite state Markov chain.





Example



- Lebesgue measure on I_l, I_r is invariant for T_0.
- $\phi_\epsilon \xrightarrow{L^1} \alpha Leb|_{I_l} + (1-\alpha) Leb|_{I_r}$, with $\frac{\alpha}{1-\alpha} = \lim_{\epsilon\to 0} \frac{Leb(H_{r,\epsilon})}{Leb(H_{l,\epsilon})}$.

Properties of invariant densities

- Bounded variation [LY73].
- Regular-singular decomposition [Bal07, BS08].

$$\phi_\epsilon = \phi_\epsilon^{reg} + \phi_\epsilon^{sal},$$

ϕ_ϵ^{reg} is Lipschitz.
$\phi_\epsilon^{sal} = \sum_{u_\epsilon} s_{u_\epsilon} H_{u_\epsilon}$, is the sum of step functions along postcritical orbits.
s_{u_ϵ} is the *jump* of ϕ_ϵ at u_ϵ,
H_u is a Heaviside function.

- Equal measure of holes.

$$\mu_\epsilon(H_{\epsilon,l}) = \mu_\epsilon(H_{\epsilon,r}).$$

- Uniformly bounded variation.

$$\sup_{0<\epsilon<\epsilon_0} \mathrm{var}(\phi_\epsilon) < +\infty.$$

- Uniform Lipschitz constants.

$$\sup_{0<\epsilon<\epsilon_0} \mathrm{Lip}(\phi_\epsilon^{reg}) < +\infty.$$

- Exponential decay of jumps along critical trajectories.
  - Small jumps inside the holes. For $* \in \{l, r\}$, $\delta > 0$ and ϵ small, the variation of ϕ_ϵ^{sal} over $H_{*,\epsilon}$ is

$$\mathrm{var}_{H_{*,\epsilon}}(\phi_\epsilon^{sal}) < \delta.$$

  - Continuity of ϕ_* at h_*.

Conclusions

- Limit invariant densities are found for a class of one-dimensional, expanding, piecewise C^2 metastable systems with finitely many components, that are not required to have Markov partitions.
- The result agrees with linear response heuristics, and complements works of [KL09], where eigenvalues of the transfer operator associated to T_ϵ are approximated. Invariant densities correspond to eigenvectors of eigenvalue 1.

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References

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