Quenched stochastic stability for eventually expanding-on-average random interval map cocycles

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Abstract

The paper [FGTQ14] established fibrewise stability of random absolutely continuous invariant measures (acims) for cocycles of random Lasota-Yorke maps under a variety of perturbations, including "Ulam's method", a popular numerical method for approximating acims. The expansivity requirements of [FGTQ14] were that the cocycle (or powers of the cocycle) should be "expanding on average" before applying a perturbation, such as Ulam's method. In the present work we make a significant theoretical and computational weakening of the expansivity hypotheses of [FGTQ14], requiring only that the cocycle be eventually expanding on average, and importantly, *allowing the perturbation to be applied after each single step of the cocycle*. The family of random maps that generate our cocycle need not be close to a fixed map and our results can handle very general driving mechanisms. We provide a detailed numerical example of a random Lasota-Yorke map cocycle with expanding and contracting behaviour and illustrate the extra information carried by our fibred random acims, when compared to annealed acims or "physical" random acims.

1 Introduction

The use of numerical approximation schemes in the study of dynamical systems has continued to evolve, benefiting from growing computer power as well as progressively more

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refined methods for analyzing and visualizing relevant features of the dynamics. These exciting advances call for increasingly powerful theory to ensure such numerical schemes indeed represent or well approximate features of interest for the underlying dynamical system.

The calculation of statistical and transport properties has become possible, and increasingly popular [Hsu87, Fro01, KGA+01, Pad05, BM12, FPG14, KNC+14, WRK15], wherein finite rank approximations are made to transfer operators induced by the underlying dynamical system. The credibility of such schemes depends on both the computational feasibility of implementation, and the robustness of the transfer operator dynamics to the types of perturbations inherent in the approximations. The Ulam method [Ula60] is a Galerkin–type projection scheme, and it has gained prominence as a simple and effective way of modeling dynamical systems via Markov models [Hsu87, DJ99, DFJ01, Fro01, DFHP09, FP09, BS13, FPG14]; it can be implemented from model simulations or sufficiently rich observed data. Its robustness is supported by an extensive rigorous literature, beginning in 1976 with a proof of convergence to acims by Li [Li76] in the context of one-dimensional dynamical systems (*Lasota-Yorke maps* [LY73]). Since then, numerous generalizations have followed, extending the rigorous analysis to more general perturbations, uniformly hyperbolic dynamics, higher dimensions, random dynamical systems, open dynamical systems, and nonuniformly expanding dynamics [Kel82, Fro95, DZ96, BK97, Fro99, BM01, Mur01, Bah06, Mur10, BFGTM14].

This numerical method has been successfully applied, in combination with modern developments in dynamical systems, to a wide variety of physical and biological problems; a small sample includes drug design, transport in dynamical astronomy, the identification large scale features of oceanic and atmospheric flows, such as oceanic gyre and eddies and atmospheric vortices, and the tracking of floating plastic garbage in the ocean. [DS04, DJK+05, FPET07, FSM10, FHR+12, MHN12, VSEF12]. This has been a major motivation to pursue investigations regarding convergence properties of the Ulam scheme beyond the autonomous setting, where one single map or vector field dictates the dynamics of the system. The non-autonomous setting considered in this paper allows for the incorporation of random or deterministic external factors which drive the nonlinear dynamics in the original state space.

Mathematically, our "random" driving dynamics will be controlled by an ergodic, invertible, probability preserving base map $\sigma : (\Omega, \mathbb{P}) \circlearrowleft$. Let I be an interval; we form a skew product $\tau : I \times \Omega \circlearrowleft$ with measurable fibrewise dynamics $\tau(x, \omega) = (f_\omega(x), \sigma(\omega))$. We denote $I_\omega = I \times \{\omega\} \subset I \times \Omega$ so that $f_\omega : I_\omega \to I_{\sigma\omega}$. Random or time-dependent orbits of length n beginning at driving configuration ω are produced by the concatenation $f_\omega^{(n)} := f_{\sigma^{n-1}\omega} \circ \cdots \circ f_{\sigma\omega} \circ f_\omega$.

We are concerned with invariant measures μ of τ. The non-autonomous analogue of invariant measures are *random invariant measures*; instead of the invariance condition $\mu = \mu \circ f^{-1}$ for a single map f, random invariant measures satisfy $\mu_{\sigma\omega} = \mu_\omega \circ f_\omega^{-1}$ for $\mathbb{P}$-a.e. ω, where each μ_ω is a probability measure on I_ω. In terms of the skew product τ, by standard disintegration, for $A \subset I \times \Omega$, we can write $\mu(A) = \int_\Omega \mu_\omega(A) \, d\mathbb{P}(\omega)$, considering μ_ω as a probability measure on $I \times \Omega$, supported on $I_\omega \times \{\omega\}$; μ is invariant for τ in the usual sense. We are particularly interested in the situation where μ_ω has a density h_ω with respect to Lebesgue measure; then we say $\{\mu_\omega\}$ is a *random absolutely continuous invariant measure* or *random acim*. Let $\mathcal{P}_\omega$ denote the Perron–Frobenius operator for f_ω, mapping $L^1(I_\omega) \to L^1(I_{\sigma\omega})$ and $\mathcal{P}_\omega^{(N)} = \mathcal{P}_{\sigma^{N-1}\omega} \circ \cdots \circ \mathcal{P}_\omega$. We focus on the situation where there is a unique random acim, with the important physical property that $\lim_{N\to\infty} \mathcal{P}^{(N)}_{\sigma^{-N}\omega} g = h_\omega$ for sufficiently regular densities g and $\mathcal{P}_\omega h_\omega = h_{\sigma\omega}$. Thus, the densities h_ω describe the distribution of orbits at configuration ω, having started at some arbitrary regular distribution g far in the past.

The random acim $\{\mu_\omega\}_{\omega\in\Omega}$ also encodes a non-random physical measure or SRB measure μ [Buz00], which can be constructed from forward time limits of convex combinations of δ-measures along random orbits in I: $\mu := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \delta_{f_\omega^{(i)}(x)}$ for *Leb* a.e. $x \in I$ and $\mathbb{P}$ a.e. $\omega \in \Omega$, where weak convergence is meant. Alternatively, $\mu = \int \mu_\omega \, d\mathbb{P}(\omega)$ and one can take Birkhoff averages of sufficiently regular observables $\phi : I \to \mathbb{R}$ along these trajectories for *Leb* a.e. initial condition $x \in I$ and obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \phi(f_\omega^{(i)} x) = \int \phi(x) h_\omega(x) \; dx \; d\mathbb{P}(\omega). \tag{1}$$

Our results imply that Ulam's method can be used to gain access to this physical measure μ for *general* driving σ (the case where σ is a Bernoulli or Markov shift has been treated in [Fro99]). Despite the importance of the SRB measure μ, we will show that significant temporal information is lost in the integration with respect to $\mathbb{P}$ in (1), and that individual h_ω can be very different from $\int h_\omega \; d\mathbb{P}(\omega)$ (see Section 6 and Figure 1). Finally, we note that when σ is a Bernoulli process, it is natural to form an *annealed* operator $\bar{\mathcal{P}} := \int_\Omega \mathcal{P}_\omega \; d\mathbb{P}(\omega)$. Any τ-invariant measure, absolutely continuous with respect to $Leb \times \mathbb{P}$, has the form $\nu \times \mathbb{P}$ where $d\nu/d(Leb)$ is a fixed point of $\bar{\mathcal{P}}$ [Mor88]. In the Bernoulli setting the non-random probability measure ν is called the annealed invariant measure of the random dynamical system and is also a physical measure in the sense above. When σ is not Bernoulli $\nu \times \mathbb{P}$ is not τ-invariant in general, and there is no known interpretation for $\bar{\mathcal{P}}$; to emphasise this point we compute ν in Figure 1.

1.1 Statement of main results

For our formal results and to ensure absolute continuity we impose some conditions on the fibre maps, namely $\mathbb{P}$–a.e. f_ω are piecewise C^{1+Lip}, with finitely many branches $N_b(\omega)$, satisfying $\int_\Omega \log^+ N_b(\omega) \, d\mathbb{P}(\omega) < \infty$. The first and second derivatives are bounded uniformly above and below: $\Lambda^{-1} \le |f_\omega'| \le \Lambda$ and $|f_\omega''| \le K$ for constants $\Lambda, K < \infty$. We assume expansion on average: let $\lambda(\omega) = \operatorname{ess\,inf}_{x\in I} |f_\omega'(x)|$, $f_\omega^{(N)} = f_{\sigma^{N-1}\omega} \circ \cdots \circ f_{\sigma\omega} \circ f_\omega$ and $\lambda_N(\omega) = \operatorname{ess\,inf}_{x\in I} |f_\omega^{(N)\prime}|$. We assume the existence of a finite $N = N_0$ such that

$$\int_\Omega \log \lambda_{N_0}(\omega) \, d\mathbb{P}(\omega) > 0. \tag{Exp}$$

Finally, to guarantee uniqueness of the random acim, we impose a *covering* condition (introduced by Buzzi in [Buz99]): for every sub-interval $J \subset I$ and a.e. $\omega \in \Omega$, there exists $n_\omega \in \mathbb{N}$ such that $f_\omega^{(n)}(J) = I$. Under the above conditions we say that $\{f_\omega\}_{\omega\in\Omega}$ is an **admissible random Lasota–Yorke map**, as studied in [Buz00, FGTQ14].

The main result of [FGTQ14] guarantees that random acims for random Lasota-Yorke maps are stable under perturbations, including those caused by the numerical error associated to the Ulam scheme, provided sufficient expansion holds, on average, prior to each application of the Ulam scheme; specifically, one needs to apply the Ulam scheme to $f_\omega^{(m)}$ for m at least N_0. The flexibility of the random framework already allows systems that experience periods of contraction, interspersed with expansion, but when one-step average expansion is slow and N_0 is large, the stability ensured by [FGTQ14] may be expensive to obtain.

In this paper, we relax the requirement of $m \geq N_0$ *to* $m = 1$*, and show stability of the Ulam scheme in the case of 'eventually expanding-on-average' random Lasota–Yorke maps, greatly facilitating the computation of the random acims and opening the possibility of computational access to the physical measure (1).* The intricacies involved in this generalisation are already evident and non-trivial in the autonomous setting: while Li's result for convergence of acims in the case of strongly expanding maps goes back to the 1970s, the result covering all piecewise C^2 (eventually) expanding interval maps was only established in 1997 by Blank and Keller [BK97].

One of the major obstacles in this extension, both in the autonomous and non-autonomous settings, is the technical difficulty associated with the presence of so-called periodic turning points (PTPs). Roughly speaking, control of the statistical properties of maps is possible because expansion has a smoothing effect. On the other hand, *turning points* (discontinuities in the map or its derivative) have the opposite effect, inducing large discontinuities in probability densities under the action of the (Perron–Frobenius) transfer operator. PTPs are problematic because the irregularities that they induce compound recurrently along periodic orbits, and this may occur faster than the expanding dynamics can smooth them away. In the random setting, orbits of turning points can be arbitrarily complicated. On the other hand, if problematic orbits occur only rarely, then the mathematical technology of random systems allows their impact to be controlled. Our techniques are motivated by [BK97], wherein the neighbourhoods of (now random) turning points are treated separately to the "smooth" parts of the dynamics. The main difficulties arise in keeping the "bad" and "good" parts of the dynamics from contaminating each other when the Ulam approximation is applied. We will say that a sequence $\{f_{\sigma^n\omega}\}_{n=0}^{\infty}$ has no recurrent turning points if each random orbit $\{f_\omega^{(n)}(y)\}_{n=0}^{\infty}$ encounters at most one discontinuity of an $f_{\sigma^j\omega}$ or $f'_{\sigma^j\omega}$, and we assume that $\mathbb{P}$-a.e. sequences have no recurrent turning points.

While we have been specifically discussing Ulam's method up until this point, Ulam's method can be viewed as a specific type of stochastic perturbation that is applied to the Perron–Frobenius operator $\mathcal{P}_\omega^{(N)}$ of $f_\omega^{(N)}$, for some $N \geq 1$; importantly we show in this work that taking $N = 1$ suffices. Let $\mathcal{P}_{\omega,\epsilon} = Q_\epsilon \mathcal{P}_\omega$ where Q_ϵ is a stochastic perturbation[1] (such as Ulam-type, but in any case corresponding to perturbations of orbits of size $\leq \epsilon$). We work in the subspace of bounded variation functions on I and assume the existence of constants C_ϵ such that

$$\operatorname{var}(Q_\epsilon h) \leq \operatorname{var}(h) + C_\epsilon \|h\|_{L^1} \tag{2}$$

for all $h \in L^1(I)$. We define the **spread** of an operator $Q : BV \to BV$ as

$$\operatorname{spread}(Q) = \inf\{\epsilon \;:\; \operatorname{supp}(Q\mathbf{1}_J) \subset J_\epsilon \;\forall J \subset I\},$$

where J_ϵ denotes the ϵ neighbourhood of J inside I.

Examples of suitable Q_ϵ include: (i) Ulam–type perturbations $Q_\epsilon = \mathbb{E}(\cdot \mid \mathcal{B}_\epsilon)$ (where $\mathcal{B}_\epsilon$ is the σ–algebra generated by partitioning I into uniform subintervals of length ϵ); (ii) convolution type perturbations (with a kernel support in $[-\epsilon, \epsilon]$); (iii) static perturbations where f_ω is replaced with $f_\omega + \xi$ (ξ drawn randomly from $[-\epsilon, \epsilon]$ and $Q_\epsilon h(x) = h(x - \xi)$). See [FGTQ14] for further details.

The main result of this paper is fibrewise convergence of the stochastically perturbed random acims as the perturbation size ϵ goes to 0:

Theorem 1.1. *Let $\sigma : (\Omega, \mathcal{F}, \mathbb{P}) \circlearrowleft$ be an ergodic, invertible, measure preserving transformation, $\{f_\omega\}_{\omega\in\Omega}$ an admissible random Lasota–Yorke map and $\{Q_\epsilon\}$ a family of stochastic perturbations satisfying (2), such that $\lim_{\epsilon\to 0} \|Q_\epsilon - Id\|_{BV\to L^1} = 0$ and spread$(Q_\epsilon) \leq \epsilon$. Let $\{h_\omega\}$ denote the densities of the (unique) random acims for $\{f_\omega\}$. Then, there is an ϵ_0 such that each cocycle generated by $\mathcal{P}_{\omega,\epsilon}$ ($\epsilon < \epsilon_0$) admits a unique random acim with random density $F_\epsilon : \Omega \to BV(I)$ and for $\mathbb{P}$ a.e. $\omega \in \Omega$, $\lim_{\epsilon\to 0} F_\epsilon(\omega) = h_\omega$ in L^1.*

1.2 Convergence of Ulam's method

Ulam's method involves specific stochastic perturbations derived from a partition $P_k = \{B_1, \ldots, B_k\}$ of I into k equally sized subintervals with associated σ-algebra $\mathcal{B}_k$. Let $Q_k = \mathbb{E}(\cdot \mid \mathcal{B}_k)$; that is, $Q_k h = \sum_{j=1}^k \frac{\int_{B_j} h\,dx}{Leb(B_j)} \mathbf{1}_{B_j}$. Then $\operatorname{spread}(Q_k) = \frac{1}{k}$, $\|(Q_k - Id)\|_{BV\to L^1} \leq 1/k$ and $\|Q_k\|_{BV\to BV} = 1$ (in particular, in equation (2) the constants $C_\epsilon = 0$). Put $\mathcal{P}_{\omega,k} = Q_k \circ \mathcal{P}_\omega$ (abusing notation slightly to index $Q_\epsilon = Q_k$ when $\epsilon = 1/k$).

Corollary 1.2. *Let $\{f_\omega\}$ be an admissible random Lasota-Yorke map. For large enough k the Ulam random cocycles generated by $\{\mathcal{P}_{\omega,k}\}$ admit unique random acims. These converge fibrewise in L^1 to the random acim for $\{f_\omega\}$ as $k \to \infty$.*

Figure 1 illustrates Ulam's method for the approximation of (i) the random densities h_ω for two ω-fibres, (ii) the physical measure, and (iii) the acim for the "annealed" Perron–Frobenius operator, for an expanding on average dynamical system (details in Section 6).

[1] $\|Q_\epsilon\|_{L^1\to L^1} = 1$, $h \geq 0 \Rightarrow Q_\epsilon h \geq 0$.

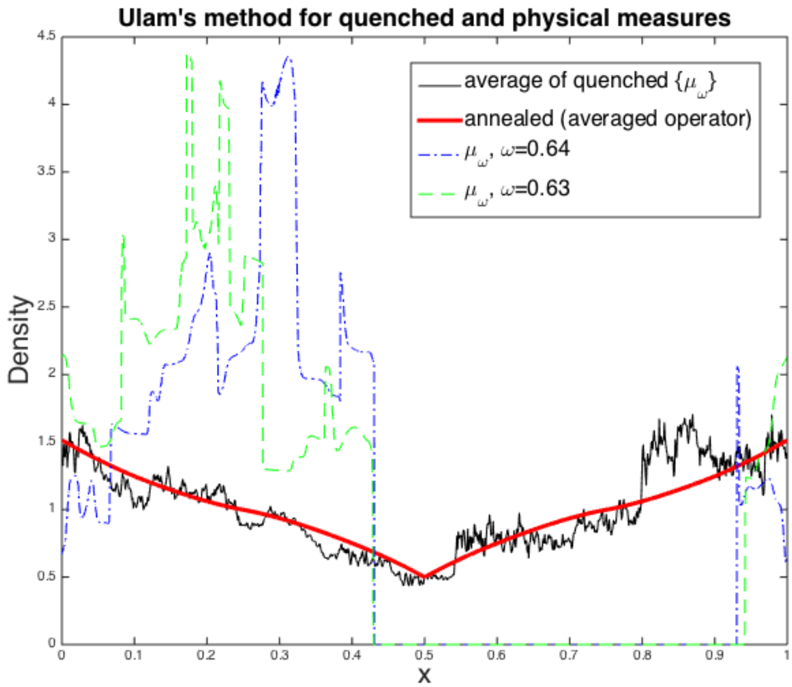

Figure 1: Comparison of random measures for an eventually expanding on average dynamical system. Ulam's method with $k = 10^5$ is used to calculate two random acims μ_ω. These are computed as $\mathcal{P}^{(200)}_{\sigma^{-200}\omega,k}\mathbf{1}$ and are displayed for $\omega = 0.63, \omega = 0.64$. The physical measure (1) is obtained by averaging over $\omega \in \Omega$ (see (12)). These measures contrast with the "annealed" density, computed as a fixed point of the averaged operator $\int_\Omega \mathcal{P}_\omega \, d\mathbb{P}(\omega)$.

1.3 Structure of the paper

Sections 2, 3, 4 contain technical details, establishing weak and strong forms of random Lasota–Yorke inequalities. The proof of Theorem 1.1 follows in Section 5. The results are illustrated in Section 6 with a numerical example.

2 The inequality (LYw)

We first obtain a "weak" Lasota–Yorke inequality:

$$\|\mathcal{P}_{\omega,\epsilon}(h)\|_{BV} \leq C(\omega)\|h\|_{BV}, \tag{LYw}$$

where $\int \log C(\omega) d\mathbb{P}(\omega) < \infty$ and $\|f\|_{BV} := \operatorname{var}(f) + \|f\|_{L^1}$. This rough estimate will allow us to control the large '$B(\omega)$' terms appearing in the strong Lasota–Yorke inequality later on.

Following Buzzi [Buz00], let $0 = a_0(\omega) < a_1(\omega) < \cdots < a_{N_b(\omega)}(\omega) = 1$ be the partition of I into intervals of differentiability of f_ω (recall that $N_b(\omega)$ is the number of branches of f_ω). Let $\xi_i = (f\big|_{[a_{i-1}(\omega),a_i(\omega)]})^{-1}$. Then, the rough bound

$$\operatorname{var}(\mathcal{P}_\omega(h)) \le \sum_{i=1}^{N_b(\omega)} \operatorname{var}\left(\frac{h}{|f'_\omega|} \circ \xi_i \cdot \mathbf{1}_{f([a_{i-1}(\omega),a_i(\omega)])}\right),$$

together with the well known facts that $\operatorname{var}(hg) \le \operatorname{var}(h)\|g\|_\infty + \operatorname{var}(g)\|h\|_\infty$ and $\|h\|_\infty \le \operatorname{var}(h) + \|h\|_{L^1}$ imply (see [Buz00, Lemma 1.2] for full details).

$$\operatorname{var}(\mathcal{P}_\omega(h)) \le 6\tilde{C}(\omega)\operatorname{var}(h) + 4\tilde{C}(\omega)\|h\|_{L^1}, \tag{3}$$

where $\tilde{C}(\omega) = \max(1, N_b(\omega)/\lambda(\omega)) \cdot \max(1, \operatorname{var}(1/f'_\omega)) \cdot \max(1, 1/\lambda(\omega))$. Note that $|(1/f'_\omega)'| \le K\,\Lambda^2$ so that each $1/f'_\omega$ is Lipschitz, and $\operatorname{var}(1/f'_\omega)$ is bounded by $K\Lambda^2 + 2N_b(\omega)\Lambda$. Thus, letting $C(\omega) = 6\tilde{C}(\omega)$, we get $\int_\Omega \log C(\omega) d\mathbb{P}(\omega) < \infty$ and

$$\|\mathcal{P}_\omega(h)\|_{BV} \le C(\omega)\|h\|_{BV}, \tag{4}$$

as required.

3 Construction of splitting into good and bad pieces

The stronger Lasota–Yorke inequality is obtained by a splitting of $\mathcal{P}_{\omega,\epsilon} = \tilde{\mathcal{P}}_{\omega,1} + \tilde{\mathcal{P}}_{\omega,2}$, where $\tilde{\mathcal{P}}_{\omega,1}$ acts on "good" parts of fibres, and $\tilde{\mathcal{P}}_{\omega,2}$ acts on "bad parts", containing turning points of the maps f_ω. The construction relies on a random decomposition of blocks of fibres

$$I \times \{\omega, \dots, \sigma^{N_1-1}\omega\} = Z(\omega) \cup Y(\omega).$$

The construction is done in two steps: first a "skeleton" TP^β of "fibrewise β-sufficient" turning points established, and then these points are "fattened" to give ("bad") intervals comprising $Y(\omega)$.

Define $N_1 = m_0 N_0$ where m_0 is chosen to satisfy

$$\log(9(m_0N_0 + 1)\Lambda^{2N_0}) + 2 < m_0 \int_\Omega \log \lambda_{N_0}(\omega)\, d\mathbb{P}(\omega). \tag{m0}$$

3.1 The skeleton TP^β of turning points

A point $(x, \omega) \in I \times \Omega$ will be called a **turning point** if f_ω or f'_ω is discontinuous at x. A set $S_\omega \subset I_\omega$ will be called a *fibrewise β–sufficient turning point set* if $\{x : (x, \omega)$ is a turning point$\} \subset S_\omega$ and for each connected component J of $I_\omega \setminus S_\omega$ and $y \in J$ we have

$$\operatorname{var}_J(f'_\omega) \le \beta\,|f'_\omega(y)| \qquad \text{and} \qquad \operatorname{var}_J(1/f'_\omega) \le \beta\left|\frac{1}{f'_\omega(y)}\right|.$$

(Since each f_ω is piecewise C^2 this can be guaranteed by ensuring that the maximum distance between turning points is no more than $\beta/(K\Lambda^2)$.)

Push forward of turning points Consider the sets $TP_{\sigma^k\omega}$ of all turning points which fall in $I_{\sigma^k\omega}$ for $k = 0, \ldots, N_1 - 1$; since each map f_ω is piecewise C^2, each such set is finite. For each $x \in TP_{\sigma^k\omega}$ augment $TP_{\sigma^{k+j}\omega}$ with $(f^{(j)}_{\sigma^k\omega}(x), \sigma^{k+j}\omega)$ for $j = 0, \ldots, N_1 - 1 - k$. The collection of all such points obtained in this way is

$$TP' := \bigcup_{k=0}^{N_1-1} \bigcup_{j=0}^{N_1-1-k} f^{(j)}_{\sigma^k\omega}(TP \cap I_{\sigma^k\omega}).$$

Note that if x is a point of discontinuity of $f_{\sigma^k\omega}$ then the future orbit of x gives rise to two sets of contributions to TP': via $\lim_{y\to x^-} f^{(j)}_{\sigma^k\omega}(y)$ and $\lim_{y\to x^+} f^{(j)}_{\sigma^k\omega}(y)$; both orbit fragments are included in TP' (despite the abuse of notation).

Pull back elements of TP' and augment First, restrict to the fibre $I_{\sigma^{N_1-1}\omega}$: if the set $TP' \cap I_{\sigma^{N_1-1}\omega}$ is not fibrewise β–sufficient then it can be[2] augmented with finitely many points to ensure fibrewise β–sufficiency on $I_{\sigma^{N_1-1}}$.

Next, the fibres $I_{\sigma^k\omega}$, $k = 0, \ldots, N_1 - 2$: Suppose that TP^β has been formed by augmenting TP' with points from $I \times \{\sigma^{k+1}\omega, \ldots, \sigma^{N_1-1}\omega\}$ to ensure fibrewise β–sufficiency on those fibres and $f^{-1}_{\sigma^j\omega}(TP^\beta \cap I_{\sigma^{j+1}\omega}) \subset TP^\beta$ $(j = k+1, \ldots, N_1 - 2)$. First augment TP^β with $f^{-1}_{\sigma^k\omega}(TP^\beta \cap I_{\sigma^{k+1}\omega})$ and then as many supplementary points as needed to ensure that TP^β is β–sufficient on the fibre $I_{\sigma^k\omega}$. Repeating this construction until $k = 0$ concludes the formation of TP^β. Put

$$\delta(\omega) := \frac{1}{2} \min_{0\le k<N_1} \min\{|x - y| \ : \ x \neq y \in I_{\sigma^k\omega} \cap TP^\beta\}. \tag{5}$$

Summarising the recursive construction:

1. TP^β contains all the turning points in $I \times \{\omega, \ldots, \sigma^{N_1-1}\omega\}$;
2. if $(a, b) \subset I_{\sigma^k\omega}$ $(0 \le k < N_1 - 1)$ is a connected component of $I_{\sigma^k\omega} \setminus TP^\beta$ then $f_{\sigma^k\omega}(a, b) \cap TP^\beta = \emptyset$;
3. the set TP^β is fibrewise β–sufficient on the fibres $I_{\sigma^k\omega}$ $(k = 0, \ldots, N_1 - 1)$;
4. if $x, y \in TP^\beta \cap I_{\sigma^k\omega}$ $(k < N_1)$ then $|x - y| \ge 2\,\delta(\omega)$.

[2] If S is a turning point set and a connected component of $I_{\sigma^k\omega} \setminus S$ has length Δ then it can be subdivided evenly into $\lfloor 1 + \Delta/(\beta/K\Lambda^2)\rfloor$ pieces such that if the break points are added to S then the resulting set is fibrewise β–sufficient.

Fattening the turning point set to form $Y(\omega)$

We now enlarge TP^β to cover it by intervals. The complement of these intervals will be the "good" parts of the space, and will comprise a family of intervals with minimum length $\delta(\omega)$ such that pseudo-orbits $(x, \sigma^k\omega) \mapsto f_{\sigma^k\omega}(x) + \epsilon$ $(k < N_1)$ which begin in the "good" parts remain therein when ϵ is small enough.

Fix

$$\epsilon(\omega) := \frac{\delta(\omega)}{2}\frac{\Lambda - 1}{2\Lambda^{N_1}}. \tag{6}$$

The construction of $Y(\omega)$ is recursive, beginning on the last fibre: form

$$Y_{\sigma^{N_1-1}\omega} = \cup_{x \in TP^\beta \cap I_{\sigma^{N_1-1}\omega}} (x - \epsilon(\omega), x + \epsilon(\omega)) \times \{\sigma^{N_1-1}\omega\}.$$

Suppose now that $Y_{\sigma^{k+1}\omega}$ has been constructed, and if $J = (a, b)$ let $J_\epsilon = (a - \epsilon, b + \epsilon) \cap I$. Let

$$Y_{\sigma^k\omega} = \left(\cup_{J \in Y_{\sigma^{k+1}\omega}} f^{-1}_{\sigma^k\omega}(J_{\epsilon(\omega)})\right) \bigcup \left(\cup_{x \in (TP^\beta \cap I_{\sigma^k\omega}) \setminus f^{-1}_{\sigma^k\omega}(TP^\beta \cap I_{\sigma^{k+1}\omega})} (x - \epsilon(\omega), x + \epsilon(\omega))\right) \times \{\sigma^k\omega\}.$$

Repeat inductively until $k = 0$.

Now define

$$Y(\omega) := \cup_{k=0}^{N_1-1} Y_{\sigma^k\omega} \qquad \text{and} \qquad Z(\omega) := (I \times \{\omega, \ldots, \sigma^{N_1-1}\omega\}) \setminus Y(\omega). \tag{7}$$

Clearly $Y(\omega)$ covers the β–sufficient turning point set TP^β. Notice that every interval in $Y_{\sigma^k\omega}$ contains an element of TP^β and has length bounded by $2\epsilon(\omega)\Lambda^{N_1}/(\Lambda - 1) = \delta(\omega)/2$ and every connected component of $Z(\omega)$ is an interval of length at least $\delta(\omega)$[3]. Moreover,

Lemma 3.1. *Let $C_0 := 4\Lambda^{N_1}/(\Lambda - 1)$ (a constant independent of ω, but depending on N_1) and let $\{x_k\}_{k=0}^{N_1-1}$ be pseudo-orbit segments with $|f_{\sigma^k\omega}(x_k) - x_{k+1}| < \delta(\omega)/C_0$. We have*

$$x_j \in Y_{\sigma^j\omega} \qquad \Rightarrow \qquad x_k \in Y_{\sigma^k\omega} \quad 0 \le k \le j$$

and

$$x_l \in Z(\omega) \qquad \Rightarrow \qquad x_j \in Z(\omega) \quad N_1 > j \ge l.$$

Proof. Let $\{x_k\}_{k=0}^{N_1-1}$ be an $\epsilon(\omega)$ pseudo-orbit sequence (compare definition (6)). Suppose that $x_j \in J$ where J is a connected component of $Y_{\sigma^j\omega}$ and $j \ge 1$. Then $f_{\sigma^{j-1}\omega}(x_{j-1}) \in J_{\epsilon(\omega)}$ so that $x_{j-1} \in Y_{\sigma^{j-1}\omega}$. Repeat inductively for $k = j-1, j-2, \ldots, 1$. On the other hand, suppose that $x_l \in Z(\omega)$ but $x_j \in Y(\omega)$ for some $l < j < N_1$. Then $x_j \in Y_{\sigma^j\omega}$, implying that $x_l \in Y_{\sigma^l\omega}$, a contradiction. □

Corollary 3.2. *With notation as in Lemma 3.1, if spread$(Q) < \epsilon(\omega)$ (compare (6)) then*

$$\mathcal{P}_{\sigma^{j+1}\omega}((Q\mathcal{P}_{\sigma^j\omega}\mathbf{1}_{Z(\omega)})\,\mathbf{1}_{Y(\omega)}) = 0.$$

[3] By Property 4 following the definition (5), elements of TP^β are at least $2\,\delta(\omega)$ apart and every component of $Y(\omega)$ is an interval which fattens a point of TP^β into an interval of length no more than $\delta(\omega)/2$.

4 The inequality (LYs)

For fixed N_1, β and ω we have a random set $Y(\omega) \subset I \times \{\omega, \sigma\omega, \dots, \sigma^{N_1-1}\omega\}$ which encloses the turning points of $\{f_{\sigma^j\omega}\}_{j=0}^{N_1-1}$ and has a number of good properties (detailed in the previous section). We define two restricted operators

$$\tilde{\mathcal{P}}_{\sigma^j\omega,1}(\cdot) = \mathcal{P}_{\sigma^j\omega,\epsilon}(\cdot\, \mathbf{1}_{Z_{\sigma^j\omega}}) \qquad \text{and} \qquad \tilde{\mathcal{P}}_{\sigma^j\omega,2}(\cdot) = \mathcal{P}_{\sigma^j\omega,\epsilon}(\cdot\, \mathbf{1}_{Y_{\sigma^j\omega}})$$

where $Y(\omega) = \cup_{j=0}^{N_1-1} Y_{\sigma^j\omega}$ and each $Z_{\sigma^j\omega} = I_{\sigma^j\omega} \backslash Y_{\sigma^j\omega}$. Iterates are defined in the obvious way. We establish two families of strong Lasota–Yorke inequalities, valid for all $0 \le j < j+k \le N_1$ and all $\epsilon < \epsilon(\omega)$:

$$\mathrm{var}(\tilde{\mathcal{P}}^{(k)}_{\sigma^j\omega,1} h) \le 3\,\lambda_k(\sigma^j\omega)^{-1}\,\mathrm{var}(h) + B_1(\omega)\|h\|_{L^1} \tag{LYs1}$$

and

$$\mathrm{var}(\tilde{\mathcal{P}}^{(k)}_{\sigma^j\omega,2} h) \le 3\,\lambda_k(\sigma^j\omega)^{-1}\,\mathrm{var}(h) + B_2(\omega)\|h\|_{L^1}, \tag{LYs2}$$

where $B_1(\omega)$ and $B_2(\omega)$ are measurable and finite $\mathbb{P}$ a.e. and $h \in BV(I_{\sigma^j\omega})$. The operator with a subscript 1 is the restriction of $\mathcal{P}_\epsilon$ to a "good" set $Z(\omega)$, well separated from turning points, and the operator with a subscript 2 is restricted to the "bad" set $Y(\omega)$ where all discontinuities occur.

These inequalities are combined to prove a fibre-dependent, but "strong" Lasota–Yorke inequality. The dependence on β is removed during the proof.

Theorem 4.1. *Suppose that $\mathbb{P}$ a.e. ω has no recurrent turning points and let N_1 be fixed by equation (m0). Then there are measurable $\delta(\omega) > 0$ and $B(\omega) < \infty$ ($\mathbb{P}$ a.e.) such that*

$$\mathrm{var}(\mathcal{P}^{N_1}_{\omega,\epsilon} h) \le \alpha(\omega)\ \mathrm{var}(h) + B(\omega)\,\|h\|_{L^1} \tag{LYs}$$

when spread$(Q_\epsilon) < \delta(\omega)/C_0$ (see Lemma 3.1) and

$$\alpha(\omega) := 9(N_1+1)\Lambda^{2N_0} \prod_{k=0}^{m_0-1} \lambda_{N_0}(\sigma^{kN_0}\omega)^{-1}.$$

Note that $\int \log(\alpha(\omega))d\mathbb{P} < -2$, because of (m0). In obtaining a random Lasota-Yorke inequality, the strong inequality (LYs) will be applied for most ω, with the weak inequality (LYw) used when $\delta(\omega)$ is too small.

4.1 The inequality (LYs1)

This part follows [BK97, §3.2]. The first ingredient (extending [BK97, Lemmas 3.5 & 3.6]) is the construction of a stochastic operator $\tilde{Q}_{\omega,\epsilon,k}$, with uniformly controlled variation bounds, such that instead of applying the perturbation Q_ϵ after each map, or a restriction thereof, one can apply the $\tilde{Q}_{\omega,\epsilon,k}$ at the start, and then the transfer operators of the unperturbed system. The second part replaces [BK97, Lemma 3.7] with the key being a localisation argument that works because mass cannot leak out of the connected components of $Z(\omega)$ (for a large measure set of $\omega \in \Omega$). For convenience of notation, in what follows we suppress ω, ϵ from the notation in $\tilde{Q}$.

Lemma 4.2. *Let $\beta > 0$ be given and $\{T_j\}_{j=0}^k$ be a family of invertible C^1 transformations of $\mathbb{R}$ such that*

$$\mathrm{var}(T_j') \leq \beta |T_j'(y_j)| \qquad \textit{and} \qquad \mathrm{var}(1/T_j') \leq \beta |1/T_j'(y_j)|$$

for fixed $\{y_0, \ldots, y_k\} \subset \mathbb{R}$. Let P_j be the transfer opertors corresponding to T_j and let $Q_j : BV(\mathbb{R}) \circlearrowleft$ be such that

$$\mathrm{var}(Q_j h) \leq \mathrm{var}(h) + C_j \|h\|_{L^1} \tag{8}$$

for fixed finite contants C_j. Then there exists $\tilde{Q}_k : BV(\mathbb{R}) \circlearrowleft$ such that

$$Q_k P_k Q_{k-1} P_{k-1} \cdots Q_0 P_0 = P_k P_{k-1} \cdots P_0 \tilde{Q}_k$$

and

$$\mathrm{var}(\tilde{Q}_k h) \leq \left(1 + \frac{3\beta}{2}\right)^{2k} \mathrm{var}(h) + \tilde{C}_k \|h\|_{L^1}$$

(the constant $\tilde{C}_k$ depends on $C_0, \ldots, C_k, \Lambda$ and β).

Proof. The key estimate is

$$\begin{aligned} \mathrm{var}(P_j h) &\leq \mathrm{var}(1/T_j') \|h\|_\infty + \|1/T_j'\|_\infty \mathrm{var}(h) \\ &\leq \beta |1/T_j'(y_j)| (\mathrm{var}(h)/2) + \beta |1/T_j'(y_j)| \mathrm{var}(h) = (1 + 3\beta/2) 1/|T_j'(y_j)| \mathrm{var}(h). \end{aligned}$$

Similarly, $\mathrm{var}(P_j^{-1} h) \leq (1 + 3\beta/2)|T_j'(y_j)| \mathrm{var}(h)$. Now put

$$\tilde{Q}_0 = P_k^{-1} Q_k P_k \qquad \text{and} \qquad \tilde{Q}_j = P_{k-j}^{-1} \tilde{Q}_{j-1} Q_{k-j} P_{k-j}, j = 1, \ldots, k.$$

Then

$$\begin{aligned} \mathrm{var}(\tilde{Q}_0 h) &\leq (1 + 3\beta/2)\, |T_k'(y_k)| \,\mathrm{var}(Q_k P_k h) \\ &\leq (1 + 3\beta/2)^2 \mathrm{var}(h) + (1 + 3\beta/2)|T_k'(y_k)| C_k \|h\|_{L^1} \\ &\leq (1 + 3\beta/2)^2 \mathrm{var}(h) + (1 + 3\beta/2\,)\Lambda\, C_k \|h\|_{L^1} \end{aligned}$$

and so on. The bounds on $\mathrm{var}(\tilde{Q}_j)$ proceed inductively, as in [BK97]. □

The next lemma shows how to localise Lemma 4.2.

Lemma 4.3. *Fix ω, and $\beta > 0$. Let $\{Z_j\}_{j=0}^k$ be a sequence of subintervals of I such that each $Z_j \times \{\sigma^j\omega\} \cap TP^\beta$ contains at most one point (and if the point is y_j then y_j is not a turning point). Suppose also that each $[f_{\sigma^j\omega}(Z_j)]_{\epsilon(\omega)} \subset Z_{j+1}$ and that $\{Q_j\}_{j=0}^{k-1}$ is a family of operators $Q_j : BV(I_{\sigma^{j+1}\omega}) \circlearrowleft$ satisfying (8) and spread$(Q_j) < \epsilon(\omega)$. Then there is a $\tilde{Q}_{k,\omega} : BV(I_\omega) \circlearrowleft$ and constant $\tilde{C}_{k,\omega}$ such that*

$$\mathcal{P}_{\omega,\epsilon}^{(k)} h = Q_{k-1}\mathcal{P}_{\sigma^{k-1}\omega} \cdots Q_0 \mathcal{P}_\omega h = \mathcal{P}_\omega^{(k)} \tilde{Q}_{k,\omega} h$$

and

$$\mathrm{var}(\tilde{Q}_{k,\omega} h) \le (1 + 3\,\beta)^{2k}\ \mathrm{var}(h) + \tilde{C}_{k,\omega}\, \|h\|_{L^1}$$

for all $h \in BV(Z_0)$, with $\tilde{C}_{k,\omega}$ bounded for $k \le N_1$.

Proof. First, embed $I_{\sigma^j\omega}$ in $\mathbb{R}$ with the map $\pi(y, \sigma^j\omega) = y$. If $L : BV(I_{\sigma^j\omega}) \to BV(I_{\sigma^l\omega})$ then $\pi^* L$ acting on $BV(\mathbb{R})$ is defined by $\pi^* L g(x) = [L(g \circ \pi)](x, \sigma^l\omega)$. By the setup on Z_j, each Z_j intersects either one or two open connected components of $Z(\omega)$. Let A_j be this (union of) component(s). For each $j = 0, \ldots, k-1$ define $T_j : \mathbb{R} \to \mathbb{R}$ such that $T_j|_{\pi(A_j)} = \pi \circ f_{\sigma^j\omega}|_{A_j}$, but T_j is C^1, with linear extensions outside of $\pi(A_j)$. Note from the construction of TP^β that each $T_j^{-1}\pi(A_{j+1}) \subseteq \pi(A_j)$. If P_j is the transfer operator for T_j then $P_j g = \pi^* \mathcal{P}_{\sigma^j\omega} g$ when $\mathrm{supp}(g) \subset \pi(A_j)$. The conditions on the sequence of intervals $\{Z_j\}$ ensure that $\mathcal{P}_{\omega,\epsilon}^{(j)} h$ is supported on Z_j when h is supported on Z_0 and hence that each

$$(\pi^* Q_j) P_j \ldots (\pi^* Q_0) P_0 h = \pi^* \mathcal{P}_{\omega,\epsilon}^{(j+1)} h$$

for such h. Each T_j satisfies the conditions of Lemma 4.2 but with 2β in place of β and $y_j = \pi(A_j \times \{\sigma^j\omega\} \cap TP^\beta)$ (when the intersection is non-empty, and any point of $\pi(A_j)$ when it is not). Let $\tilde{Q}_k : BV(\mathbb{R}) \circlearrowleft$ be the operator from the lemma applied to the sequence $\{\pi^* Q_j\}$, and define $\tilde{Q}_{k,\omega} h(y, \omega) = [\tilde{Q}_k(h \circ \pi)](y)$. The constants $\tilde{C}_{k,\omega}$ are the $\tilde{C}_k$ from applying Lemma 4.2 with 2β in place of β. □

We now obtain a Lasota–Yorke inequality for functions which are supported in the good set. Fix β to satisfy

$$(1 + 3\,\beta)^{2N_1}(1 + N_1\,\beta\,\Lambda^{2N_1}) < 1.5.$$

Lemma 4.4. *Let N_1 be fixed by* (m0), *β as above and let spread$(Q_\epsilon) < \epsilon(\omega)$. There is a constant $B_0(\omega)$ such that for each k with $j + k \le N_1$*

$$\mathrm{var}(\mathcal{P}_{\sigma^j\omega,\epsilon}^{(k)} h) \le 1.5\,\lambda_k(\omega)^{-1}\ \mathrm{var}(h) + B_0(\omega)\, \|h\|_{L^1}$$

when $h \in BV(I_{\sigma^j\omega})$ is supported on a connected interval $Z_j \subset Z(\omega)$.

Proof. For each k let Z_{j+k} be the component of $Z(\omega)$ containing $f_{\sigma^{j+k-1}\omega} Z_{j+k-1}$. Since $\epsilon \le \epsilon(\omega)$, each $(f_{\sigma^{j+k-1}\omega} Z_{j+k-1})_{\epsilon(\omega)} \subset Z_{j+k}$ (Lemma 3.1). Q_ϵ has spread bounded by $\epsilon(\omega)$, so

Lemma 4.3 applies, giving $\tilde{Q}_{k,\sigma^j\omega}$ with the stated properties. [The proofs above reveal that each $\tilde{C}_{k,\sigma^j\omega} \leq \tilde{C}_{N_1,\omega}$.] Standard estimates give

$$\mathrm{var}(\mathcal{P}^{(k)}_{\sigma^j\omega} g) \leq \lambda_k(\omega)^{-1}\ \mathrm{var}(g) + \mathrm{var}(1/f^{(k)}_{\sigma^j\omega}{}') \, \|g\|_\infty.$$

The β–sufficiency condition implies that each $\mathrm{var}_{f^{(l)}_{\sigma^j\omega}(Z_j)}(1/f'_{\sigma^{j+l}\omega}) \leq \beta\,\Lambda$, so a standard induction gives $\mathrm{var}(1/f^{(k)}_{\sigma^j\omega}{}') \leq k\,\beta\Lambda^k$. Combining with $\|g\|_\infty \leq \mathrm{var}(g)/2$,

$$\mathrm{var}(\mathcal{P}^{(k)}_{\sigma^j\omega} g) \leq (\lambda_k(\omega)^{-1} + k\,\beta\,\Lambda^k/2)\ \mathrm{var}(g) \leq (1 + k\,\beta\,\Lambda^{2k}/2)\lambda_k(\omega)^{-1}\ \mathrm{var}(g).$$

Applying with $g = \tilde{Q}_{k,\sigma^j\omega} h$ gives

$$\mathrm{var}(\mathcal{P}^{(k)}_{\sigma^j\omega} g) \leq \underbrace{(1+3\,\beta)^{2k}(1 + k\,\beta\,\Lambda^{2k}/2)}_{\leq 1.5}\ \lambda_k(\omega)^{-1}\ \mathrm{var}(h) + \underbrace{(1 + k\,\beta\,\Lambda^{2k}/2)\,\Lambda^k \tilde{C}_{k,\sigma^j\omega}}_{B_0(\omega)\text{ when } k = N_1}\ \|h\|_{L^1}$$

(since $k \leq N_1$ and the choice of β).

□

Proof of (LYs1)*:* Without loss of generality we establish for $j = 0$. Let $h \in BV(I_\omega)$. Then $Z_\omega = Z(\omega) \cap I_\omega$ is a union of intervals of length at least $\delta(\omega)$ and by iterated application of equation (14)

$$\sum_{Z \in Z_\omega} \mathrm{var}(h\,\mathbf{1}_Z) \leq 2\ \mathrm{var}(h) + \frac{2}{\delta(\omega)}\,\|h\|_{L^1}. \tag{9}$$

By Lemma 3.1, when spread(Q_ϵ) is less than $\epsilon(\omega)$,

$$\mathrm{supp}\left(\mathcal{P}_{\sigma^k\omega,\epsilon}\mathbf{1}_{I_{\sigma^k\omega}\cap Z(\omega)}\right) \subseteq I_{\sigma^{k+1}\omega} \cap Z(\omega).$$

This implies that each $\tilde{\mathcal{P}}^{(k)}_{\omega,1} h = \mathcal{P}^{(k)}_{\omega,\epsilon} h$ when h is supported on Z_ω. In particular,

$$\tilde{\mathcal{P}}^{(k)}_{\omega,1} h = \mathcal{P}^{(k)}_{\omega,\epsilon}(h\,\mathbf{1}_{Z_\omega}) = \sum_{Z \in Z_\omega} \mathcal{P}^{(k)}_{\omega,\epsilon}(h\,\mathbf{1}_Z).$$

By Lemma 4.4, when $Z \in Z_\omega$,

$$\mathrm{var}(\mathcal{P}^{(k)}_{\omega,\epsilon}(h\,\mathbf{1}_Z)) \leq 1.5\,\lambda_k^{-1}(\omega)\ \mathrm{var}(h\,\mathbf{1}_Z) + B_0(\omega)\,\|h\,\mathbf{1}_Z\|_{L^1}.$$

Then

$$\begin{aligned}
\operatorname{var}(\tilde{\mathcal{P}}^{(k)}_{\omega,1}h) &= \operatorname{var}(\mathcal{P}^{(k)}_{\omega,\epsilon}(h\,\mathbf{1}_{Z_\omega}))\\
&\le \sum_{Z\in Z_\omega} \operatorname{var}(\mathcal{P}^{(k)}_{\omega,\epsilon}(h\,\mathbf{1}_{Z}))\\
&\le \sum_{Z\in Z_\omega} \left[1.5\,\lambda_k^{-1}(\omega)\ \operatorname{var}(h\,\mathbf{1}_Z) + B_0(\omega)\,\|h\,\mathbf{1}_Z\|_{L^1}\right]\\
&\le 1.5\,\lambda_k^{-1}(\omega) \sum_{Z\in Z_\omega} \operatorname{var}(h\,\mathbf{1}_Z) + B_0(\omega)\,\|h\|_{L^1}\\
&\le 3\,\lambda_k^{-1}(\omega)\ \operatorname{var}(h) + \underbrace{\left[\frac{3\,\Lambda^{N_1}}{\delta(\omega)} + B_0(\omega)\right]}_{=B_1(\omega)} \|h\|_{L^1},
\end{aligned}$$

where the last inequality uses (9). This defines (B_1) and establishes (LYs1). ☐

4.2 The inequality (LYs2)

The next two lemmas complement Lemma 4.4 by treating functions whose support is contained in the bad set. This proceeds in a similar way to (LYs1), but has additional complexities because of turning points and the construction of $Y(\omega)$. Let β be fixed as in Lemma 4.4.

Lemma 4.5. *Let $y_0 \in TP^\beta$, $k < N_1$ and $y_j := f^{(j)}_\omega(y_0)$ ($j = 1, \ldots, k$). Let $Y_0 \subset I_\omega$ be the connected component of Y_ω containing y_0. If spread(Q_ϵ) $< \epsilon(\omega)$ and none of $\{y_j\}_{j=0}^{k-1}$ is a turning point then there is a measurable $B_0(\omega) < \infty$ such that*

$$\operatorname{var}(\tilde{\mathcal{P}}^{(k)}_{\omega,2}(h\mathbf{1}_{Y_0})) \le 1.5\,\lambda_k(\omega)^{-1}\ \operatorname{var}(h\,\mathbf{1}_{Y_0}) + B_0(\omega)\,\|h\,\mathbf{1}_{Y_0}\|_{L^1}$$

for $h \in BV(I_\omega)$.

Proof. First, for each $j = 1, \ldots, k$ let each $Y_j \subset I_{\sigma^j\omega}$ be the connected component of $Y(\omega)$ containing y_j. Note from the construction of $Y(\omega)$ that each $f_{\sigma^j\omega}(Y_j) \subset (Y_{j+1})_{\epsilon(\omega)}$. Consequently, the support of each $Q_\epsilon\mathcal{P}_{\sigma^j\omega}(\mathbf{1}_{Y_j})$ intersects each adjacent $Z \in Z(\omega)$ in a subinterval of length at most $2\,\epsilon(\omega) < \delta(\omega)/2$ (compare with equation (6)). Since each such Z has length at least $\delta(\omega)$, this support cannot extend into another component of $Y_{\sigma^{j+1}\omega}$ and

$$\mathbf{1}_{Y(\omega)}\,(\mathcal{P}_{\sigma^j\omega,\epsilon}\mathbf{1}_{Y_j}) = \mathbf{1}_{Y_{j+1}}\,(\mathcal{P}_{\sigma^j\omega,\epsilon}\mathbf{1}_{Y_j}) = \mathbf{1}_{Y_{j+1}}\,(Q_\epsilon\mathcal{P}_{\sigma^j\omega}\mathbf{1}_{Y_j}).$$

Hence

$$\tilde{\mathcal{P}}^{(k)}_{\omega,2}h = Q_\epsilon\,\mathcal{P}_{\sigma^{k-1}\omega}\mathbf{1}_{Y_{k-1}}\,Q_\epsilon\,\mathcal{P}_{\sigma^{k-2}\omega}\mathbf{1}_{Y_{k-2}}\,Q_\epsilon\,\mathcal{P}_{\sigma^{k-3}\omega}\cdots\mathbf{1}_{Y_1}Q_\epsilon\,\mathcal{P}_\omega\,h$$

if supp(h) $\subset Y_0$. Now, for each $j = 1, \ldots, k-1$ define Q_{j-1} acting on $BV(I_{\sigma^j\omega})$ by $Q_{j-1}g = \mathbf{1}_{Y_j}\,Q_\epsilon g$ and $Q_{k-1} = Q_\epsilon$. Then

$$\tilde{\mathcal{P}}^{(k)}_{\omega,2}(\mathbf{1}_{Y_0}h) = Q_{k-1}\mathcal{P}_{\sigma^{k-1}\omega}Q_{k-2}\mathcal{P}_{\sigma^{k-2}\omega}\cdots Q_0\mathcal{P}_\omega(\mathbf{1}_{Y_0}h)$$

for each $h \in BV(I_\omega)$. Now Lemma 4.2 can be applied exactly as in the proof of Lemma 4.3 to produce a $\tilde{Q}$ such that $\tilde{\mathcal{P}}_{\omega,2}^{(k)}(\mathbf{1}_{Y_0}h) = \mathcal{P}_{\omega,\epsilon}^{(k)}\,\tilde{Q}(\mathbf{1}_{Y_0}h)$ (and $\tilde{Q}$ has the same properties as in the conclusion of Lemma 4.3). The remainder of the proof now proceeds as in Lemma 4.4, including the choice of $B_0(\omega)$. □

Lemma 4.6. *Let all hypotheses be as in Lemma 4.5, except that now suppose that y_j is a turning point for some $j < k$. Then*

$$\operatorname{var}(\tilde{\mathcal{P}}_{\omega,2}^{(k)}(h\mathbf{1}_{Y_0})) \leq 3\,\lambda_k(\omega)^{-1}\,\operatorname{var}(h\,\mathbf{1}_{Y_0}) + 2\,B_0(\omega)\,\|h\,\mathbf{1}_{Y_0}\|_{L^1}$$

for $h \in BV(I_\omega)$.

Proof. First of all clarify notation by putting $y_l^+ = f_\omega^l(y_0)$ for $l \leq j$ and $y_l^+ = \lim_{y\to y_j^+} f_{\sigma^j\omega}^{l-j}(y)$ for $j < l \leq k$. Let Y_l^+ be the connected component of (the closure of) $Y(\omega)$ containing y_l^+ for each l, and construct similar sequences $\{y_l^-\}$ and $\{Y_l^-\}$ as $y \to y_j^-$. [Note that if y_l is a point of continuity of $f_{\sigma^j\omega}$ but discontinuity of the derivative then all $y_l^+ = y_l^-$, and if y_j is a discontinuity of the map then $y_l^+ = y_l^-$ and $Y_l^+ = Y_l^-(= Y_l)$ for $l \leq j$. In particular, $Y_0 = Y_0^{\pm}$.] Let $Y_j^{\pm} = Y_j = [a,b]$. Then, similar to Lemma 4.5:

$$\begin{aligned}
\tilde{\mathcal{P}}_{\omega,2}^{(j)}(\mathbf{1}_{Y_0}h) &= Q_\epsilon\,\mathcal{P}_{\sigma^{j-1}\omega}\mathbf{1}_{Y_{j-1}}\,Q_\epsilon\,\mathcal{P}_{\sigma^{j-2}\omega}\mathbf{1}_{Y_{j-2}}\,Q_\epsilon\,\cdots\mathcal{P}_\omega\,(\mathbf{1}_{Y_0}h)\\
\tilde{\mathcal{P}}_{\omega,2}^{(j+l)}(\mathbf{1}_{Y_0}h) &= Q_\epsilon\,\mathcal{P}_{\sigma^{j+l-1}\omega}\mathbf{1}_{Y_{j+l-1}^-}\,Q_\epsilon\,\mathcal{P}_{\sigma^{j+l-2}\omega}\mathbf{1}_{Y_{j+l-2}^-}\,Q_\epsilon\,\cdots\mathcal{P}_{\sigma^j\omega}\,(\mathbf{1}_{[a,y_j]}\tilde{\mathcal{P}}_{\omega,2}^{(j)}(\mathbf{1}_{Y_0}h))\\
&\quad + Q_\epsilon\,\mathcal{P}_{\sigma^{j+l-1}\omega}\mathbf{1}_{Y_{j+l-1}^+}\,Q_\epsilon\,\mathcal{P}_{\sigma^{j+l-2}\omega}\mathbf{1}_{Y_{j+l-2}^+}\,Q_\epsilon\,\cdots\mathcal{P}_{\sigma^j\omega}\,(\mathbf{1}_{[y_j,b]}\tilde{\mathcal{P}}_{\omega,2}^{(j)}(\mathbf{1}_{Y_0}h)).
\end{aligned}$$

The argument now proceeds as in Lemma 4.5, applied independently to each of the two products of operators. □

Proof of (LYs2)*:* Without loss of generality we establish for $j = 0$. Let $h \in BV(I_\omega)$. Then $Z_\omega = Z(\omega) \cap I_\omega$ is a union of intervals of length at least $\delta(\omega)$ and

$$h\,\mathbf{1}_{Y_\omega} = \sum_{Y_0\in Y_\omega} h\,\mathbf{1}_{Y_0}.$$

By iterated application of equation (16), successively removing intervals $Z \in I_\omega \setminus Y_\omega$,

$$\sum_{Y_0\in Y_\omega}\operatorname{var}(h\,\mathbf{1}_{Y_0}) \leq \operatorname{var}(h) + \sum_{Z\in Z_\omega}\frac{2}{\delta(\omega)}\,\|h\,\mathbf{1}_Z\|_{L^1} \leq \operatorname{var}(h) + \frac{2}{\delta(\omega)}\,\|h\|_{L^1}. \tag{10}$$

Let $h \in BV$. Then, each $Y_0 \in Y_\omega$ is of a type which is covered by either Lemma 4.5 or 4.6. Hence, each

$$\operatorname{var}(\tilde{\mathcal{P}}_{\omega,2}^{(k)}(h\,\mathbf{1}_{Y_0})) \leq 3\,\lambda_k(\omega)^{-1}\operatorname{var}(h\,\mathbf{1}_{Y_0}) + 2\,B_0(\omega)\,\|h\,\mathbf{1}_{Y_0}\|_{L^1}.$$

Then

$$\begin{aligned}
\operatorname{var}(\tilde{\mathcal{P}}^{(k)}_{\omega,2} h) &= \operatorname{var}(\mathcal{P}^{(k)}_{\omega,2}(h\,\mathbf{1}_{Y_\omega})) \\
&\le \sum_{Y_0 \in Y_\omega} \operatorname{var}(\mathcal{P}^{(k)}_{\omega,2}(h\,\mathbf{1}_{Y_0})) \\
&\le \sum_{Y_0 \in Y_\omega} \left[3\,\lambda_k^{-1}(\omega)\ \operatorname{var}(h\,\mathbf{1}_{Y_0}) + 2\,B_0(\omega)\,\|h\,\mathbf{1}_{Y_0}\|_{L^1}\right] \\
&\le 3\,\lambda_k^{-1}(\omega) \sum_{Y_0 \in Y_\omega} \operatorname{var}(h\,\mathbf{1}_{Y_0}) + 2\,B_0(\omega)\,\|h\|_{L^1} \\
&\le 3\,\lambda_k^{-1}(\omega)\ \operatorname{var}(h) + \underbrace{\left[\frac{6\,\Lambda^{N_1}}{\delta(\omega)} + 2\,B_0(\omega)\right]}_{=B_1(\omega)} \|h\|_{L^1},
\end{aligned}$$

where the last inequality uses (10). This defines (B_2) and establishes (LYs2). □

4.3 Proof of Theorem 4.1

The calculation combines (LYs1) and (LYs2). Let β be chosen as in Lemma 4.4 and spread$(Q_\epsilon) < \delta(\omega)/C_0$. By Corollary 3.2, each

$$\tilde{\mathcal{P}}_{\sigma^k\omega,2}\tilde{\mathcal{P}}^k_{\omega,1} = 0$$

$(k = 1, \ldots, N_1 - 1)$. Hence

$$\mathcal{P}^{(N_1)}_{\omega,\epsilon} = (\tilde{\mathcal{P}}_{\sigma^{N_1-1}\omega,1} + \tilde{\mathcal{P}}_{\sigma^{N_1-1}\omega,2}) \cdots (\tilde{\mathcal{P}}_{\omega,1} + \tilde{\mathcal{P}}_{\omega,2}) = \sum_{n=0}^{N_1} \tilde{\mathcal{P}}^{N_1-n}_{\sigma^n\omega,1}\tilde{\mathcal{P}}^n_{\omega,2}.$$

Each term in this sum can be controlled by a combination of (LYs1) and (LYs2):

$$\tilde{\mathcal{P}}^{N_1-n}_{\sigma^n\omega,1}\tilde{\mathcal{P}}^n_{\omega,2} h \le 3^2\,\lambda_{N_1-n}(\sigma^n\omega)^{-1}\,\lambda_n(\omega)^{-1}\operatorname{var}(h) + (3\,\lambda_{N_1-n}(\sigma^n\omega)^{-1}\,B_2(\omega) + B_1(\sigma^n\omega))\,\|h\|_{L^1}.$$

Put $B(\omega) := B_1(\omega) + \sum_{n=1}^{N_1-1}(3\,\lambda_{N_1-n}(\sigma^n\omega)^{-1}\,B_2(\omega) + B_1(\sigma^n\omega)) + B_2(\omega)$. Then

$$\begin{aligned}
\operatorname{var}(\mathcal{P}^{(N_1)}_{\omega,\epsilon} h) &= \sum_{n=0}^{N_1} \operatorname{var}(\tilde{\mathcal{P}}^{N_1-n}_{\sigma^n\omega,1}\tilde{\mathcal{P}}^n_{\omega,2} h) \\
&\le 9(N_1+1) \max_{0\le n\le N_1} \left\{\lambda_{N_1-n}(\sigma^n\omega)^{-1}\,\lambda_n(\omega)^{-1}\right\} \operatorname{var}(h) + B(\omega)\,\|h\|_{L^1} \\
&\le \underbrace{9(N_1+1)\Lambda^{2N_0} \prod_{k=0}^{m_0-1} \lambda_{N_0}(\sigma^{kN_0}\omega)^{-1}}_{\alpha(\omega)} \operatorname{var}(h) + B(\omega)\|h\|_{L^1},
\end{aligned} \tag{LYs}$$

using equation (17). □

5 The random Lasota–Yorke inequality and proof of the main result

Theorem 5.1. *Let the dynamical conditions of Section 1 hold, and suppose there are no recurrent turning points $\mathbb{P}$-a.e. Then, there is an $\epsilon_0 > 0$ such that when spread$(Q_\epsilon) < \epsilon_0$ the following random Lasota–Yorke inequality holds*

$$\operatorname{var}(\mathcal{P}^{(N)}_{\omega,\epsilon}(h)) \leq \tilde{\alpha}(\omega) \operatorname{var}(h) + \tilde{B}(\omega)\|h\|_{L^1}, \tag{LY}$$

for some fixed $N \in \mathbb{N}$, with $\int \log \tilde{\alpha}(\omega) d\mathbb{P} < 0$ and $\tilde{B}(\omega)$ measurable.

Proof. We combine information from strong and weak inequalities, (LYs) and (LYw), to get (LY).

Let $N := N_1$ be fixed by (m0) and let $C(\omega)$ be given by (LYw). Choose γ such that if Ω' has $\mathbb{P}(\Omega') > 1 - \gamma$ then

$$\int_{\Omega\setminus\Omega'} |\log \lambda_{N_0}| d\mathbb{P} \leq 1/m_0 \qquad \text{and} \qquad \int_{\Omega\setminus\Omega'} |\log C(\omega)|\, d\mathbb{P} \leq 1.$$

Since $\delta(\omega)$ is defined, measurable and positive a.e., there is a $\delta_0 > 0$ such that

$$\mathbb{P}(\delta \geq \delta_0) \geq 1 - \gamma/2.$$

Define the set of good ω as $\Omega_G = \{\omega \ : \ \delta(\omega) \geq \delta_0\}$ and put $\epsilon_0 := \delta_0/C_0$ (see Lemma 3.1). Then Theorem 4.1 applies when $\omega \in \Omega_G$ and spread$(Q_\epsilon) < \epsilon_0$. Referring to (LYs), choose $K > 0$ such that $\mathbb{P}(B(\omega) > K) < \gamma/2$ and set $\Omega' := \Omega_G \cap \{B(\omega) \leq K\}$. Then $\mathbb{P}(\Omega') > 1 - \gamma$. Moreover, with $\alpha(\omega)$ from (LYs)

$$\begin{aligned}
\int_{\Omega'} \log \alpha(\omega)\ d\mathbb{P} &= \mathbb{P}(\Omega')\ \log(9(N_1+1)\Lambda^{2N_0}) + \int_{\Omega'} \log \prod_{k=0}^{m_0-1} \lambda_{N_0}(\sigma^{kN_0}\omega)^{-1}\ d\mathbb{P} \\
&\leq \log(9(N_1+1)\Lambda^{2N_0}) - \sum_{k=0}^{m_0-1} \int_{\sigma^{-kN_0}\Omega'} \log \lambda_{N_0}\, d\mathbb{P} \\
&\leq \log(9(N_1+1)\Lambda^{2N}) - \sum_{k=0}^{m_0-1} \int_{\Omega} \log \lambda_{N_0}\, d\mathbb{P} + \sum_{k=0}^{m_0-1} \int_{\Omega\setminus\sigma^{-kN_0}\Omega'} |\log \lambda_{N_0}|\, d\mathbb{P} \\
&< -2 + m_0\,(1/m_0) = -1,
\end{aligned}$$

by (m0) and the fact that each $\mathbb{P}(\Omega \setminus \sigma^{-kN_0}\Omega') = 1 - \mathbb{P}(\Omega') < \gamma$. Then, by the choice of γ,

$$\int_{\Omega'} \log \alpha(\omega)\ d\mathbb{P} + \int_{\Omega\setminus\Omega'} \log C(\omega)\ d\mathbb{P} < -1 + 1 = 0. \tag{11}$$

Letting α, B be as in Theorem 4.1 and $C(\omega)$ as in (LYw) put

$$(\tilde{\alpha}(\omega), \tilde{B}(\omega)) = \begin{cases} (\alpha(\omega), B(\omega)) & \omega \in \Omega', \\ (C(\omega), C(\omega)) & \text{otherwise.} \end{cases}$$

Thus, $\log \tilde{B}$ is integrable and $\int_\Omega \log \tilde{\alpha}\, d\mathbb{P} < 0$ (by (11)). We apply (LYs) when $\omega \in \Omega'$ and (LYw) otherwise. Then

$$\mathrm{var}(\mathcal{P}_{\omega,\epsilon}^{(N_1)}(h)) \le \tilde{\alpha}(\omega)\, \mathrm{var}(h) + \tilde{B}(\omega) \|h\|_{L^1}$$

and the proof is complete. □

Proof of Theorem 1.1

Let ϵ_0, N be as in Theorem 5.1. Put $\mathcal{L}_\omega = \mathcal{P}_\omega^{(N)}$ and $\mathcal{L}_{\omega,\epsilon} = \mathcal{P}_{\omega,\epsilon}^{(N)}$. The random Lasota–Yorke inequality in Theorem 5.1 holds uniformly for $\mathcal{L}_\omega$ and $\mathcal{L}_{\omega,\epsilon}$ for all $\epsilon < \epsilon_0$.

Due to the covering property, the random acim $\{h_\omega\}$ is unique for the original cocycle $\mathcal{P}_\omega$ [Buz00, Buz99]. Proceeding as in the the proof of [FGTQ14, Theorem 2.4], one has that, for sufficiently small ϵ, $\mathcal{L}_{\omega,\epsilon}$ also has a unique random acim (see also [Buz00, Proposition 2.1]). Specifically, there are unique (for $\epsilon < \epsilon_0$) $F, F_\epsilon : \Omega \to BV(I)$ with $\|F_\cdot(\omega)\|_{L^1} = 1$, $F_\cdot(\omega) \ge 0$ and such that $\mathcal{L}_\omega F(\omega) = F(\sigma^N \omega)$, $\mathcal{L}_{\omega,\epsilon} F_\epsilon(\omega) = F_\epsilon(\sigma^N \omega)$. The same proof as [FGTQ14, Theorem 3.7] gives fibrewise convergence of the random equivariant functions for $\mathcal{L}_{\omega,\epsilon}$ to those of $\mathcal{L}_\omega$ as $\epsilon \to 0$; that is, $F_\epsilon(\omega) \xrightarrow{L^1} F(\omega)$ for $\mathbb{P}$–a.e. ω. These are densities of random absolutely continuous invariant measures (acims). Because $\mathcal{L}_{\sigma\omega} \mathcal{P}_\omega = \mathcal{P}_{\sigma^N \omega} \mathcal{L}_\omega$, the random densities $G(\omega) := \mathcal{P}_{\sigma^{-1}\omega} F(\sigma^{-1}\omega)$ are also $\mathcal{L}$–equivariant. Hence $F = G$. In particular, $F(\sigma\omega) = G(\sigma\omega) = \mathcal{P}_\omega F(\omega)$, so F is $\mathcal{P}_\omega$–equivariant, and thus $F(\omega)$ coincides fibrewise with the densities h_ω. Similarly, the F_ϵ are the unique equivariant densities for $\{\mathcal{P}_{\omega,\epsilon}\}_{\omega\in\Omega}$, and the proof is complete.

6 Example

We illustrate the results with a system that exhibits alternating periods of expansion and contraction, while remaining sufficiently expanding on average. Let $I = [0, 1]$ and

$$f_1(x, \omega) = 2.1\,(x - 2\omega) \pmod 1, \qquad f_2(x, \omega) = 0.5\,(x - 2(\omega - 0.5)) \pmod 1$$

where $\omega \in \Omega := S^1$, $\mathbb{P}$ is Lebesgue measure and $\sigma(\omega) = \omega + \rho \pmod 1$ for irrational ρ. Put

$$f_\omega(x) = \begin{cases} f_1(x, \omega) & \text{if } \omega \in [0, 1/2), \\ f_2(x, \omega) & \text{if } \omega \in [1/2, 1). \end{cases}$$

Then $\int_\Omega \log \lambda(\omega)\, d\mathbb{P}(\omega) = 0.5(\log(2.1) + \log(0.5)) \approx 0.0244$ and $N_b(\omega) \leq 4$, so $\{f_\omega\}$ is an admissible random Lasota-Yorke map with $K = 0$ and $\Lambda = 2.1$. It remains to check that the set of recurrent turning points is $\mathbb{P}$-trivial. The only turning points of $\{f_\omega\}$ arise at discontinuities of $f_1(\cdot, \omega)$ or $f_2(\cdot, \omega)$. If x is such a discontinuity point, then $f_\cdot(x_+, \omega) = 0$ and $f_\cdot(x_-, \omega) = 1$ (where $f(x_\pm, \omega) = \lim_{y \to x^\pm} f(y, \omega)$). It therefore suffices to check for recurrence to 0 and 1.

Proposition 6.1. *For each n the set $R_n := \{\omega \;:\; f_\omega^{(n)}(\{0,1\}) \cap \{0,1\} \neq \emptyset\}$ is finite. In particular, $\mathbb{P}\{\omega : \exists x \in I s.t. \{f_\omega^{(n)}(x)\}_{n=0}^\infty$ visits two turning points $\} \leq \sum_{j=0}^\infty \sum_{n=1}^\infty \mathbb{P}(f^{-j} R_n) = 0$.*

Proof. For each (x, ω) there are integers $l_1 \in \{-2, -1, 0, 1, 2, 3\}$, $l_2 \in \{-1, 0\}$ such that

$$\begin{pmatrix} f_\omega(x) \\ \sigma(\omega) \end{pmatrix} = A(\omega) \begin{pmatrix} x \\ \omega \end{pmatrix} + b(\omega)$$

where

$$A_1 = \begin{pmatrix} 2.1 & -4.2 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.5 & -1 \\ 0 & 1 \end{pmatrix}, \qquad b_1 = \begin{pmatrix} l_1 \\ \rho + l_2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0.5 + l_1 \\ \rho + l_2 \end{pmatrix}$$

and $A(\omega) = A_j, b(\omega) = b_j$ when $f_\omega = f_j$ $(j = 1, 2)$. If $x_n = f^{(n)}(\omega)$ and $\omega_n = \sigma^n(\omega)$ then

$$\begin{pmatrix} x_n \\ \omega_n \end{pmatrix} = A(\omega_{n-1}) \cdots A(\omega_0) \begin{pmatrix} x_0 \\ \omega_0 \end{pmatrix} + \sum_{i=0}^{n-1} A(\omega_{n-1}) \cdots A(\omega_{i+1}) b(\omega_i).$$

Thus, for each n it is possible to write

$$x_n = \alpha_n\, x_0 + \beta_n\, \omega_0 + \gamma_n$$

where there are only finitely many possible choices of γ_n (depending on the (l_1, l_2) pairs defining $b_\cdot$). Additionally, $\beta_n < 0$ for all n (an easy induction). In particular, $\omega_0 = \frac{\alpha_n x_0 + \gamma_n - x_n}{-\beta_n}$. Setting each of x_0, x_n to be 0 or 1 shows that the set of possible ω_0 comprising R_n is finite. If $f_\omega^{(m)}(x)$ and $f_\omega^{(m+n)}(x)$ are both turning points then $\sigma^{m+1}(\omega) \in R_n$, completing the proof. $\square$

By way of example, consider $\rho = \frac{1}{10\sqrt{2}}$. Orbits of σ alternately spend 7 (or 8) iterates in $[0, 1/2)$, followed by 7 (or 8) iterates in $[1/2, 1)$. This gives rise to alternating periods of contraction and expansion, but a random Lasota-Yorke inequality holds for the transfer operator associated to f_ω, as well as stochastic perturbations satisfying

$$\mathrm{var}(Qh) \leq \mathrm{var}(h) + C\, \|h\|_{L^1}$$

when Q has small enough spread. We use an Ulam-type perturbation, where k is fixed, $\mathcal{B}_k$ is the σ-algebra generated by uniform subintervals $\{[i/k, (i+1)/k)\}_{i=0}^{k-1}$ and $Q(k) = \mathbb{E}(\cdot|\mathcal{B}_k)$. Then $\mathrm{var}(Q(k)h) \leq \mathrm{var}(h)$ (so $C = 0$) and $\mathrm{spread}(Q(k)) = \frac{1}{k}$.

We have approximated the random invariant density by pushing forward Lebesgue measure with the sequence of operators

$$\mathcal{P}_{\omega,k}^{(N)} := Q(k) \circ \mathcal{P}_{\sigma^{N-1}\omega} \circ Q(k) \circ \mathcal{P}_{\sigma^{N-2}\omega} \cdots \circ Q(k) \circ \mathcal{P}_{\omega}.$$

The results after 500 timesteps[4] are displayed in Figure 2 for $k = 10^p$ ($p = 2, 3, 4, 5, 6$). This implements Ulam's method, because the expectation with respect to a partition into subintervals is applied after *every* step of the dynamics. Notice that the coarser resolution pictures are very different to the finer ones, revealing a complicated local structure.

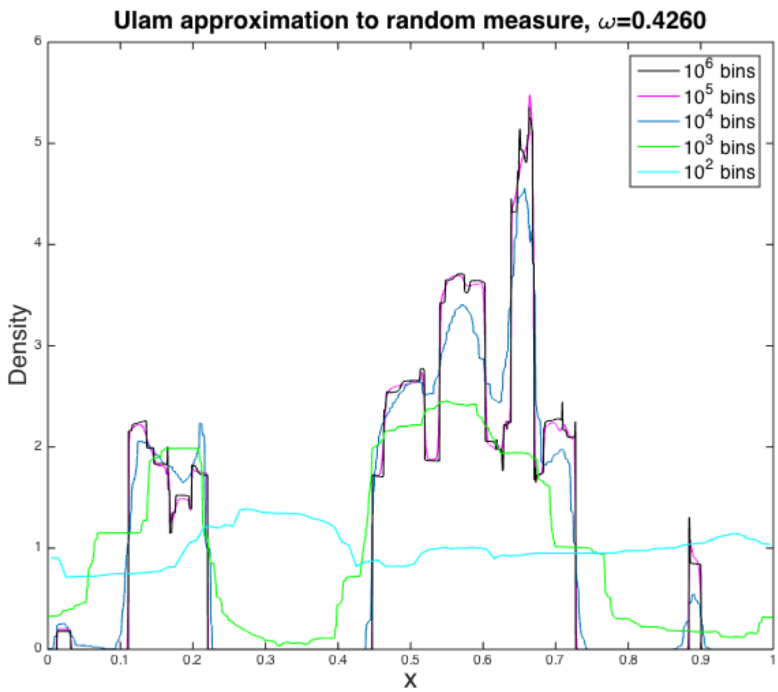

Figure 2: Approximations to the density of the quenched random measure $\mu_\omega|_{\omega=0.4260}$, computed via Ulam's method at differing levels of accuracy.

The densities in Figure 2 are supported on the fibre I_ω where $\omega = 501\rho \pmod 1 \approx 0.4260$. Indeed, the next few ωs are $\{0.4260, 0.4968, 0.5675, 0.6382, 0.7089\}$ so two subsequent iterations of the dynamics are expanding (via f_1, since $\omega \in [0, 1/2)$), followed by several contracting maps. The density from Figure 2, together with the next five densities are shown in Figure 3. Note in particular the increased irregularity under the contracting maps, illustrating the complexity of the random dynamics.

[4]These approximations rely on quasicompactness to ensure convergence of $\mathcal{P}_{\omega,\epsilon}^{(N)}\mathbf{1}$ for large N. Repeating the experiments with $\mathcal{P}_{\sigma^{-n_0}\omega,\epsilon}^{(N+n_0)}\mathbf{1}$ gives the same results, suggesting that convergence has been achieved.

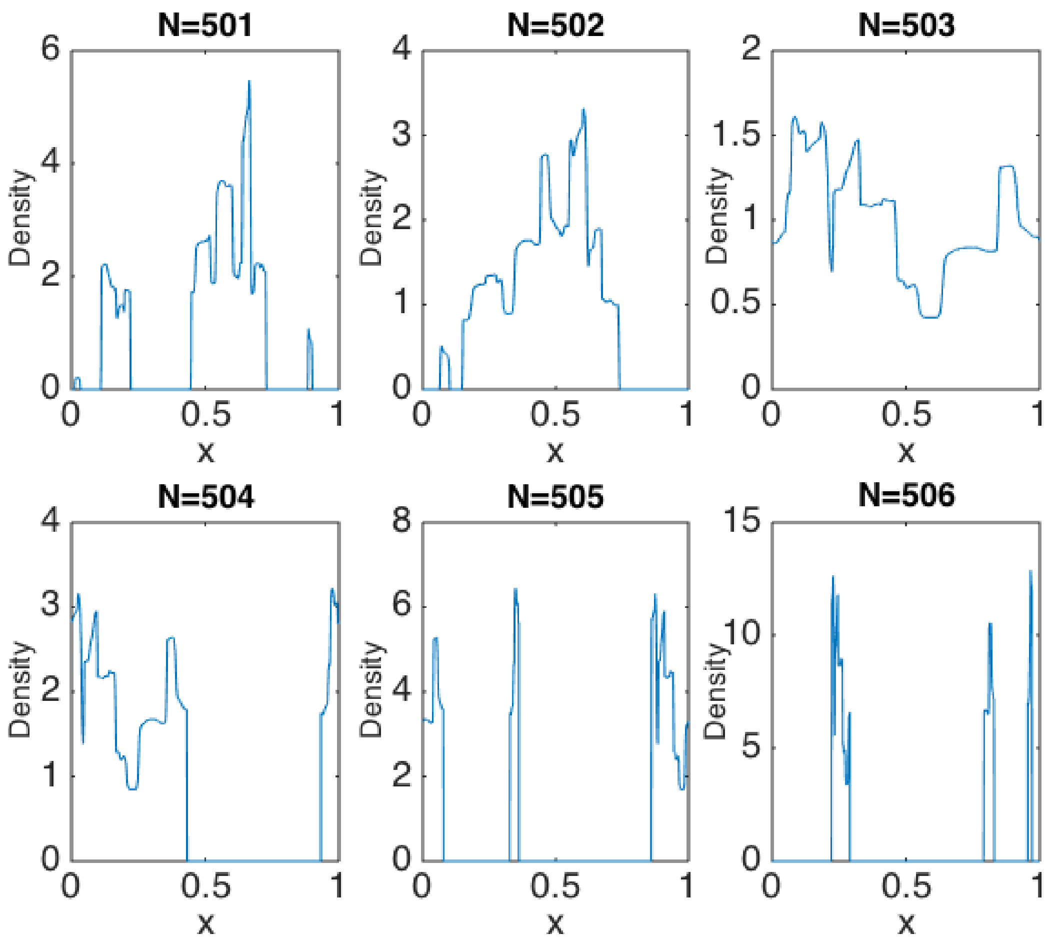

Figure 3: Several consecutive quenched random measures μ_ω, computed by Ulam's method with $k = 10^5$ subintervals and supported on fibres I_ω, $\omega = \sigma^N 0$.

Ulam's method can be used to gain computational access to the *physical measure* [Buz00] for the random system (see equation (1)). Due to ergodicity of $(\tau, I \times \Omega)$, this measure can be approximated via a long random orbit. Alternatively, the quenched random densities $h_\omega = \frac{d\mu_\omega}{dx}$ on fibres I_ω can be averaged over Ω. Relying on ergodicity of σ, we offer an approximation to the density of the physical measure as

$$\frac{1}{N_1 - N_0} \sum_{t=N_0+1}^{N_1} h^{(m)}_{\sigma^t \omega_0} \tag{12}$$

where $h^{(k)}_\omega$ is the Ulam approximation to h_ω using k–uniform subintervals[5], $N_0 = 500$ and $N_1 = 10^4$. Such an average is shown in Figure 4 with a $k = 10^5$ subintervals, along with a

[5] In practice, we iterate with $h^{(k)}_{\sigma\omega} = \mathcal{P}_{\omega,k} h^{(k)}_\omega$.

comparison of a random orbit of length 10^8 (both are shown as histograms over 1000 uniform bins). Qualitative agreement is evident between the two methods.

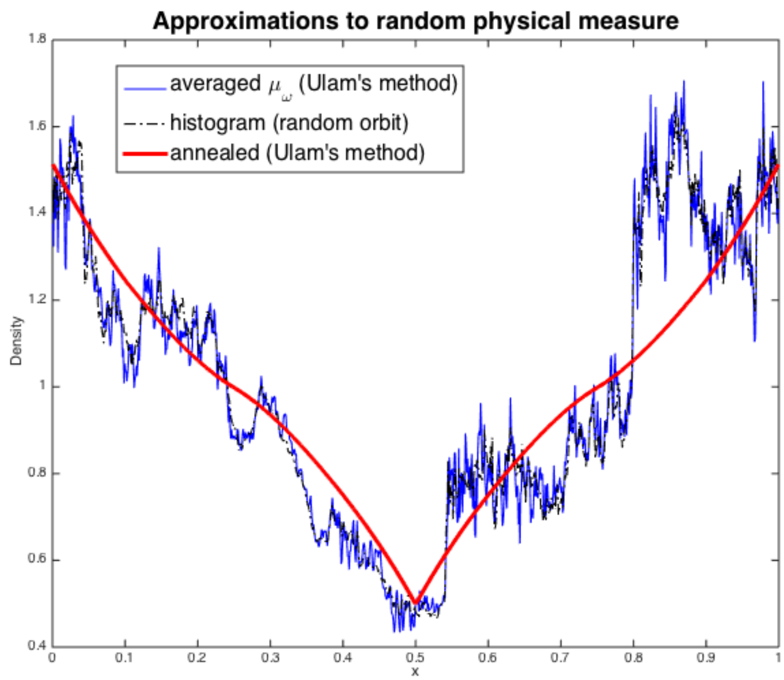

Figure 4: Physical measure of random Lasota–Yorke maps temporally-averaged Ulam approximations of the quenched invariant measures μ_ω as per equation (12) ($k = 10^5$ subintervals, $N_0 = 500 \ldots N_1 = 10^4$) (solid blue). Histogram over a random orbit of length 10^8, drawn as histograms over 1000 bins (dash-dot black). Ulam approximation to density of averaged operator, computed according to (13) (dashed red).

These experiments also provide an ideal illustration of the loss of information inherent in approximating a random dynamical system via an *averaged transfer operator*. The averaged operator is

$$\bar{\mathcal{P}} := \int_\Omega \mathcal{P}_\omega \, d\mathbb{P}(\omega),$$

and its fixed points can be interpreted as physical densities when the base dynamics is IID [Ohn83, Bal97]. This is sometimes called the *annealed* case. In non-IID, but σ–ergodic, cases Ulam's method gives an approximation

$$\bar{\mathcal{P}}_k := \frac{1}{N} \sum_{t=0}^{N-1} \mathcal{P}_{\sigma^t \omega, k}. \tag{13}$$

Figure 4 includes a comparison with the fixed point of this averaged operator, calculated via

Ulam's method as a fixed point of the approximation to $\bar{\mathcal{P}}_k$ with $k = 500$ subintervals and $N = 5000$; calculating with higher k and N showed no visible changes.

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A Variation estimates

Let $h \in BV(\mathbb{R})$, let $|b-a| < \delta$ and let $y \in [a,b]$ be such that $|h(y)| \le \frac{\int_a^b |h|}{b-a}$. Then

$$|h(a)| \le |h(a) - h(y)| + \frac{\|h\|_{L^1[a,b]}}{\delta} \qquad \text{and} \qquad |h(b)| \le |h(y) - h(b)| + \frac{\|h\|_{L^1[a,b]}}{\delta}.$$

Hence

$$|h(a)|+|h(b)| \le \mathrm{var}_{[a,b]}(h)+\frac{2\,\|h\|_{L^1[a,b]}}{\delta} \qquad \text{and} \qquad \max\{|h(a)|,|h(b)|\} \le \mathrm{var}_{[a,b]}(h)+\frac{\|h\|_{L^1[a,b]}}{\delta}.$$

Let $b - a < \delta$ and let $c \in (-\infty, a) \cup (b, \infty)$. The facts above imply the following estimates for $h \in BV(\mathbb{R})$:

$$\mathrm{var}(h\,\mathbf{1}_{[a,b]}) \le 2\,\mathrm{var}_{[a,b]}(h) + 2\,\frac{\|h\|_{L^1}}{\delta} \tag{14}$$

and

$$|h(c)| \le \begin{cases} \mathrm{var}_{[c,b]}(h) + \frac{\|h\|_{L^1[a,b]}}{\delta} & \text{if } c < a, \\ \mathrm{var}_{[a,c]}(h) + \frac{\|h\|_{L^1[a,b]}}{\delta} & \text{if } b < c. \end{cases} \tag{15}$$

Using (15):

$$\begin{aligned} \mathrm{var}(h\,\mathbf{1}_{\mathbb{R}\setminus(a,b)}) &\le \mathrm{var}_{(-\infty,a]}(h) + \mathrm{var}_{[a,b]}(h) + \mathrm{var}_{[b,\infty)}(h) + 2\,\frac{\|h\|_{L^1[a,b]}}{\delta} \\ &= \mathrm{var}(h) + 2\,\frac{\|h\|_{L^1[a,b]}}{\delta} \end{aligned} \tag{16}$$

B Estimates on $\lambda_{N_1-n}(\sigma^n\omega)\lambda_n(\omega)$

Note from the definition of $\lambda_n(\omega) = \inf_x |f_\omega^{(n)\prime}(x)|$ that for all $s, t \geq 0$

$$\lambda_s(\sigma^t\omega)\lambda_t(\omega) \leq \lambda_{s+t}(\omega) \leq \Lambda^s \lambda_t(\omega).$$

Now let $n \leq N_1$ be given and choose $m = \lfloor \frac{n}{N_0} \rfloor$ and write $n = mN_0 + t$. Then

$$\lambda_{mN_0+t}(\omega) \geq \lambda_t(\sigma^{mN_0}\omega) \prod_{k=0}^{m-1} \lambda_{N_0}(\sigma^{kN_0}\omega) \geq \Lambda^{-(N_0-t)} \prod_{k=0}^{m} \lambda_{N_0}(\sigma^{kN_0}\omega).$$

Similarly,

$$\lambda_{N_1-n}(\sigma^n\omega) \geq \left(\prod_{k=m+1}^{m_0} \lambda_{N_0}(\sigma^{kN_0}\omega) \right) \lambda_{N_1-t}(\sigma^n\omega) \geq \Lambda^{-(N_0-t)} \prod_{k=m+1}^{m_0-1} \lambda_{N_0}(\sigma^{kN_0}\omega).$$

Combining these,

$$\lambda_{N_1-n}(\sigma^n\omega)\, \lambda_n(\omega) \geq \Lambda^{-2N_0} \prod_{k=0}^{m_0-1} \lambda_{N_0}(\sigma^{kN_0}\omega). \tag{17}$$

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