Multiplicative ergodic theorems for transfer operators: towards the identification and analysis of coherent structures in non-autonomous dynamical systems

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Abstract. We review state-of-the-art results on multiplicative ergodic theory for operators, with a view towards applications to the analysis of transport phenomena in non-autonomous dynamical systems, such as geophysical flows. The focus of this work is on ideas and motivation, rather than on proofs and technical aspects.

Contents

1. Introduction

An important motivation behind the recent work on multiplicative ergodic theorems is the desire to develop a mathematical theory which is useful for the study of global transport properties of real world dynamical systems, such as oceanic and atmospheric flows.

Global features of the ocean flow include large scale structures which are important for the global climate. One of the best known such structures is the Gulf Stream, which was discovered and navigated by Spanish sailors about five hundred

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years ago, at the time of the Spanish colonization of the Americas. The Gulf Stream transports water from the tropical latitudes in the warm Caribbean towards the cold North Atlantic, and it is of high importance for the global climate system. Another large scale coherent oceanic structure is the North Pacific Gyre. There, water sinks propitiating conditions for the Great Pacific Garbage Patch to be formed[1]. This patch was discovered less than twenty years ago, by Captain Charles Moore. It concentrates plastic residues mostly in the form of microplastics, at the surface and beneath it, and affects marine life in various ways. This fact highlights the importance of developing tools which are capable of analyzing three-dimensional systems.

In the atmosphere, it is well known that there exist vortices near the poles which are persistent over time scales of the order of weeks or longer. The conditions on the Antarctic polar vortex are favorable for chemical reactions responsible for the depletion of ozone, which manifest themselves in the ozone hole. Figure 1 shows streamlines of a snapshot of the wind vector field around the Antarctic polar vortex on September 13th, 2015, from `http://earth.nullschool.net/#current/wind/isobaric/10hPa/orthographic=-70.56,-81.09,288`. Both polar vortices retain cold air, especially in the winter, and change their location over time. This fact illustrates that a time-dependent (or non-autonomous) point of view is necessary for the study of such structures.

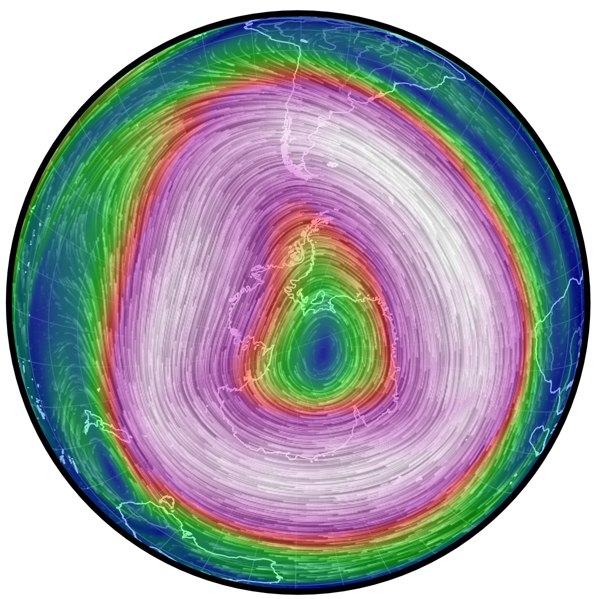

FIGURE 1. Visualization of the Antarctic polar vortex in September 2015. Snapshot taken from `http://earth.nullschool.net`.

[1]The reader may visit the following National Geographic educational article on this topic: `http://education.nationalgeographic.com/encyclopedia/great-pacific-garbage-patch/`

Two key features of these so-called *coherent structures* which make their analysis challenging, both computationally and mathematically, are: (i) The structures in question are not completely invariant, but rather interact weakly with their surroundings. Hence, especially in the three dimensional setting, it is often unclear how to identify or determine their boundaries; and (ii) The structures are not static, but change their shape and location over time.

A natural framework in which one may attempt to address these challenges is that of *non-autonomous dynamical systems*, introduced in Section 2. Such a setting deals with systems evolving under time-dependent mechanisms, and hence can conveniently account for non static features such as the coherent structures of interest. Non-autonomous dynamical systems have received considerable attention in the literature over the last few years. In fact, some instances thereof –the so-called noisy and driven systems– have been identified as important directions going forward in dynamics in the recent survey article by L.-S. Young [**97**].

One of the tools which has proved to be very successful for analyzing the long term behavior of a large class of dynamical systems is the so-called *transfer operator*, discussed in Section 3. The idea behind this approach is to take a probabilistic point of view of the system, and represent its state at a given time via a density function, describing the distribution of mass (or fluid) in the system. The transfer operator is a linear operator that encodes the action of the dynamics on this density. That is, if ρ represents the state of the system at the present time, and $\mathcal{L}$ is the transfer operator associated to the one-step dynamics, then $\mathcal{L}\rho$ describes the state of the system one time step ahead. The crucial factor is that spectral properties (eigenvalues and eigenvectors) of the transfer operator are closely related to objects of dynamical interest.

Section 4 is about *multiplicative ergodic theorems*, dating back to the work of Oseledets [**85**] in the mid 1960s. They provide time-dependent versions of spectral decompositions for composition of random (or non-autonomous) linear operators. Given the considerable success that spectral methods have enjoyed in explaining autonomous dynamical systems via transfer operators, multiplicative ergodic theorems seem to be an ideal candidate for the investigation of coherent structures in the non-autonomous setting. This was the breakthrough idea of Froyland, Lloyd and Quas [**42**], which has been extensively investigated in recent years.

The design, development, implementation and optimization of numerical and computational algorithms to identify and visualize relevant features, such as almost invariant sets and coherent structures for models of dynamical systems of physical interest, has paralleled the theoretical development for transfer operators and multiplicative ergodic theorems. This is the topic of Section 5.

Section 6 discusses recent works which have used the technology of transfer operators and multiplicative ergodic theorems to locate coherent structures in models of oceanic and atmospheric flows. A list of online resources for visualizing oceanic and atmospheric flows is presented in Appendix A.

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2. Non-autonomous dynamical systems

The difference between autonomous and non-autonomous systems is that the rules dictating the dynamics are fixed in the former, and change over time in the latter. Thus, non-autonomous dynamical systems may be used to describe the evolution of a wider class of phenomena, including systems which are forced or driven, like a planet (or even a group of planets) whose evolution is affected by external forces, such as the presence of one or more other planets, stars and perhaps other factors.

We remark that it is, at least in theory, possible to enlarge a non-autonomous system to make it autonomous, by incorporating all the external factors into it. An important benefit of considering the non-autonomous point of view is that one may focus on the feature of interest – for example, one planet – and obtain much finer conclusions about its behavior than one could get from considering a larger autonomous system, at least with the current technology.

A strategy which has been successful for extending existing theory for autonomous systems to the non-autonomous setting is to consider still objects which explain or illustrate some dynamical feature in the autonomous setting, and animate them into movies, capturing the corresponding characteristic in the non-autonomous setup. An example which has been highly studied is that of a so-called random fixed point: this is simply a point in state space which is moving over time. This approach has been used to describe the asymptotic behavior in some classes of random dynamical systems.

Non-autonomous dynamical systems are also known as random, forced and time-dependent systems, as well as skew-products or cocycles. They consist of a *driving system* $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ and an *evolution rule*. The latter consists of an indexed family of maps $\mathcal{L}_\omega := \mathcal{L}(1, \omega, \cdot)$, where $\mathcal{L} : \mathbb{T} \times \Omega \times X \to X$ is called the *generator* of the system, and is generally assumed to be a measurable function. Here, we will focus on the case of one sided discrete time systems, i.e. $\mathbb{T} = \mathbb{N} := \{0, 1, 2, \dots\}$, but other common choices are $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}, \mathbb{R}^+\}$, allowing for two sided and continuous time systems as well. Thus, the full non-autonomous dynamical systems is described by a tuple $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$. A systematic treatment of these systems is presented in the 1998 book by L. Arnold [**2**].

2.1. The driving system. The driving system encodes the external influences on the system of interest. It is modeled by a probability preserving transformation σ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That is, $\mathbb{P}(\sigma^{-1}E) = \mathbb{P}(E)$ for every measurable set $E \in \mathcal{F}$. We assume σ is invertible[2] and ergodic; that is, if $E \in \mathcal{F}$ is such that $\sigma^{-1}E = E$, then $\mathbb{P}(E) \in \{0, 1\}$.

[2]This feature can often be obtained by considering the natural extension.

This scenario is very flexible. For example, it allows for modeling deterministic driving, such as quasi-periodic forcing, e.g. $\Omega = S^1$, $\mathbb{P} = \text{Leb}$, $\sigma(\omega) = \omega + \alpha \pmod 1, \alpha \notin \mathbb{Q}$. Furthermore, the setting also allows for handling a stochastic framework, in which the external forcing can reflect the influence of noise, for example by setting $\Omega = [-\epsilon, \epsilon]^{\mathbb{Z}}$, $\mathbb{P}$ a product of uniform measures, and σ the shift. In addition, this setting generalizes the autonomous case, which can be recovered by choosing $\Omega = \{\omega_0\}, \mathbb{P} = \delta_{\omega_0}, \sigma = \text{Id}$.

2.2. The evolution rule. For each element $\omega \in \Omega$ there is a corresponding evolution operator $\mathcal{L}_\omega : X \to X$. Here X is the state space where the non-autonomous dynamical system evolves. A useful way to think about the whole system is to regard Ω as representing time. Then, $\mathcal{L}_\omega$ corresponds to the evolution rule at the instant ω. The so-called *cocycle property* is necessary to describe the evolution rule for the cocycle after $n \in \mathbb{N}$ steps:

$$\mathcal{L}_\omega^{(n)} := \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega. \tag{1}$$

When the maps $\mathcal{L}_\omega$ are linear transformations, the cocycle is called a *linear cocycle*.

Among the most extensively studied linear cocycles is the so-called derivative cocycle, which arises from taking a differentiable map $f : M \to M$ as the base transformation, and constructing the cocycle by composing derivatives $\{Df_x\}_{x\in M}$. In this case, $\sigma = f$, $\Omega = M$, $\mathbb{P}$ must be a probability measure preserved by f, $X = T_xM \cong \mathbb{R}^{\dim M}$, and $\mathcal{L}_x = Df_x$. The chain rule implies that the matrix obtained by composition of the matrices along the cocycle exactly describes the linearization (derivative) of the evolution of the system from time 0 to time n: $D(f^n)_x = Df_{f^{n-1}x} \circ \cdots \circ Df_{fx} \circ Df_x$.

In the applications discussed in this work, two other types of evolution operators are considered. In the first one, X is finite, $X = \{B_1, \dots, B_n\}$. This choice corresponds to a partition of the state space for a non-autonomous dynamical systems of interest, with evolution rules dictated by $T_\omega : M \to M$, where $M = \cup_{i=1}^n B_j$. The evolution rule associated to $\omega \in \Omega$ is in this case a matrix, A_ω, obtained by the so-called Ulam's method [**92**]: $(A_\omega)_{ij} = \frac{m(B_i \cap T_\omega^{-1} B_j)}{m(B_i)}$. In words, the ij-th entry of A_ω is the fraction of orbits starting in B_i which fall inside B_j under one step of the dynamics T_ω, conditional on starting in B_i. This construction relies on having a reference measure m on M, which in applications can frequently be taken to be Lebesgue measure. This is a very intuitive finite-state Markovian model approximating the dynamics of the system.

Another type of evolution operator, highly relevant to the problems at hand, is given by transfer operators. These are bounded linear operators $\mathcal{L}_\omega$ acting on an infinite dimensional Banach space $X = \mathcal{B}$, which encode the evolution of densities or ensembles of particles under the dynamics T_ω. These operators will be the topic of Section 3.

3. Transfer operators and quasi-compactness

Transfer operators yield powerful tools for the study of transport in dynamical systems. Instead of representing the pointwise dynamics of the system, they encode the evolution of densities or ensembles under the dynamics, as illustrated in Figure 2. In many instances, transfer operators may be shown to be bounded linear operators, but typically have the difficulty of acting on infinite dimensional Hilbert or Banach spaces.

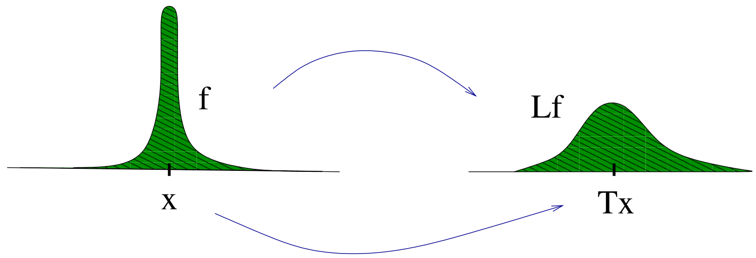

FIGURE 2. Schematic representation of the action of a transfer operator L on a density f, and its relation with the pointwise action of T.

Consider a probability space $(M, \mathcal{M}, m)$ and a non-singular[3] dynamical system $T : M \to M$. The transfer operator L associated to T is a linear operator $L : X \to X$ defined via the duality relation:

$$\int f \cdot g \circ T \, dm = \int Lf \cdot g \, dm, \text{ for every } f \in X, g \in X^*.$$

Here, X can be thought of a space such as $L^1(m)$, but many other possibilities have been used in the literature. In fact, the choice of X is often part of the challenge faced in concrete applications.

3.1. The autonomous setting. A concept which has been a key to the current understanding of transport properties of dynamical systems is that of *quasi-compactness*. This property was investigated in the work of Ionescu Tulcea and Marinescu [**59**], and later also by Nussbaum and Hennion [**83, 57**].

A bounded linear operator L acting on a Banach space $(X, \|\cdot\|)$ is called quasi-compact if its spectral radius ρ is strictly larger than its essential spectral radius ρ_e. The *spectral radius* of L is defined as $\rho(L) := \lim_{n\to\infty} \|L^n\|^{1/n}$, and one way of defining the *essential spectral radius* of L is $\rho_e(L) := \lim_{n\to\infty} ic_X(L^n)^{1/n}$, where

$$ic_X(A) := \inf\{r > 0 : A(B_X) \text{ can be covered with finitely many balls of radius } r\},$$

and $B_X = \{x \in X : \|x\| \leq 1\}$ denotes the unit ball of X. Existence of the limit follows in both cases from submultiplicativity. It is straightforward to see that $\rho \geq \rho_e$, and also that L is a compact operator if and only if $\rho_e = 0$.

When L is quasi-compact and $\tau > \rho_e(L)$, the spectrum of L outside the disc of radius τ consists only of finitely many eigenvalues, $\gamma_1, \ldots, \gamma_l$ of finite (algebraic) multiplicity. In particular, there is a *spectral gap* between the leading eigenvalue and the rest of the spectrum. As in the case of compact operators, it may be that eigenvalues of L accumulate on the disc of radius $\rho_e(L)$.

Furthermore, if L is quasi-compact, it may be written as follows:

$$L = \sum_{i=1}^{l} (\gamma_i P_i + D_i) + R,$$

[3] $T : M \to M$ is non-singular if for every $E \in \mathcal{M}$, $m(T^{-1}(E)) = 0$ if and only if $m(E) = 0$.

where P_i is the eigen-projection and D_i the eigen-nilpotent corresponding to γ_i, [**9**, §1.3], [**63**, §III.5]. In addition, $\rho(R) < |\gamma_l|$, and the following relations hold.

$$P_iP_j = \delta_{ij}P_i, \quad P_iD_i = D_iP_i = D_i, \quad (L-\gamma_i)P_i = D_i, \quad P_iR = RP_i = 0.$$

In particular, these imply that

$$L^n = \sum_{i=1}^{l}(\gamma_i P_i + D_i)^n + R^n.$$

Thus, quasi-compactness allows for the spectral analysis of the largest magnitude eigenvalues of L, which are dynamically the most important.

When quasi-compactness holds in the context of transfer operators, the leading eigenvalue is always $\gamma_1 = 1$. The eigenvalues of large modulus and their corresponding (generalized) eigenspaces have been associated with fundamental properties of the underlying dynamical system.

3.1.1. *Peripheral eigenvalues.* The spectral decomposition for the transfer operator (acting on the space of functions of bounded variation, $X = BV$) was used by Hofbauer, Keller [**58**] and Rychlik [**88**] in the study of of piecewise smooth, piecewise expanding maps of the interval. In those works, it was shown that the multiplicity k_1 of the leading eigenvalue 1 provides the number of ergodic components of positive Lebesgue measure – these are considered the relevant ones from a *physical* point of view. A basis for the associated eigenspace may be taken as a set $\{\phi_1, \dots, \phi_k\}$ of non-negative functions with disjoint supports $\pmod m$.

Each of the ϕ_j gives rise to an *absolutely continuous invariant measure (acim)* μ_j defined by $\frac{d\mu_j}{dm} = \phi_j$, describing the long term statistical behavior of $m-$almost every trajectory in the support of ϕ_j, $M_j = supp(\phi_j)$. This is in essentially the same way that the stationary distribution of a finite-state Markov chain describes the long term statistical behavior of almost all realizations, via the law of large numbers: For $m-$almost every $x \in M_j$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) = \int f\phi_j dm.$$

Furthermore, when 1 is simple ($k_1 = 1$), the existence of unit-length eigenvalues different from 1 is the only obstruction for the system to have the so-called mixing property[4]. In fact, if $|\gamma_2| < 1$, then the exponential decay of correlations property holds: Let μ be the measure given by $\frac{d\mu}{dm} = \phi_1$. Then, for every $f \in X, g \in X^*$,

$$\Big| \int f \cdot g \circ T^N d\mu - \int f d\mu \int g d\mu \Big| \leq C\|f\|_X \|g\|_{X^*} |\gamma_2|^N.$$

Finer statistical properties, such as central limit theorems, weak invariant principles and laws of iterated logarithms have also been established from spectral properties. The seminal work of Keller and Liverani [**65**] shows stability properties of the spectrum in this and more general settings.

Thanks to the great amount of work devoted to the study of transfer operators in the last decades, these properties are now known to hold in much more general settings than piecewise smooth expanding maps in BV. They essentially rely on establishing the quasi-compactness property in an adequate Banach space. Examples include higher dimensional (piecewise) expanding systems, as well as some

[4]The system is mixing if for every $A, B \in \mathcal{M}$, $\lim_{N\to\infty} m(A \cap T^{-N}B) = m(A)m(B)$.

(piecewise) hyperbolic systems. The book by V. Baladi [**9**] contains references up to the year 2000. More recent work in this direction includes [**15, 12, 56, 27, 10**].

3.1.2. *Sub-unit eigenvalues.* In the 1990's, it was suggested by a group of applied mathematicians, including Dellnitz, Junge, Deuflhard and Schütte, that sub-unit eigenvalues of the transfer operators together with their associated eigenspaces also encompass important dynamical information [**26, 28, 25**]. For example, they can encode the location of so-called metastable regions or almost invariant components.

It remained for some time unknown whether sub-unit isolated eigenvalues could be found in some well studied dynamical systems settings. For instance, the first explicit example of an analytic expanding map of the circle with a sub-unit eigenvalue larger than the essential spectral radius was only found in 2004 [**67**]. Since then, the development of the corresponding theory clarifying the connection between metastability and spectral properties of the transfer operator has received considerable attention [**66, 52, 49, 45, 31, 7, 55, 90**].

This approach has been successfully used by Froyland and collaborators in the study of models of oceanic flow, for example to detect gyres in the Southern ocean [**46**].

3.1.3. *The Koopman operator and a duality relation.* The Koopman or composition operator $\mathcal{K} : X^* \to X^*$, given by $g \mapsto g \circ T$, is the dual of the transfer operator in that $\int \mathcal{L}f \cdot g dm = \int f \cdot g \circ T dm$ for every $f \in X, g \in X^*$.

The Koopman operator is receiving considerable attention in the study of dynamical systems because direct information regarding the evolution of the Koopman operator can be recovered from data measurements [**20**]. For example, suppose T describes the evolution of a fluid over one unit of time, and g is a function measuring temperature, or a similar observable quantity, at different moments in time and along a large number of sensors placed in different spatial locations. Then, each set of measurements taken after j units of time yields information regarding the corresponding power $\mathcal{K}^j = g \circ T^j$.

Despite being closely related, the transfer operator and Koopman operator points of view of the dynamics have so far been considered mostly separately in the literature. An approach which combines the dual perspectives is discussed in Section 4.3.

3.2. The non-autonomous setting. Two common approaches to the study of non-autonomous systems are the *annealed* and *quenched* points of view. In the annealed case, the focus is on identifying properties of an averaged system which describe global features of the non-autonomous dynamics. This approach has been successful in the study of systems which are driven by a random iid process, such as deterministic systems perturbed by noise [**13, 65**].

The independence property in the driving system allows for the construction of an annealed transfer operator, $\bar{\mathcal{L}} := \int_\Omega \mathcal{L}_\omega d\mathbb{P}(\omega)$, which in some sense represents the behavior of the non-autonomous system. However, when the driving process is not iid, the operator $\bar{\mathcal{L}}$ does not necessarily reflect dynamical properties of the underlying non-autonomous system. In fact, it is not known whether in this more general case there exists an annealed operator reflecting relevant dynamical properties.

The quenched perspective seeks to understand the system by considering individual realizations of the driving process. Early works combining this approach with transfer operators technology include [**69, 68, 8, 11, 22, 21**].

Buzzi [**22**] investigated non-autonomous compositions of piecewise smooth interval maps, under fairly general assumptions on expansion and *distortion*. He established the existence of at least one and at most finitely many so-called *random absolutely continuous invariant measures (acims)*, $\mu = \{\mu_\omega\}_{\omega\in\Omega}$, each with an associated basin $B(\mu) \subset M$ of positive Lebesgue measure, and whose union covers $M \pmod m$. The probability measures μ_ω have associated densities $\phi_\omega = \frac{d\mu_\omega}{dm}$ satisfying the equivariance relation $\mathcal{L}_\omega \phi_\omega = \phi_{\sigma\omega}$. This condition ensures that random acims also satisfy a law of large numbers type condition: For $m-$almost every $x \in B(\mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T_\omega^{(n)} x) = \int \int f d\mu_\omega d\mathbb{P}(\omega).$$

In subsequent work [**21**], Buzzi identified conditions ensuring exponential decay of random correlations, as follows:

$$\Big| \int f \cdot g \circ T_\omega^{(N)} d\mu_\omega - \int f d\mu_\omega \int g d\mu_{\sigma^n\omega} \Big| \leq C(\omega) \|f\|_{BV} \|g\|_\infty |\gamma_2|^N.$$

Limit theorems have also been investigated in the non-autonomous setting. Kifer [**70**] introduced a time-dependent centering term, and obtained a version of the central limit theorem, as well as other limit laws. Without the time-dependent centering, this type of limit theorems have only been established in cases where all maps preserve a common invariant measure [**4, 82, 1**].

A key component of the recent progress in the study of transport in non-autonomous systems has been the formulation of a non-autonomous version of the quasi-compactness property, going back to Thieullen [**91**]. A non-autonomous dynamical systems $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ is called quasi-compact if its *maximal Lyapunov exponent*,

$$\Lambda(\mathcal{R}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|_X,$$

is strictly larger than its *index of compactness*,

$$\kappa(\mathcal{R}) := \lim_{n\to\infty} \tfrac{1}{n} \log \mathrm{ic}_X(\mathcal{L}_\omega^{(n)}),$$

where ic_X was defined in Section 3. The fact that these limits exist and are constant $\mathbb{P}-$almost everywhere is a consequence of (i) submultiplicativity of both $\|\cdot\|_X$ and $ic_X(\cdot)$, (ii) ergodicity of the driving system σ and (iii) Kingman's sub-additive ergodic theorem [**71**]. As in the autonomous case, it is straightforward to show that $\Lambda(\mathcal{R}) \geq \kappa(\mathcal{R})$.

The non-autonomous quasi-compactness property has allowed for the possibility of extending the use of transfer operator technology to the study of coherent structures, which are the analogue of metastable regions in the non-autonomous setting. This approach has been facilitated by the so-called multiplicative ergodic theorems, discussed in Section 4.

4. Multiplicative ergodic theorems and coherent structures

Multiplicative ergodic theorems give rise to a time-dependent decomposition which plays the role of a spectral decomposition in the non-autonomous setup. The so-called *Oseledets splitting* has been used to describe the dominant transport features of the dynamics via non-linear, time-varying modes called *coherent structures*, which decay slowly over time. The rate of decay of each structure is measured by an asymptotic exponential rate, called Lyapunov exponent. This is in contrast with the one-step contraction (or expansion) factor, which an eigenvalue would yield.

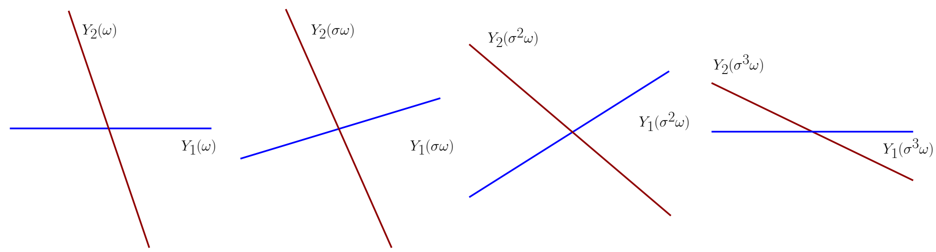

FIGURE 3. Schematic representation of the evolution of Oseledets splittings under a non-autonomous dynamical system.

Intuitively, one may think of each of the components of the splitting as describing one evolving feature of the system, such as a vortex, or an eddy. However, in practice, recovering the actual spatial features from the decomposition requires significantly more work, and there is still considerable research effort taking place in that direction.

The study of multiplicative ergodic theorems covers the cases of finite as well as infinite dimensional underlying spaces X. The finite dimensional case includes settings such as discrete time finite-state Markov chains. The infinite dimensional setting is especially relevant for analyzing more general models of physical phenomena, including the framework of transfer operators. Another distinction among multiplicative ergodic theorems considers whether the evolution rule is invertible or not. While both cases have been investigated in the literature from the start, it is only recently that results guaranteeing a splitting have been established for non-invertible, more precisely semi-invertible, cocycles.

4.1. Definitions and convention. Consider a Banach space X. The *Grassmannian* of X, $\mathcal{G} = \mathcal{G}(X)$, is the set of closed linear subspaces of X, which have closed complements. Included in $\mathcal{G}(X)$ are all finite dimensional and finite codimensional subspaces $\mathcal{G}_k(X) = \{E \subset X, \dim E = k < \infty\}$ and $\mathcal{G}^k(X) = \{F \subset X, \operatorname{codim} F = k < \infty\}$. In particular, when X is a finite dimensional space, $\mathcal{G}(X)$ is the set of all linear subspaces of X.

The Grassmannian $\mathcal{G}(X)$ can be endowed with a metric, defined by $d_{\mathcal{G}}(E,F) := d_H(E \cap B_X, F \cap B_X)$ where B_X denotes the unit ball in X and d_H the Hausdorff distance, $d_H(A,B) := \max\{\sup_{a\in A} \inf_{b\in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|a-b\|\}$. Some fundamental properties of $\mathcal{G}(X)$ are established in [**63**] and, in the context of separable X[5], in [**53**, Appendix B].

An *Oseledets splitting* for a linear cocycle $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ consists of:

(I) A sequence of isolated (exceptional) *Lyapunov exponents* $\infty > \lambda_1 > \cdots > \lambda_l > \kappa \geq -\infty$, where the index $l \geq 1$ is allowed to be finite or countably infinite; and

(II) A family of ω-dependent splittings $X = \bigoplus_{j=1}^{l} Y_j(\omega) \oplus V(\omega)$, where $\dim Y_j(\omega) < \infty$ for every $1 \leq j \leq l$, and $V(\omega) \in \mathcal{G}(X)$,

[5]A Banach space is called *separable* if it has a countable dense subset.

satisfying the following conditions:

(i) Measurability:
The maps $\omega \mapsto Y_j(\omega)$ and $\omega \mapsto V(\omega)$ are $(\mathcal{F}, \mathcal{B}(\mathcal{G}))$ measurable, where $\mathcal{B}(\mathcal{G})$ is the Borel σ-algebra of the Grassmannian $\mathcal{G}(X)$.

(ii) Equivariance:
For each $1 \leq j \leq l$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,
$$\mathcal{L}_\omega Y_j(\omega) = Y_j(\sigma\omega) \text{ and } \mathcal{L}_\omega V(\omega) \subseteq V(\sigma\omega).$$

(iii) Growth rates:
For every $1 \leq j \leq l$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,
$$\lim_{n\to\infty} \tfrac{1}{n} \log \|\mathcal{L}_\omega^{(n)} y\| = \lambda_j \text{ for every } y \in Y_j(\omega) \setminus \{0\}, \text{ and}$$
$$\lim_{n\to\infty} \tfrac{1}{n} \log \|\mathcal{L}_\omega^{(n)} v\| \leq \kappa \text{ for every } v \in V(\omega).$$

(iv) Tempered projections:
Let $\Pi_j(\omega) = \Pi_{Y_j(\omega)\|\oplus_{i\neq j} Y_i(\omega)\oplus V(\omega)}$ and $\Pi'_j(\omega) = \Pi_{\oplus_{i\neq j} Y_i(\omega)\oplus V(\omega)\|Y_j(\omega)}$[6]. Then,
$$\lim_{n\to\pm\infty} \tfrac{1}{n} \log \|\Pi_j(\sigma^n\omega)\| = 0 = \lim_{n\to\pm\infty} \tfrac{1}{n} \log \|\Pi'_j(\sigma^n\omega)\|.$$

It is worth mentioning that, in the context of invertible cocycles, the original work of Oseledets and many subsequent works state, instead of property (iv), a condition on sub-exponential decay of angles between subspaces along trajectories. For instance, for $j' \neq j$,
$$\lim_{n\to\pm\infty} \log \sin \angle(Y_j(\sigma^n\omega), Y_{j'}(\sigma^n\omega)) = 0.$$
Property (iv) implies this alternative statement, when the notion of angle between subspaces, $\angle(E, F)$, (or sine thereof) is extended to Banach spaces, for example as the Grassmannian distance between them, $d_{\mathcal{G}}(E, F)$. This follows from the fact that if $d_{\mathcal{G}}(E, F) < \delta$, then for every $e \in E$ of norm 1, there exists $f \in F$ so that $\|e - f\| < \delta$. In particular, since $\Pi_{E\|F}(e - f) = e$, then $\|\Pi_{E\|F}\| > \delta^{-1}$.

Throughout this section, we will say that the *multiplicative ergodic theorem* holds for the non-autonomous dynamical systems $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ if $\mathcal{R}$ has an Oseledets splitting. However, we note that the term is used in a more general way in the literature.

4.2. Brief history.

4.2.1. *Finite dimensional systems.* Multiplicative ergodic theorems (METs) were first discovered in the mid 1960s by Oseledets [**85**] in the context of finite-dimensional linear cocycles. The existence of an Oseledets splitting was established for non-autonomous dynamical systems of the form $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathbb{R}^n, A)$, where $A : \Omega \to GL(n, \mathbb{R})$ is an invertible matrix valued cocycle satisfying the integrability conditions $\int_\Omega \log^+ \|A(\omega)\| d\mathbb{P}(\omega) < \infty$ and $\int_\Omega \log^+ \|A^{-1}(\omega)\| d\mathbb{P}(\omega) < \infty$.

Furthermore [**85**] addressed the non-invertible case, where $A : \Omega \to M_{n\times n}(\mathbb{R})$ satisfies $\int_\Omega \log^+ \|A(\omega)\| d\mathbb{P}(\omega) < \infty$, and also $\sigma : \Omega \to \Omega$ is allowed to be non-invertible. In this setting, a coarser conclusion was provided, for $\mathbb{P}$-a.e. $\omega \in \Omega$: Aside from the sequence of Lyapunov exponents, there exists a measurable family of ω-dependent *filtrations*, $X = V_1(\omega) \supset V_2(\omega) \cdots \supset V_l(\omega) \supset \{0\}$, which are

[6]When the spaces $E, F \in \mathcal{G}(X)$ have trivial intersection, $E \cap F = \{0\}$, the operator $\Pi_{E\|F}$ denotes *projection onto E along F*; that is, $\Pi_{E\|F}(e + f) = e$ for every $e \in E, f \in F$.

equivariant in the sense that $A(\omega)V_j(\omega) \subseteq V_j(\sigma\omega)$, and satisfy the growth rate condition $\lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(\omega)v\| \leq \lambda_j$, for every $v \in V_j(\omega)$.

In 1979, Raghunathan [**86**] provided an alternative proof of the multiplicative ergodic theorem of Oseledets relying on the singular value decomposition (SVD). This approach has turned out to be useful in much more general settings, and is also at the base of numerical methods for approximation of Oseledets splittings, discussed in Section 5.

One of the major motivations for the development of multiplicative ergodic theorems has been their power in establishing the existence of *stable and unstable manifolds* in dynamical systems. The book by Barreira and Pesin [**14**] includes a detailed treatment of the multiplicative ergodic theorem in the setup of smooth ergodic theory.

Further generalizations of the multiplicative ergodic theorem have been established in other settings, including semi-simple Lie groups [**60**], non-positively curved spaces [**62**] and isometry groups of metric spaces [**61**].

In 2010, Froyland, Lloyd and Quas [**42**] extended Oseledets' theorem to the case of *semi-invertible* cocycles. They established the multiplicative ergodic theorem for finite dimensional linear cocycles $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathbb{R}^n, A)$ under the assumptions of invertibility of σ and integrability of $\log^+ \|A(\omega)\|$, without any invertibility hypotheses on the matrices $A(\omega)$.

4.2.2. *Infinite dimensional systems.* The first multiplicative ergodic theorems in the infinite dimensional case are due to Ruelle [**87**] and Mañé [**76**]. They showed the existence of Oseledets splittings for injective, compact operators acting on separable Hilbert and Banach spaces, respectively. Ruelle treated discrete as well as continuous time systems, and had in mind applications to partial differential equations arising in the study of hydrodynamic turbulence. Mañé treated discrete time systems and aimed for applications to parabolic semilinear equations and delay functional differential equations.

Thieullen [**91**] extended Mañé's result to cocycles of non-autonomous quasi-compact operators on Banach spaces, satisfying a continuity property called $\mathbb{P}$-continuity. In the early 1990s, related results were also established in the context of stochastic partial differential equations [**89**] and stochastic linear delay equations [**77**].

Lian and Lu [**75**] established a version of the multiplicative ergodic theorem which relaxed the continuity condition of [**91**] to a (strong) measurability condition, at the cost of requiring the Banach space X to be separable.

Having in mind applications to the detection of coherent structures in oceanic and atmospheric dynamics, the recent works of Froyland, Lloyd, Quas and González-Tokman [**43, 53, 54**] extend infinite dimensional versions of the multiplicative ergodic theorem to the setting of semi-invertible cocycles. Based on [**91**], Froyland, Lloyd and Quas established a multiplicative ergodic theorem in the context of $\mathbb{P}$-continuous cocycles [**43**]. Building on [**75**] and [**30**], Quas and González-Tokman replaced the continuity condition by a strong measurability condition, together with the requirement that the Banach space X (and its dual X^*) is separable [**53, 54**]. A related volume-based approach to the multiplicative ergodic theorem on Banach spaces is presented in [**16**].

4.3. Identification of Oseledets splittings and coherent structures. The breakthrough paper of Froyland, Lloyd and Quas [**42**] made the connection between the hierarchical structure provided by multiplicative ergodic theorems and coherent structures in non-autonomous dynamical systems. It was for the first time then that ideas such as the second Oseledets space encoding information regarding a dynamical structure which, while changing its location over time, retains a large portion of its *mass* as it evolves were put on firm mathematical grounds. It is also worth remarking that the theoretical advances of [**42**] were followed by a numerical demonstration of their applicability in non-autonomous models of fluid flow in [**44**], a paper in which the term *coherent sets* was coined.

Next, we present a recent, general method for the identification of Oseledets splittings in the context of linear cocycles on Banach spaces, following [**54**]. Applications of the theory to the identification of coherent structures in geophysical flows are discussed in Section 6. We recall that a singular value decomposition (SVD)[7] of a real $d \times d$ matrix M consists of a diagonal matrix D, and orthogonal matrices U and V such that $M = UDV^T$, where V^T is the transpose of V. The diagonal entries s_j of D are the singular values of M, which will be considered in decreasing order $s_1 \geq s_2 \geq \cdots \geq s_d$. The column vectors of U and V are called left and right singular vectors of M, respectively. If v_j denotes the j^{th} column vector of V, then $Mv_j = s_j u_j$. Hence, the vectors $v_1, \dots, v_d$ form an orthonormal basis of $\mathbb{R}^d$ which is transformed by M into the orthogonal basis $s_1 u_1, \dots, s_d u_d$.

It is worth remarking that the case of top singular values and singular vectors of a matrix M are easy to visualize. The top singular value is the norm of M and the associated right singular vector(s) is (are) the unit length vectors v for which the norm of M is realized, $\|Mv\| = \|M\|$. The corresponding left singular vector(s) is (are) the normalized images of the right singular vector(s). Similarly, in the non-autonomous setting, the top Oseledets space and top Lyapunov exponent correspond to the fastest growing vectors and maximal exponential growth rate for the composition of matrices along the cocycle, respectively.

In the finite dimensional setting, the approach of Raghunathan [**86**] gives rise to a method for identifying the Lyapunov exponents and components of the components of the Oseledets filtration, $\bigoplus_{i\geq j} Y_i(\omega)$, by considering SVDs of the matrices $A^{(n)}(\omega)$. Indeed, the Lyapunov exponents are $\lambda_j = \lim_{n\to\infty} \frac{1}{n} \log s_{d_j}(A^{(n)}(\omega))$, where $s_k(M)$ denotes the k^{th} largest singular value of M, and $d_j = 1 + m_1 + \cdots + m_{j-1}$, with $m_0 = 0$ and for each $i \geq 1$, $m_i = \dim Y_i(\omega)$ is the multiplicity of λ_i. Furthermore, $\bigoplus_{i\geq j} Y_i(\omega) = \lim_{n\to\infty} \bigoplus_{i\geq j} E_i(A^{(n)}(\omega))$ for each j, where $E_j(M)$ denotes the m_j-dimensional space spanned by right singular vectors of M with singular values $s_{d_j}, \dots, s_{d_j+m_j-1}$.

The SVD decomposition is not available in the infinite-dimensional Banach space setup, because it relies on the notion of orthogonality. However, an inductive algorithm to identify Oseledets spaces building on the essential properties of SVDs, which is valid in both finite and infinite dimensional settings, is presented in [**54**]. To overcome the lack of the notion of orthogonality, the algorithm also involves the so-called dual cocycle $\mathcal{R}^* = (\Omega, \mathcal{F}, \mathbb{P}, \sigma^{-1}, X^*, \mathcal{L}^*)$, as well as different notions of volume in Banach spaces. Here X^* is the dual of X, and $\mathcal{L}^*_\omega$ is defined as the adjoint to $\mathcal{L}_{\sigma^{-1}\omega}$, $\mathcal{L}^*_\omega := (\mathcal{L}_{\sigma^{-1}\omega})^*$. In particular, $\mathcal{L}^{*(n)}_{\sigma^n\omega} = (\mathcal{L}^{(n)}_\omega)^*$. While the analysis is

[7]The SVD of a matrix M can be obtained from the eigen-decomposition of the symmetric matrices MM^T and M^TM. It is not always unique, although the singular values as well as the spaces spanned by singular vectors associated to a fixed singular value are uniquely defined, just as eigenspaces are.

significantly more technical than in the finite-dimensional case, Lyapunov exponents can also be obtained from the maximal exponential growth rates of volumes of parallelepipeds of increasing dimensions.

An algorithm to identify Oseledets splittings is as follows. Let $V_1^{(n)}(\omega) = X$, and for each $l > 1$, and $n \in \mathbb{N}$, let $k_l = m_0 + \cdots + m_{l-1}$, and $\theta_1^{(n)}, \ldots, \theta_{k_l}^{(n)} \in S_{X^*}$ be unit vectors in X^* such that the volume of the parallelepiped with sides $(\mathcal{L}_\omega^{(n)})^* \theta_1^{(n)}, \ldots, (\mathcal{L}_\omega^{(n)})^* \theta_{k_l}^{(n)}$ is (nearly) maximal over all possible choices of k_l-tuples in S_{X^*}. Let $V_l^{(n)}(\omega) \subset X$ be the annihilator of the linear span of $(\mathcal{L}_\omega^{(n)})^* \theta_1^{(n)}, \ldots, (\mathcal{L}_\omega^{(n)})^* \theta_{k_l}^{(n)}$,

$$V_l^{(n)}(\omega) = \big\langle (\mathcal{L}_\omega^{(n)})^* \theta_1^{(n)}, \ldots, (\mathcal{L}_\omega^{(n)})^* \theta_{k_l}^{(n)} \big\rangle^{\perp}.$$

It is shown in [**54**] that the sequence $V_l^{(n)}(\omega)$ has a limit $V_l(\omega)$ as n approaches infinity. This provides an Oseledets filtration, just like in the finite-dimensional approach of Raghunathan.

Calling $V_l^*(\omega)$ the corresponding Oseledets filtration for the dual cocycle $\mathcal{R}^*$ and letting $W_l(\omega) = (V_l^*(\omega))^{\perp}$, one obtains a collection of equivariant spaces complementary to the $V_l(\omega)$, $X = V_l(\omega) \oplus W_l(\omega)$. The Oseledets splitting is recovered by letting $Y_l(\omega) = V_l(\omega) \cap W_{l+1}(\omega)$.

It is worth noting that the multiplicative ergodic theorems discussed in Section 4.2 can be applied to Koopman operator cocycles, duals of transfer operator cocycles, whenever the necessary hypotheses hold[8].

The level sets associated to vectors in the dominant Oseledets spaces of transfer operator cocycles (or their finite-time approximations) have been used to locate coherent structures in models of non-autonomous dynamical systems, for example, by thresholding singular vectors at optimal values [**48**]. Thus, the algorithm discussed above may assist in validating existing singular-value decomposition based methods, and perhaps also yield new insights for the identification of coherent structures in applications.

4.4. Stability. A topic of paramount interest for applications is that of stability: How much do the behavior and properties of a dynamical system change under perturbations? The answer to this question is highly relevant for modeling, both analytically and numerically, as well as for accounting for noise and random effects which may affect the evolution of the system.

In our setting, the stability question translates into whether Oseledets splittings and Lyapunov exponents are stable under perturbations. One of the difficulties of this problem is that for stability to hold, non-trivial conditions have to be imposed on the non-autonomous dynamical systems. Indeed, without extra hypotheses, there are examples where Lyapunov exponents undergo discontinuous changes under perturbations. This is in contrast with the case of autonomous systems, where spectral stability properties of linear operators are known to hold in very broad generality [**63**].

A topic complementary to stability is bifurcation theory, which investigates qualitatively significant changes which can occur when a system undergoes small perturbations. For a treatment of bifurcations in non-autonomous dynamical systems, we refer the reader to the monograph [**72**].

[8]Quasi-compactness and integrability conditions for $\mathcal{K}$ would follow directly from the analogue conditions for $\mathcal{L}$. However, separability of X^* does not follow from separability of X.

4.4.1. *Finite dimensional systems.* The question of stability of Lyapunov exponents for invertible matrix cocycles was treated by Young in [**96**], and later by Ledrappier and Young in [**73**]. They showed stability of Lyapunov exponents when sufficiently random perturbations are considered. In a similar context of invertible matrix cocycles, Ochs showed that stability of Lyapunov exponents is equivalent to stability of the associated Oseledets splitting [**84**]. Recently, Froyland, González-Tokman and Quas [**37**] established a finite-dimensional stochastic stability result for Lyapunov exponents and Oseledets splittings associated to semi-invertible matrix cocycles. Further results concerning stability of Lyapunov exponents in different settings are presented in the recent book by Viana [**94**], and in the forthcoming research monograph by Duarte and Klein [**32**].

Negative stability results for finite-dimensional cocycles include [**3**], which shows that an arbitrarily small L^p perturbation can make all Lyapunov exponents collapse to one point. Another mechanism for instability of Lyapunov exponents has been shown in the context of derivative cocycles associated to area-preserving C^1 diffeomorphisms [**17**]. The idea behind such a method is attributed to Mañé, and it relies on carefully interchanging vectors corresponding to directions of strongest and weakest expansion on *non-uniformly hyperbolic* cocycles, where there is no uniform lower bound for the angles between Oseledets spaces associated to different Lyapunov exponents.

4.4.2. *Infinite dimensional systems.* In the autonomous case, the work of Keller and Liverani [**65**] guarantees stability of isolated eigenvalues and their associated eigenspaces for quasi-compact operators. This technology may be applied in the setting of random iid perturbations of a non-random map, to obtain results for the annealed (averaged) operators. However, the stability problem remains largely open in the general non-autonomous case.

Most of the stability results available in the quenched setup deal with random perturbations of a fixed (non-random) initial map. Baladi, Kondah and Schmitt [**11**] show stability of the random invariant measure for random perturbations of a smooth expanding map. In a similar setting, Baladi obtains stability results for more general equilibrium states in [**8**].

Bogenschütz [**18**] shows stability of Lyapunov exponents and Oseledets splittings under strong hypotheses on the cocycle and the perturbations. An interesting application of this work addresses the case of stability of the splitting for non-autonomous perturbations of an autonomous system. A recent work has employed multiplicative ergodic theorems to investigate small, quenched random perturbations in the context of partially expanding maps on the torus [**81**].

A stability result in the context of transfer operators of random maps has been established in [**36**]. The main result shows stability of the top Oseledets space –the one corresponding to the random invariant measure– under perturbations including model and numerical approximation errors, as well as noisy perturbations of the underlying maps. A relevant feature of this setting, which makes the analysis more tractable than in the case of second and higher components of the Oseledets splitting, is that the leading Lyapunov exponent is always zero and, in fact, there exists a random fixed point.

5. The numerical and computational aspects

The so-called Ulam's method [**92**], introduced in Section 2.2, has provided an intuitive and effective way of analyzing dynamical systems numerically via transfer operators. Figure 4 shows an approximation of the density of the acim for an interval map relying on Ulam's method (blue), and on averages taken along a long trajectory (red).

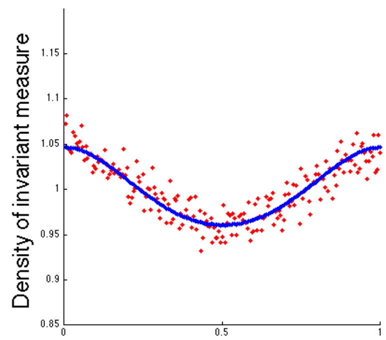

FIGURE 4. Two approximations of the density of the acim of an interval map, obtained via Ulam's method (blue) and by taking averages over a long trajectory (red).

The rigorous convergence properties of this approximation method, as finer and finer grids are considered, are still an open question in general, even in the autonomous case. This is a particular instance of the stability question discussed in Section 4.4. In some settings, including quite general classes of one-dimensional maps, the scheme has been shown to well approximate acims as well as rates of exponential correlation decay. Some works in this direction are [**74, 29, 33, 78, 65, 64, 79, 19, 80, 5, 6**]. Ulam's method was investigated the case of iid and Markovian composition of one-dimensional maps in [**34**].

In the non-autonomous case, efficient numerical algorithms to identify Lyapunov exponents and Oseledets spaces for finite-dimensional (matrix) cocycles have been developed in the last decade by Ginelli and coauthors [**51**], Wolfe and Samelson [**95**] and Froyland, Hüls, Morriss and Watson [**40**]. These approaches rely on very efficient routines built into standard mathematical software, including the QR decomposition and singular-value decomposition algorithms. A comparison of these methods in different model scenarios is presented in [**40**].

Computational algorithms relying on Ulam's method have been used to identify almost invariant sets and coherent structures in autonomous and non-autonomous dynamical systems, for example in the context of geophysical flows, as will be detailed in Section 6. Further numerical methods have been proposed to test for regularity properties of the transfer operator spectrum, with the aim of distinguishing between true and spurious eigenfunctions in [**35**].

The computation of Ulam matrices with high resolution can be a costly algorithm, as it requires to sample all regions of the state space at a high resolution, in fact finer than grid size. On the other hand, once this computation is done, the matrix can be stored for later calculations in the autonomous case. In the non-autonomous setting, a similar procedure is possible, with the additional requirement that, in principle, one has to compute one Ulam matrix for each ω. For these reasons, the quest for more efficient numerical algorithms for the computational analysis of dynamical systems is still a topic of current research efforts [**47, 41**].

6. Applications to oceanic and atmospheric flows

An early application of transfer operator methods to the analysis of geophysical flows was the work of Froyland and collaborators [**46**], where they detect the two major oceanic gyres near Antarctica, the Ross and Wedell seas. The seasonal variability as well as three-dimensional features of these sub polar gyres were subsequently investigated in [**24**].

A related approach has also been used to study "origin, dynamics and evolution of ocean garbage patches from observed surface drifters" in [**93**], and to investigate the global connectedness of the ocean surface in [**50**].

The non-autonomous transfer operator technology has been used to detect and track an eddy of the Agulhas ring, moving around the coast of Africa for several months [**38, 39**]. This analysis was done in two and three dimensions, and showed that such coherent structures are not only displaced horizontally parallel to the surface, but also change along the vertical direction. The outcomes of this approach yield significantly (about 15%) more coherent structures than alternative essentially two-dimensional methods, based on relative vorticity or the so-called Okubo-Weiss parameter.

In the atmosphere, the tools have been used to detect the Antarctic polar vortex and track its evolution over a two-week period [**48**].

A related non-autonomous dynamical systems approach, considering random attractors, has been investigated in stochastic models of climate dynamics in [**23**].

Appendix A. Online resources for the visualization of oceanic and atmospheric flows

Besides the illustrations available in the references provided, there also exist wonderful online resources to visualize oceanic and atmospheric flows. Non-autonomous visualizations of the ocean currents from the years 2005–2007 are available at: `http://svs.gsfc.nasa.gov/cgi-bin/details.cgi?aid=3827`.

An interactive website which contains (autonomous) approximations of ocean currents and wind from recent data is `http://earth.nullschool.net`; see also `http://earth.nullschool.net/about.html` for details. These visualizations are built from data which are constantly being updated. Hence, some features which were visible at the time of writing this work may have changed at the time of reloading the website.

Some of the views available are:

- The Gulf Stream:
  `http://earth.nullschool.net/#current/ocean/surface/currents/orthographic=-41.40,29.75,351`
- The Agulhas rings:
  `http://earth.nullschool.net/#current/ocean/surface/currents/orthographic=21.28,-18.45,718`

- Oceanic eddies around Australia:
`http://earth.nullschool.net/#current/ocean/surface/currents/orthographic=-222.20,-26.19,1024`
- Winds at surface level:
`http://earth.nullschool.net/#current/wind/surface/level/orthographic=-98.32,31.17,512`
- The Jet stream:
`http://earth.nullschool.net/#current/wind/isobaric/250hPa/equirectangular`
- The Antarctic polar vortex:
`http://earth.nullschool.net/#current/wind/isobaric/10hPa/orthographic=-189.44,-72.33,322`
- The Arctic polar vortex:
`http://earth.nullschool.net/#current/wind/isobaric/10hPa/orthographic=-135.96,59.60,322`

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