A semi-invertible operator Oseledets theorem

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Abstract

Semi-invertible multiplicative ergodic theorems establish the existence of an Oseledets splitting for cocycles of non-invertible linear operators (such as transfer operators) over an invertible base. Using a constructive approach, we establish a semi-invertible multiplicative ergodic theorem that for the first time can be applied to the study of transfer operators associated to the composition of piecewise expanding interval maps randomly chosen from a set of cardinality of the continuum. We also give an application of the theorem to random compositions of perturbations of an expanding map in higher dimensions.

1 Introduction

1.1 Motivation and History

Oseledets' proof, in 1965, of the multiplicative ergodic theorem is a milestone in the development of modern ergodic theory. It has been applied to differentiable dynamical systems to establish the existence of Lyapunov exponents and plays a crucial role in the construction of stable and unstable manifolds. It also has substantial applications in the theory of random matrices, Markov chains, etc.

The proof has been generalized in many directions by a number of authors (including Ruelle [39], Mañé [36], Ledrappier [26], Raghunathan [38], Kaimanovich [25] and many others). In the original version, one has an ergodic measure-preserving system $\sigma\colon \Omega \to \Omega$ and for each $\omega \in \Omega$, a corresponding matrix $A(\omega) \in M_d(\mathbb{R})$. Under suitable integrability conditions on the norms of the matrices it is shown that over almost every point, ω, of Ω, there is a measurably-varying collection of subspaces $(V_i(\omega))_{1\le i\le k}$, with a decreasing sequence of characteristic exponents λ_i such that (i) the subspaces are *equivariant* - that is, $A(\omega)(V_i(\omega)) \subset V_i(\sigma\omega)$; and (ii) that vectors in $V_i(\omega)$ (typically) expand

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at rate λ_i under sequential applications of the matrices $A(\sigma^j\omega)$ along the orbit. That is,

$$\lim_{n\to\infty}\frac{1}{n}\log\|A(\sigma^{n-1}\omega)\cdots A(\omega)v\| = \lambda_i.$$

More specifically and of crucial significance for this article, Oseledets' multiplicative ergodic theorem was proved in two versions: an invertible version and a non-invertible version.

In the invertible version the following is assumed: σ is an invertible transformation of Ω; the matrices $A(\omega)$ are each invertible and $\int \log^+\|A(\omega)\|\,d\mathbb{P}(\omega)$ and $\int \log^+\|A(\omega)^{-1}\|\,d\mathbb{P}(\omega)$ are both finite. The conclusion of the theorem is then that there is for almost every ω a measurable *splitting* of $\mathbb{R}^d$:

$$\mathbb{R}^d = Y_1(\omega)\oplus Y_2(\omega)\oplus\ldots\oplus Y_l(\omega) \tag{1}$$

such that for all $v \in Y_i(\omega)\setminus\{0\}$

$$\lim_{n\to\infty}\frac{\log\|A^{(n)}(\omega)v\|}{n} = \lambda_i; \tag{2}$$

$$\lim_{n\to-\infty}\frac{\log\|A^{(n)}(\omega)v\|}{n} = \lambda_i, \tag{3}$$

where $A^{(n)}$ denotes the matrix cocycle $A(\sigma^{n-1}\omega)\cdots A(\omega)$ for $n > 0$ whereas for $n < 0$, it is $A(\sigma^{-n}\omega)^{-1}\cdots A(\sigma^{-1}\omega)^{-1}$.

In the non-invertible version of the theorem, σ is no longer assumed to be invertible and there is no assumption on the invertibility of the matrices $A(\omega)$. In this case there is a weaker conclusion: rather than a splitting of $\mathbb{R}^d$ one obtains a *filtration*: A decreasing sequence of subspaces of $\mathbb{R}^d$

$$\mathbb{R}^d = V_1(\omega)\supset V_2(\omega)\supset\ldots\supset V_l(\omega) \tag{4}$$

such that for all $v \in V_i(\omega)\setminus V_{i+1}(\omega)$ (defining $V_{l+1}(\omega)$ to be $\{0\}$), (2) holds.

In [18], Froyland, Lloyd and Quas refined the dichotomy between invertible and non-invertible versions of the theorem, introducing a third class of versions of the theorem: *semi-invertible* multiplicative ergodic theorems. For semi-invertible ergodic theorems the underlying dynamical system is assumed to be invertible, but no assumption is made on the invertibility of the matrices. The conclusion of the theorem in this category is that there is again a splitting of the vector space (instead of a filtration) and that for all $v \in Y_i(\omega)\setminus\{0\}$, (2) holds (but not (3) which does not make sense in this context).

Our motivation for considering semi-invertible multiplicative ergodic theorems comes from application-oriented studies of rates of mixing due to Dellnitz, Froyland and collaborators [12, 11, 17]. Given a measure-preserving dynamical system it is called (strong-) mixing if $\mu(A\cap T^{-n}B)\to\mu(A)\mu(B)$ for all measurable sets A and B. This is an asymptotic independence property for any measurable sets under evolution.

An equivalent formulation of mixing is that $\int f\cdot g\circ T^n\,d\mu$ should converge to $\int f\,d\mu\int g\,d\mu$ for all L^2 functions f and g. Clearly nothing is lost if one demands that the functions should have zero integral.

Relaxing the assumption that μ is an invariant measure, one may take μ to be some ambient measure (e.g. Lebesgue measure in the case that T is a smooth map of a manifold or subset of $\mathbb{R}^d$). A key tool in this study is the *Perron- Frobenius Operator* or *transfer operator*, $\mathcal{L}$, acting on $L^1(\mu)$. This is the pre-dual of the operator of composition with T acting on L^∞, the so-called Koopman operator, so that $\int f \cdot g \circ T \, d\mu = \int \mathcal{L}f \cdot g \, d\mu$ for all $f \in L^1$ and $g \in L^\infty$). In many cases one can give a straightforward expression for $\mathcal{L}f$. It is not hard to check from the definition that $\mathcal{L}f = f$ if and only if f is the density of an absolutely continuous invariant measure for T.

One might naively ask for the rate of convergence of $\int f \cdot g \circ T^n \, d\mu$ to 0 if indeed the system is mixing, but simple examples show that there is no uniform rate of convergence: one can construct in any non-trivial mixing system, functions f and g such that the rate of convergence is arbitrarily slow. One does however obtain rates of mixing if one places suitable restrictions on the class of 'observables' f and g for which one computes $\int f \cdot g \circ T^n \, d\mu$. It turns out that an important reason that the Perron-Frobenius operator is so useful is that if one restricts the function f to a suitable smaller Banach space of observables $B \subset L^1$, then in many cases $\mathcal{L}$ maps B to B; and better still $\mathcal{L}$ is a quasi-compact operator on B, so that the spectrum of $\mathcal{L}$ consists of a discrete set of values outside the essential spectral radius each corresponding to eigenvalues of $\mathcal{L}$ with finite-dimensional eigenspaces. Given this one can relate the rate of mixing of the dynamical system (restricted to a suitable class of observables) to the spectral properties of the operator $\mathcal{L}$ restricted to the Banach space B. It is a key fact for our purposes that the Perron-Frobenius operators $\mathcal{L}$ that one works with are almost invariably non-invertible.

Ulam's method takes this one step further, replacing the operator $\mathcal{L}$ by a finite rank approximation. In works of Froyland [15, 16] and Baladi, Isola and Schmitt [3], the relationship between the finite rank approximations of $\mathcal{L}$ and the original Perron-Frobenius operators is studied. This turns out to be remarkably effective and this is a good technique for computing invariant measures numerically (see for example work of Dellnitz and Junge [12], Froyland[14], Keane, Murray and Young [28]). Keller and Liverani [29] showed that exceptional eigenvalues of $\mathcal{L}$ (those outside the essential spectral radius) persist under approximation of $\mathcal{L}$.

In a development of Ulam's method, [12] and later [16] related the large sub-unit eigenvalues and corresponding eigenvectors of the finite rank approximation of $\mathcal{L}$ to properties of the underlying system. In particular they showed that these exceptional eigenvectors give rise to global features inhibiting mixing of the system (whereas the essential spectral radius is related to local features inhibiting mixing of the system). For a cartoon example, one can consider a map of the interval $[-1, 1]$ in which the left sub-interval $[-1, 0]$ and right sub-interval $[0, 1]$ are almost invariant (that is only a small amount of mass leaks from one to the other under application of the map) but within each subinterval there is rapid mixing- see work of González-Tokman, Hunt and Wright [23] and Dellnitz, Froyland and Sertl [11]. In this case one observes eigenvalues that are close to 1, where the eigenfunction takes values close to 1 on one sub-interval and close to -1 on the other sub-interval. In applied work Dellnitz, Froyland and collaborators[21, 10] make use of these exceptional eigenvectors to analyse the ocean and locate regions with poor mixing, called gyres.

The current work (and its predecessors [18] and [19]) is motivated by extending the program of Dellnitz and Froyland to the case of forced dynamical systems (or equivalently random dynamical systems), that is systems of the form $T(\omega, x) = (\sigma(\omega), T_\omega(x))$. Again as a cartoon example, one can consider the effect of the moon on the oceans: the moon evolves autonomously (and *invertibly*), whereas the evolution of the ocean is affected by the position of the moon. Relating this to the context of the multiplicative ergodic theorem, think of the dynamical system $\sigma\colon \Omega \to \Omega$ as being the autonomous dynamics of the moon and the ω-dependent matrix to be a map on a Banach space of densities in the ocean. The aim is, once again, to identify and study the second and subsequent exceptional eigenspaces with a view to understanding obstructions to mixing. The importance of the semi-invertible multiplicative ergodic theorems here (the underlying base dynamics is invertible but the Perron-Frobenius operators are non-invertible) are that the obstructions to mixing, the $V_2(\omega)$, appear here as finite-dimensional subspaces rather than the finite-codimensional subspaces that one would obtain from the standard multiplicative ergodic theorems. This program has been demonstrated to work in practice for driven cylinder flows in an article of Froyland, Lloyd and Santitissadeekorn [20].

In all three works, this paper and its two predecessors, [18] and [19], the goal is to prove a semi-invertible multiplicative ergodic theorem and apply it to as general a class of random dynamical systems as possible. In all three papers, the starting point was a pair of multiplicative ergodic theorems: an invertible and a non-invertible; and then to derive, using the pair of ergodic theorems as *black boxes*, a semi-invertible ergodic theorem.

[18] dealt with the original Oseledets context of $d \times d$ real matrices (and used Oseledets' original theorem [37] as the basis). [19] dealt with the case of an operator-valued multiplicative ergodic theorem where the map $\mathcal{L}\colon \omega \mapsto \mathcal{L}(\omega)$ is (almost)-continuous with respect to the operator norm (using a theorem of Thieullen [42] as a basis). The current paper deals with the case of an operator-valued multiplicative ergodic theorem where the map $\omega \mapsto \mathcal{L}(\omega)$ is measurable with respect to a σ-algebra related to the strong operator topology (using a Theorem of Lian and Lu [34] as a basis).

The applications to random dynamical systems have become progressively more general through the sequence of works: [18] applied to finite-dimensional approximations of random dynamical systems (using the Ulam scheme) as well as dealing exactly with some dynamical systems satisfying an extremely strong jointly Markov condition. [19] applied to one-dimensional expanding maps. However, since the set of Perron-Frobenius operators of C^2 expanding maps acting on the space of functions of bounded variation is uniformly discrete, the conditions of the theorem restricted the authors to studying random dynamical systems with at most countably many maps. In the current paper, Lian and Lu's result allows us to weaken the continuity assumption to strong measurability (defined below). Essentially, this amounts to checking continuity of $\omega \mapsto \mathcal{L}_{T_\omega} f$

for a *fixed* f. The cost, however, is that the Banach space on which the transfer operators act is now required to be separable (which the space of functions of bounded variation, used in [19], is not). In order to apply the semi-invertible ergodic theorem to random one-dimensional expanding maps, we make substantial use of recent work of Baladi and Gouëzel [2] who used a family of local Sobolev norms to study Perron-Frobenius operators of (higher-dimensional) piecewise hyperbolic maps; see also Thomine [43] for a specialization in the context of expanding maps. While Baladi and Gouëzel were working with a single map, we show that the Perron-Frobenius operators on the Banach spaces that they construct depend in a suitable way for families of expanding one-dimensional maps allowing us to apply our semi-invertible multiplicative ergodic theorem (making essential use also of an idea of Buzzi [7]). We also point out that, to our knowledge, it was not even known whether an Oseledets filtration existed in this setting.

Another feature of the proofs is the way in which the semi-invertible theorem is proved from the invertible and non-invertible theorems. The essential issue is that the non-invertible theorem provides equivariant families of finite co-dimensional subspaces $V_i(\omega)$ (being the set of vectors that expand at rate λ_i or less). One is then attempting to build an equivariant family of (finite-dimensional) vector spaces $Y_i(\omega)$ so that $V_{i+1}(\omega) \oplus Y_i(\omega) = V_i(\omega)$.

In [18] this was done in a relatively natural way (by pushing forward the orthogonal complement of $V_i(\sigma^{-n}\omega) \ominus V_{i+1}(\sigma^{-n}\omega)$ under $A(\sigma^{-1}\omega) \cdots A(\sigma^{-n}\omega)$ and taking a limit as n tends to infinity).

In [19], the proof exploited the structure of the proof given by Thieullen. Specifically Thieullen first proved the invertible multiplicative ergodic theorem and then obtained the non-invertible theorem as a corollary by building an inverse limit Banach space (reminiscent of the standard inverse limit constructions in ergodic theory). The finite-co-dimensional family $V_i(\omega)$ was obtained by projecting the corresponding subspaces from the invertible theorem onto their zeroth coordinate. In [19] it was proved that applying the same projection to the finite-dimensional complementary family yielded the $Y_i(\omega)$ spaces. This proof, while relatively simple, is problematic for applications as there appears to be no sensible way to computationally work with these inverse limit spaces. We see this proof technique as non-constructive. This non-constructive proof technique should probably apply with a high degree of generality.

In the current paper we come back much closer to the scheme applied in [18]. The same non-constructive techniques that were used by Thieullen to obtain the non-invertible theorem from the invertible theorem were used by Doan in his thesis [13] to obtain a non-invertible version of the result of Lian and Lu [34]. Starting from the non-constructive existence proof of the finite co-dimensional subspaces we obtain a constructive proof of the finite-dimensional $Y_i(\omega)$ spaces. We see this as being likely to lead to computational methods although we have not implemented these at the current time.

1.2 Statement of Results and structure of paper

The context of Lian and Lu's multiplicative ergodic theorem is that of strongly measurable families of operators.

If X is a separable Banach space, then $L(X)$ will denote the set of bounded linear maps from X to X. The strong operator topology on $L(X)$ is the topology generated by the sub-base consisting of sets of the form $\{T\colon \|T(x)-y\| < \epsilon\}$. The strong σ-algebra $\mathcal{S}$ is defined to be the Borel σ-algebra on $L(X)$ generated by the strong operator topology. Appendix A develops a number of basic results about strong-measurability, including the following useful characterization: A map $\mathcal{L}\colon \Omega \to L(X)$ is strongly measurable if for each $x \in X$, the map $\Omega \to X$, $\omega \mapsto \mathcal{L}(\omega)(x)$ is measurable with respect to the σ-algebra on Ω and the Borel σ-algebra on X.

Of course the 'strong operator topology' is very much coarser than the norm topology on $L(X)$ - checking continuity in the strong operator topology can be done one x at a time. This is the essential difference between the result of Thieullen and that of Lian and Lu: for a given function f, $\mathcal{L}(\omega)f$ and $\mathcal{L}(\omega')f$ are close if T_ω and $T_{\omega'}$ are close enough, but the operators $\mathcal{L}_{T_\omega}$ and $\mathcal{L}_{T_{\omega'}}$ are, in many interesting cases, uniformly far apart. (An exception to this is the setting of smooth expanding analytic maps.)

For convenience we state our main results here, even though some of the terms in the statement have yet to be defined. These correspond to Theorems 2.10 and 3.19 in the body of the paper.

Our new semi-invertible multiplicative ergodic theorem is the following (for simplicity we state the version in which there are finitely many exceptional exponents; a corresponding version holds if there are countably many exponents which then necessarily converge to κ^*).

Theorem A. *Let σ be an invertible ergodic measure-preserving transformation of the Lebesgue space $(\Omega, \mathcal{F}, \mathbb{P})$. Let X be a separable Banach space. Let $\mathcal{L}\colon \Omega \to L(X)$ be a strongly measurable family of mappings such that $\log^+ \|\mathcal{L}(\omega)\| \in L^1(\mathbb{P})$ and suppose that the random linear system $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ is quasi-compact (i.e. the analogue of the spectral radius, λ^*, is larger than the analogue of the essential spectral radius, κ^*).*

Then there exists $1 \leq l \leq \infty$ and a sequence of exceptional Lyapunov exponents $\lambda^* = \lambda_1 > \lambda_2 > \ldots > \lambda_l > \kappa^*$ (or in the case $\lambda = \infty$, $\lambda^* = \lambda_1 > \lambda_2 > \ldots$; $\lim_{n\to\infty} \lambda_n = \kappa^*$).

For $\mathbb{P}$-almost every ω there exists a unique measurable equivariant splitting of X into closed subspaces $X = V(\omega) \oplus \bigoplus_{j=1}^{l} Y_j(\omega)$ where the $Y_j(\omega)$ are finite-dimensional. For each $y \in Y_j(\omega) \setminus \{0\}$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} y\| = \lambda_j$. For $y \in V(\omega)$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} y\| \leq \kappa^*$.

The application to random piecewise expanding systems is as follows:

Theorem B. *Let σ be an invertible ergodic measure-preserving transformation of the Lebesgue space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $\omega \in \Omega$, let T_ω be a random expanding* *dynamical system acting on $X_0 \subset \mathbb{R}^d$. Assume further that $\omega \mapsto T_\omega$ is Borel-measurable, the $C^{1+\alpha}$ norm of T_ω is uniformly bounded above, the maps T_ω have a derivative that is uniformly bounded away from 1, and that some integrability conditions are satisfied.*

Suppose that either $d = 1$ (Lasota-Yorke case); or $d > 1$ and the maps T_ω are C^2, have a common branch partition and belong to a sufficiently small neighbourhood of a Cowieson map.

Then there exist a separable, reflexive Banach space X containing the C^∞ functions supported on X_0 for which the map $\omega \mapsto \mathcal{L}_\omega$ given by the transfer operator associated to T_ω is strongly measurable, a quantity $1 \leq l \leq \infty$, a sequence of exceptional exponents $0 = \lambda_1 > \ldots > \lambda_l > \kappa^*$, (or if $l = \infty$, then $0 = \lambda_1 > \lambda_2 > \ldots > \kappa^*$; $\lim_{n\to\infty} \lambda_n = \kappa^*$), and a family of finite-dimensional equivariant subspaces $(Y_i(x))_{1\leq i\leq l}$ satisfying the conclusions of Theorem A.

The main motivation behind our search for semi-invertible Oseledets theorems has been to provide a general framework in which it is possible to identify low-dimensional spaces that are responsible for impeding mixing in infinite-dimensional dynamical systems. Following Dellnitz, Froyland and collaborators we want to extract information not simply from the exceptional Lyapunov exponents, but rather from the corresponding Lyapunov subspaces.

It is important to note that exponential decay of correlations is not assumed. Our work applies, for instance, to an example of Buzzi in [6] (Example 3). Buzzi's example (which works by essentially having two copies of the interval and a pair of maps each of which acts as doubling on each interval and then simply permutes the intervals) fails to have exponential decay of correlations, but it is still quasi-compact. In our context this will be reflected in the fact that the top exceptional Lyapunov subspace has multiplicity 2 rather than 1. In fact, the structure of this top subspace exactly illustrates the goal of our work because the Oseledets space will consist of a constant function and a function which is 1 on one of the intervals and -1 on the other, thereby indicating the source of non-mixing.

In addition, there are examples in the existing literature showing the possibilities of having more than one Oseledets space; that is, $l \geq 2$. In the random setting, there is an example by Froyland, Lloyd and Quas, [18, Theorem 5.1]; in the deterministic case, there is one by Keller and Rugh [30, Theorem 1]. In fact, it is *a priori* possible to have all sorts of combinations for number of exceptional Lyapunov exponents ($1 \leq l \leq \infty$) and multiplicities ($1 \leq m_1, \ldots, m_l < \infty$), in a similar way that square matrices may have different Jordan normal forms.

In section 2 we give the proof of the semi-invertible multiplicative theorem. In section 3 we introduce the fractional Sobolev spaces (as used in Baladi and Gouëzel) and study the continuity properties of the map sending a Lasota-Yorke map to its Perron-Frobenius operator. We then adapt the proof given by Baladi and Gouëzel of quasi-compactness for a single map to the situation of a random composition of one-dimensional expanding maps (using results of Hennion and Buzzi) to show that the theorem of section 2 applies in this context. We also present an application of Theorem A to piecewise expanding maps in higher dimensions, building on work of Cowieson [9]. Section 4 summarizes possible directions for future work.

The paper has three appendices: Appendix A contains results about strong measurability. Appendix B contains results about the Grassmannian of a separable Banach space. Appendix C collects some results from ergodic theory: a useful characterization of tempered maps and a Hennion type theorem for random linear systems.

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2 Oseledets splittings for random linear systems.

2.1 Preliminaries

We start by introducing some notation about random dynamical systems.

Definition 2.1. *A* ***separable strongly measurable random linear system*** *is a tuple $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ such that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, σ is a probability preserving transformation of $(\Omega, \mathcal{F}, \mathbb{P})$, X is a separable Banach space, and the generator $\mathcal{L} : \Omega \to L(X)$ is a strongly measurable map (see Definition A.3). We use the notation $\mathcal{L}_\omega^{(n)} = \mathcal{L}(\sigma^{n-1}\omega) \circ \cdots \circ \mathcal{L}(\omega)$.*

Definition 2.2. *The* ***index of compactness*** *(or Kuratowski measure of non-compactness) of a bounded linear map $A : X \to X$ is*

$$\|A\|_{ic(X)} = \inf\{r > 0 : A(B_X) \text{ can be covered by finitely many balls of radius } r\},$$

where B denotes the unit ball in X.

Definition 2.3. *Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ be a separable strongly measurable random linear system. Assume that $\int \log^+ \|\mathcal{L}_\omega\| d\mathbb{P}(\omega) < \infty$. For each $\omega \in \Omega$, the* ***maximal Lyapunov exponent*** *for ω is defined as*

$$\lambda(\omega) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|,$$

whenever the limit exists. For each $\omega \in \Omega$, the ***index of compactness*** *for ω is defined as*

$$\kappa(\omega) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|_{ic(X)},$$

whenever the limit exists. Whenever we want to emphasize the dependence on $\mathcal{R}$, we will write $\lambda_{\mathcal{R}}(\omega)$ and $\kappa_{\mathcal{R}}(\omega)$.

Lemma 2.4. *Let $\mathcal{R}$ be as in Definition 2.3. $\lambda(\omega)$ is well defined for $\mathbb{P}$-almost every ω. The function $\omega \mapsto \lambda(\omega)$ is measurable and σ-invariant.*

Proof. The sequence of functions $\{\log \|\mathcal{L}_\omega^{(n)}\|\}_{n\in\mathbb{N}}$ is subadditive. That is,

$$\log \|\mathcal{L}_\omega^{(m+n)}\| \leq \log \|\mathcal{L}_{\sigma^n\omega}^{(m)}\| + \log \|\mathcal{L}_\omega^{(n)}\|.$$

Since the composition of strongly measurable maps is strongly measurable by Lemma A.5, and the sets $L_r(X) = \{A \in L(X) : \|A\| \leq r\}$ are $\mathcal{S}$ measurable by Lemma A.2(1), then the map $\omega \mapsto \|\mathcal{L}_\omega^{(n)}\|$ is measurable. Measurability of the map $\omega \mapsto \lim_{n\to\infty} \sup \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|$ follows. By Kingman's subadditive theorem [33], the limit $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|$ exists for $\mathbb{P}$-almost every $\omega \in \Omega$, and it is σ-invariant. $\square$

Lemma 2.5. *Let $\mathcal{R}$ be as in Definition 2.3. The index of compactness is finite, submultiplicative and measurable, when $L(X)$ is equipped with the strong σ-algebra $\mathcal{S}$. Thus, $\kappa(\omega)$ is well defined for $\mathbb{P}$-almost every ω. The function $\omega \mapsto \kappa(\omega)$ is measurable and σ-invariant.*

Proof. The index of compactness is bounded by the norm. Submultiplicativity is straightforward to check. To show $\mathcal{S}$-measurability of the index of compactness, we present the argument given in Lian and Lu [34]. Let $\{x_i\}_{i\in\mathbb{N}}$ be a dense subset of X and $\{y_j\}_{j\in\mathbb{N}}$ be a dense subset of $B(X)$. Let U be the (countable) set of finite subsets of $\{x_i\}_{i\in\mathbb{N}}$. Let $U = \bigcup_{i\in\mathbb{N}} U_i$. Then, one can check that

$$\{A \in L(X) : \|A\|_{ic(X)} < r\} = \bigcup_{n=2}^{\infty} \bigcup_{i=i}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{x\in U_i} \{A : \|A(y_j) - x\| < (1 - 1/n)r\},$$

(see [34, Lemma 6.5] for a proof.) Hence, $A \mapsto \|A\|_{ic(X)}$ is $\mathcal{S}$-measurable. Thus, $\mathbb{P}$-almost everywhere existence, measurability and σ-invariance of κ follow just like in the proof of Lemma 2.4. $\square$

Remark 2.6. *If $\mathcal{R}$ has an ergodic base, then λ and κ are $\mathbb{P}$-almost everywhere constant. We call these constants $\lambda^*(\mathcal{R})$ and $\kappa^*(\mathcal{R})$, or simply λ^* and κ^* if $\mathcal{R}$ is clear from the context. It follows from the definitions that $\kappa^* \leq \lambda^*$. The assumption $\int \log^+ \|\mathcal{L}_\omega\| d\mathbb{P}(\omega) < \infty$ implies that $\lambda^* < \infty$.*

Definition 2.7. *A strongly measurable random linear system with ergodic base is called **quasi-compact** if $\kappa^* < \lambda^*$.*

2.2 Construction of Oseledets splitting

The following theorem was obtained by Doan [13] as a corollary of the two-sided Oseledets theorem proved by Lian and Lu [34].

Theorem 2.8 (Doan [13])**.** *Let $\mathcal{R} = (\Omega, \mathcal{F}, \sigma, \mathbb{P}, X, \mathcal{L})$ be a separable strongly measurable random linear system with ergodic base. Assume that $\log^+ \|\mathcal{L}(\omega)\| \in$*

$L^1(\Omega, \mathcal{F}, \mathbb{P})$ *and that* $\mathcal{R}$ *is quasi-compact. Then, there exists* $1 \le l \le \infty$, *numbers* $\lambda^* = \lambda_1 > \cdots > \lambda_l > \kappa^*$ *(or in the case* $l = \infty$, $\lambda_1 > \lambda_2 > \ldots > \kappa^*$; $\lim_{n\to\infty} \lambda_n = \kappa^*$*), the exceptional Lyapunov exponents of* $\mathcal{R}$, *multiplicities* $m_1, \ldots, m_l$, *and a filtration* $X = V_1(\omega) \supset \ldots \supset V_l(\omega) \supset V_{l+1}(\omega)$ *of finite-codimensional subspaces (in the case* $l = \infty$ *we have* $V_1(\omega) \supset V_2(\omega) \supset \ldots$*) defined on a full* $\mathbb{P}$*-measure,* σ*-invariant subset of* Ω *satisfying:*

1. *For every* $1 \le j \le l$, $\mathcal{L}_\omega V_j(\omega) \subseteq V_j(\sigma\omega)$, *and the codimension of* $V_{j+1}(\omega)$ *in* $V_j(\omega)$ *is* m_j. *Furthermore,* $\mathcal{L}_\omega V_{l+1}(\omega) \subseteq V_{l+1}(\sigma\omega)$.

2. *For every* $1 \le j \le l$ *and* $v \in V_j(\omega) \setminus V_{j+1}(\omega)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} v\| = \lambda_j.$$

   For every $v \in V_{l+1}(\omega)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} v\| \le \kappa^*.$$

Remark 2.9. *Combining the result of Lian and Lu [34] with Lemma B.15 and the proof of [13], we obtain that the spaces* $V_j(\omega)$ *forming the filtration given by Theorem 2.8 depend measurably on* ω.

The main result of this section is the following.

Theorem 2.10 (Semi-invertible operator Oseledets theorem)**.**
Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ *be a separable strongly measurable random linear system with ergodic invertible base. Assume that* $\log^+ \|\mathcal{L}(\omega)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ *and that* $\mathcal{R}$ *is quasi-compact. Let* $\lambda^* = \lambda_1 > \cdots > \lambda_l > \kappa^*$ *be the exceptional Lyapunov exponents of* $\mathcal{R}$, *and* $m_1, \ldots, m_l \in \mathbb{N}$ *the corresponding multiplicities (or in the case* $l = \infty$, $\lambda_1 > \lambda_2 > \ldots$ *with* $m_1, m_2, \ldots$ *the multiplicities).*

Then, up to $\mathbb{P}$*-null sets, there exists a unique, measurable, equivariant splitting of* X *into closed subspaces,* $X = V(\omega) \oplus \bigoplus_{j=1}^{l} Y_j(\omega)$, *where possibly* $V(\omega)$ *is infinite dimensional and* $\dim Y_j(\omega) = m_j$. *Furthermore, for every* $y \in Y_j(\omega) \setminus \{0\}$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} y\| = \lambda_j$, *for every* $v \in V(\omega)$, $\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} v\| \le \kappa^*$ *and the norms of the projections associated to the splitting are tempered with respect to* σ *(where a function* $f : \Omega \to \mathbb{R}$ *is called tempered if for* $\mathbb{P}$*-almost every* ω, $\lim_{n\to\pm\infty} \frac{1}{n} \log |f(\sigma^n\omega)| = 0$*).*

The proof of 2.10 occupies the rest of the section. First, we present Lemma 2.11, that allows us to choose complementary spaces in the filtration of Theorem 2.8. Then, Lemma 2.12 provides an inductive step that establishes the proof of Theorem 2.10.

Lemma 2.11 (Existence of a good complement)**.** *Let the filtration* $V_1(\omega) \supset \ldots \supset V_{l+1}(\omega)$ *be as in Theorem 2.8. Then, for every* $1 \le j \le l$, *there exist* m_j *dimensional spaces* $U_j(\omega)$ *such that the following conditions hold.*

1. *For $\mathbb{P}$-almost every $\omega \in \Omega$, $V_{j+1}(\omega) \oplus U_j(\omega) = V_j(\omega)$.*

2. *The map $\omega \mapsto U_j(\omega)$ is $(\mathcal{F}, \mathcal{B}_{\mathcal{G}}(X))$ measurable.*

   For $j = 1$, let $U_{<j}(\omega) = \{0\}$, and for $1 < j \leq l$, let $U_{<j}(\omega) = \bigoplus_{i=1}^{j-1} U_i(\omega)$. Then,

3. $\|\Pi_{U_j \| V_{j+1} \oplus U_{<j}(\cdot)}\|, \|\Pi_{V_{j+1} \| U_j \oplus U_{<j}(\cdot)}\| \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

Proof of Lemma 2.11. We proceed by induction on j. Fix some $1 \leq j \leq l$. If $j > 1$, assume the statement has been obtained for all $1 \leq j' < j$. Let $V(\omega) = V_j(\omega)$, $V_+(\omega) = V_{j+1}(\omega)$ and $k = m_j$. Also let $U_-(\omega) = \{0\}$ if $j = 1$ and $U_-(\omega) = \bigoplus_{j'<j} U_{j'}(\omega)$ if $j > 1$. Let $(x_i)_{i\in\mathbb{N}}$ be a countable dense subset of the unit sphere in X.

Let $\epsilon > 0$ be a constant to be fixed later in the proof. For $1 \leq l \leq k$, we claim that we can inductively pick measurable families of vectors $u_l(\omega) \in V_j(\omega)$ satisfying (a) $\|u_l(\omega)\| = 1$; (b) $d(u_l(\omega), W_l(\omega)) > 1-\epsilon$, where $W_l(\omega) = U_{<j}(\omega) \oplus V_+(\omega) \oplus \operatorname{span}(u_1(\omega), \ldots, u_{l-1}(\omega))$.

Assume $1 \leq l \leq k$ and that $u_1(\omega), \ldots, u_{l-1}(\omega)$ have already been constructed. Let $r_1(\omega) = \min\{i \in \mathbb{N} \colon d(x_i, W_l(\omega)) > 1 - \epsilon/2 \text{ and } d(x_i, V(\omega)) < \epsilon/2\}$. Then define subsequent terms of a sequence $(r_s(\omega))_{s\geq 1}$ by $r_s(\omega) = \min\{i \in \mathbb{N} \colon d(x_i, x_{r_{s-1}(\omega)}) < \epsilon/2^s; d(x_i, V(\omega)) < \epsilon/2^s\}$. Then $(x_{r_s(\omega)})_{s\geq 1}$ is a sequence of measurable functions pointwise convergent to a measurable function $u_l(s)$ satisfying the required properties.

To check the last condition, we let $\Pi_i = \Pi_{\operatorname{span}(u_i(\omega)) \| W_i(\omega)}$. It follows from the above that $\|\Pi_i\| \leq 1/(1-\epsilon)$. Also, $\Pi_{U_j \| V_{j+1} \oplus U_{<j}(\cdot)}$ and $\Pi_{V_{j+1} \| U_j \oplus U_{<j}(\cdot)}$ can be expressed as a finite sum of compositions of Π_i and I. The result follows. $\square$

As before, fix some $1 \leq j \leq l$ and let $V(\omega) = V_j(\omega), V_+(\omega) = V_{j+1}(\omega)$, $k = m_j = \operatorname{codim}(V_+, V)$, $\lambda = \lambda_j$ and $\mu = \lambda_{j+1}$.

Lemma 2.12. *Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ be a separable strongly measurable random linear system with ergodic invertible base. Let $\log^+ \|\mathcal{L}(\omega)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{R}$ be quasi-compact. Let $U(\omega)$ be a good complement of $V_+(\omega)$ in $V(\omega)$, as provided by Lemma 2.11. For $n \geq 0$, let $Y^{(n)}(\omega) = \mathcal{L}^{(n)}_{\sigma^{-n}\omega} U(\sigma^{-n}\omega)$. Then, for $\mathbb{P}$-almost every ω the following holds.*

1. *(Convergence) As $n \to \infty$, $Y^{(n)}(\omega)$ converges to a k-dimensional space $Y(\omega)$, which depends measurably on ω.*

2. *(Equivariant complement) $V_+(\omega) \oplus Y(\omega) = V(\omega)$. Hence, for all $y \in Y(\omega) \setminus \{0\}$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^{(n)}_\omega y\| = \lambda$. Furthermore, $\mathcal{L}_\omega Y(\omega) = Y(\sigma\omega)$.*

3. *(Uniqueness) $Y(\omega)$ is independent of the choice of $U(\omega)$.*

   Let $\tilde{Y}(\omega) = \bigoplus_{i\leq j} Y_i(\omega)$. Then,

4. *(Temperedness) The norms of projections $\Pi_1(\omega) := \Pi_{V_+ \| \tilde{Y}(\omega)}$ and $\Pi_2(\omega) := \Pi_{\tilde{Y} \| V_+(\omega)}$ are tempered with respect to σ.*

Before proceeding to the proof of Lemma 2.12, let us collect some facts that will be used in it. For $j=1$, let $U_-(\omega)=\{0\}$, and for $1<j\le l$, let $U_-(\omega)=\bigoplus_{i=1}^{j-1}U_i(\omega)$. Then, we have that $\Pi_{V\|U_-(\omega)}=\Pi_{V_+\|U\oplus U_-(\omega)}+\Pi_{U\|V_+\oplus U_-(\omega)}$. Also, by invariance of $V(\omega)$ under $\mathcal{L}_\omega$, we have that

$$\mathcal{L}_\omega\circ\Pi_{V\|U_-(\omega)}=(\Pi_{V_+\|U\oplus U_-(\sigma\omega)}+\Pi_{U\|V_+\oplus U_-(\sigma\omega)})\mathcal{L}_\omega(\Pi_{V_+\|U\oplus U_-(\omega)}+\Pi_{U\|V_+\oplus U_-(\omega)}).$$

Let

$$\begin{aligned}
\mathcal{L}_{00}(\omega)&=\Pi_{V_+\|U\oplus U_-(\sigma\omega)}\mathcal{L}_\omega\Pi_{V_+\|U\oplus U_-(\omega)},\\
\mathcal{L}_{01}(\omega)&=\Pi_{U\|V_+\oplus U_-(\sigma\omega)}\mathcal{L}_\omega\Pi_{V_+\|U\oplus U_-(\omega)},\\
\mathcal{L}_{10}(\omega)&=\Pi_{V_+\|U\oplus U_-(\sigma\omega)}\mathcal{L}_\omega\Pi_{U\|V_+\oplus U_-(\omega)},\\
\mathcal{L}_{11}(\omega)&=\Pi_{U\|V_+\oplus U_-(\sigma\omega)}\mathcal{L}_\omega\Pi_{U\|V_+\oplus U_-(\omega)}.
\end{aligned}$$

Note that by invariance of V_+, $\mathcal{L}_{01}(\omega)=0$, $\mathbb{P}$-almost surely. Therefore, $\mathcal{L}_\omega\circ\Pi_{V\|U_-(\omega)}=\mathcal{L}_{00}(\omega)+\mathcal{L}_{10}(\omega)+\mathcal{L}_{11}(\omega)$.

Let $\mathcal{L}_{00}^{(n)}(\omega)=\Pi_{V_+\|U\oplus U_-(\sigma^n\omega)}\mathcal{L}_\omega^{(n)}\Pi_{V_+\|U\oplus U_-(\omega)}$, and define operators $\mathcal{L}_{01}^{(n)}(\omega)$ and $\mathcal{L}_{11}^{(n)}(\omega)$ analogously. It is straightforward to verify the following identities.

$$\mathcal{L}_{00}^{(n)}(\omega)=\mathcal{L}_{00}(\sigma^{n-1}\omega)\dots\mathcal{L}_{00}(\omega),\tag{L_{00}}$$

$$\mathcal{L}_{11}^{(n)}(\omega)=\mathcal{L}_{11}(\sigma^{n-1}\omega)\dots\mathcal{L}_{11}(\omega).\tag{L_{11}}$$

By induction, we also have that

$$\mathcal{L}_{10}^{(n)}(\omega)=\sum_{i=0}^{n-1}\mathcal{L}_{00}^{(i)}(\sigma^{n-i}\omega)\mathcal{L}_{10}(\sigma^{n-i-1}\omega)\mathcal{L}_{11}^{(n-i-1)}(\omega).\tag{L_{10}}$$

Sublemma 2.13. *Under the assumptions of Lemma 2.12, the following statements hold.*

1. *For $\mathbb{P}$-almost every $\omega\in\Omega$,*

   $$\lim_{n\to\infty}\frac{1}{n}\log\|\mathcal{L}_{00}^{(n)}(\omega)\|\le\mu.$$

   Furthermore, for every $\epsilon>0$ and $\mathbb{P}$-almost every $\omega\in\Omega$, there exists $D_1(\omega)<\infty$ such that for every $i\ge 0$,

   $$\|\mathcal{L}_{00}^{(i)}(\sigma^{-i}\omega)\|\le D_1(\omega)e^{i(\mu+\epsilon)}.$$

   Applying this with ω replaced by $\sigma^n\omega$ we obtain

   $$\|\mathcal{L}_{00}^{(i)}(\sigma^{n-i}\omega)\|\le D_1(\sigma^n\omega)e^{i(\mu+\epsilon)}.$$

2. *For every $\epsilon>0$ and $\mathbb{P}$-almost every $\omega\in\Omega$, there exists $D_2(\omega)<\infty$ such that for every $n\in\mathbb{Z}$,*

   $$\|\mathcal{L}_{10}(\sigma^n\omega)\|\le D_2(\omega)e^{|n|\epsilon}.$$

3. For $\mathbb{P}$-almost every $\omega \in \Omega$, and every $u \in U(\omega) \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_{11}^{(n)}(\omega) u\| = \lambda.$$

In particular, $\mathcal{L}_{11}(\omega)|_{U(\omega)} : U(\omega) \to U(\sigma\omega)$ is invertible for $\mathbb{P}$-almost every ω. Furthermore, for every $\epsilon > 0$ and $\mathbb{P}$-almost every $\omega \in \Omega$, there exists $C(\omega) < \infty$ such that for every $n \geq 0$, and every $u \in U(\sigma^{-n}\omega)$ satisfying $\|u\| = 1$

$$\|\mathcal{L}_{11}^{(n)}(\sigma^{-n}\omega) u\| \geq C(\omega) e^{n(\lambda-\epsilon)}.$$

Proof.

Proof of (1)**.**

From the definition, $\|\mathcal{L}_{00}^{(n)}(\omega)\| \leq \|\Pi_{V_+\|U\oplus U_-(\sigma^n\omega)}\| \|\mathcal{L}_{\omega}^{(n)}|_{V_+(\omega)}\| \|\Pi_{V_+\|U\oplus U_-(\omega)}\|$. By Theorem 2.8, we have that $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_{\omega}^{(n)}|_{V_+(\omega)}\| = \mu$. Using that $\Pi_{V_+\|U\oplus U_-(\omega)}$ is tempered with respect to σ, which follows from Lemma 2.11(3), we get that for $\mathbb{P}$-almost every ω,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_{00}^{(n)}(\omega)\| \leq \mu.$$

The second claim follows exactly like Claim C in the predecessor paper, [18].

Proof of (2)**.**

$\|\mathcal{L}_{10}(\sigma^n\omega)\| \leq \|\Pi_{V_+\|U\oplus U_-(\sigma^{n+1}\omega)}\| \|\mathcal{L}_{\sigma^n\omega}\| \|\Pi_{U\|V_+\oplus U_-(\sigma^n\omega)}\|$. Since $\log^+ \|\mathcal{L}_\omega\|$ is integrable with respect to $\mathbb{P}$, using the Birkhoff ergodic theorem, one sees that $\lim_{n\to\pm\infty} \frac{1}{|n|} \log \|\mathcal{L}_{\sigma^n\omega}\| = 0$ for $\mathbb{P}$-almost every ω. Using again that $\Pi_{U\|V_+\oplus U_-(\omega)}$ and $\Pi_{V_+\|U\oplus U_-(\omega)}$ are tempered with respect to σ, we get that for $\mathbb{P}$-almost every ω,

$$\lim_{n\to\pm\infty} \frac{1}{|n|} \log \|\mathcal{L}_{10}(\sigma^n\omega)\| = 0.$$

Thus, the claim follows.

Proof of (3)**.**

We use the bases $\{u_1(\omega), \dots, u_k(\omega)\}$ and $\{u_1(\sigma\omega), \dots, u_k(\sigma\omega)\}$ constructed in the proof of Lemma 2.11 to express $\mathcal{L}_{11}|_{U(\omega)} : U(\omega) \to U(\sigma\omega)$ in matrix form. Recall that the norms of $\{u_1(\omega), \dots, u_k(\omega)\}$ are bounded functions of ω. Also from the proof of Lemma 2.11(1), we have that

$$\Big\| \sum_{i=1}^k a_i u_i(\omega) \Big\| \geq \frac{3^{-k}(1-5\epsilon)}{2} \max_{1\leq i\leq k} |a_i|.$$

Condition (3) of Lemma 2.11 implies that the multiplicative ergodic theorem of Oseledets [37] applies to $\mathcal{L}_{11}$. Hence, convergence of $\frac{1}{n}\log\|\mathcal{L}_{11}^{(n)}(\omega)u\|$ follows from Equation (L_{11}). Call this limit Λ. Thus, for every $\epsilon > 0$ there exists a constant $D(\omega, u) < \infty$ such that $\|\mathcal{L}_{11}^{(n)}(\omega)u\| \leq D(\omega, u)e^{n(\Lambda+\epsilon)}$.

On the one hand, by definition and invariance of $V(\omega)$, we have that

$$\begin{aligned}\mathcal{L}_{11}^{(n)}(\omega) &= \Pi_{U\|V_+\oplus U_-(\sigma^n\omega)}\mathcal{L}_\omega^{(n)}\Pi_{U\|V_+\oplus U_-(\omega)} \\ &= \Pi_{U\|V_+\oplus U_-(\sigma^n\omega)}\mathcal{L}_\omega^{(n)}|_{V(\omega)}\Pi_{U\|V_+\oplus U_-(\omega)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Lambda &= \lim_{n\to\infty}\frac{1}{n}\log^+\|\mathcal{L}_{11}^{(n)}(\omega)u\| \\ &\leq \lim_{n\to\infty}\frac{1}{n}\big(\log^+\|\Pi_{U\|V_+\oplus U_-(\sigma^n\omega)}\| + \log^+\|\mathcal{L}_\omega^{(n)}|_{V(\omega)}\| + \log^+\|\Pi_{U\|V_+\oplus U_-(\omega)}\|\big) = \lambda.\end{aligned}$$

The last equality follows from Theorem 2.8 and temperedness of $\Pi_{U\|V_+\oplus U_-(\omega)}$ with respect to σ.

On the other hand, for any $u \in U(\omega)\setminus\{0\}$,

$$\begin{aligned}\|\mathcal{L}_\omega^{(n)}u\| &= \|\mathcal{L}_{10}^{(n)}(\omega)u + \mathcal{L}_{11}^{(n)}(\omega)u\| \leq \|\mathcal{L}_{10}^{(n)}(\omega)u\| + \|\mathcal{L}_{11}^{(n)}(\omega)u\| \\ &= \Big\|\sum_{i=0}^{n-1}\mathcal{L}_{00}^{(i)}(\sigma^{n-i}\omega)\mathcal{L}_{10}(\sigma^{n-i-1}\omega)\mathcal{L}_{11}^{(n-i-1)}(\omega)u\Big\| + \|\mathcal{L}_{11}^{(n)}(\omega)u\| \\ &\leq \sum_{i=0}^{n-1}\big\|\mathcal{L}_{00}^{(i)}(\sigma^{n-i}\omega)\big\|\big\|\mathcal{L}_{10}(\sigma^{n-i-1}\omega)\big\|\big\|\mathcal{L}_{11}^{(n-i-1)}(\omega)u\big\| + \|\mathcal{L}_{11}^{(n)}(\omega)u\|.\end{aligned}$$

Let us estimate the first sum. In view of parts (1) and (2), we have that

$$\begin{aligned}&\sum_{i=0}^{n-1}\big\|\mathcal{L}_{00}^{(i)}(\sigma^{n-i}\omega)\big\|\big\|\mathcal{L}_{10}(\sigma^{n-i-1}\omega)\big\|\big\|\mathcal{L}_{11}^{(n-i-1)}(\omega)u\big\| \\ &\qquad\leq D_1(\sigma^n\omega)D_2(\omega)D(\omega,u)\sum_{i=0}^{n-1}e^{i(\mu+\epsilon)}e^{(n-i-1)\epsilon}e^{(\Lambda+\epsilon)(n-i-1)} \\ &\qquad\leq D_1(\sigma^n\omega)D_2(\omega)D(\omega,u)\sum_{i=0}^{n-1}e^{(n-1)(\max(\Lambda,\mu)+2\epsilon)}.\end{aligned}$$

Let $M > 0$ be such that $\mathbb{P}(D_1(\omega) < M) > 0$. Then, by ergodicity of σ, for $\mathbb{P}$-almost every ω, there are infinitely many n such that $D_1(\sigma^n\omega) < M$. For every such n we have that

$$\|\mathcal{L}_{10}^{(n)}(\omega)u\| \leq MD_2(\omega)D(\omega,u)ne^{(n-1)(\max(\Lambda,\mu)+2\epsilon)}.$$

Hence,

$$\liminf_{n\to\infty}\frac{1}{n}\log\|\mathcal{L}_{10}^{(n)}(\omega)u\| \leq \max(\Lambda,\mu).$$

By definition of Λ, we also know that

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_{11}^{(n)}(\omega)u\| = \Lambda.$$

Therefore,

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_{\omega}^{(n)}u\| \leq \max(\Lambda, \mu).$$

Recalling that $\mu < \lambda$, we get that $\lambda \leq \Lambda$. Combining with the argument above, we conclude that $\lambda = \Lambda$ as claimed.

The almost-everywhere invertibility of $\mathcal{L}_{11}(\omega)|_{U(\omega)}$ follows immediately.

The last statement follows as in the case of matrices in the predecessor paper [18, Lemma 8.3]. □

The following lemma will be useful in the proof of Lemma 2.12(3).

Lemma 2.14. *Assume $Y'(\omega)$ is a measurable equivariant complement of $V_+(\omega)$ in $V(\omega)$. From Theorem 2.8, for every $y' \in Y'(\omega)\backslash\{0\}$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_{\omega}^{(n)}y'\| = \lambda$. Furthermore, for every $\epsilon > 0$ and $\mathbb{P}$-almost every $\omega \in \Omega$, there exists $C'(\omega) > 0$ such that for every $n \geq 0$, and every $y' \in Y'(\omega)$ satisfying $\|y'\| = 1$,*

$$\|\mathcal{L}_{\omega}^{(n)}y'\| \geq C'(\omega)e^{n(\lambda-\epsilon)}.$$

Proof. The proof follows from the corresponding statement for matrices due to Barreira and Silva [5], as in [19, Lemma 19], with the only difference being the choice of suitable bases for $Y'(\omega)$, which in our setting may be done in a measurable way similar to that used for the proof of Lemma 2.11. □

Proof of Lemma 2.12.
Proof of (1)**.**
The proof follows closely that presented in [18, Theorem 4.1], for the case of matrices. First, we define

$$g_n(\omega) = \max_{u\in U(\omega), \|u\|=1} \frac{\|\mathcal{L}_{10}^{(n)}(\sigma^{-n}\omega)u\|}{\|\mathcal{L}_{11}^{(n)}(\sigma^{-n}\omega)u\|}.$$

Then, using the characterizations from (L_{11}) and (L_{10}) together with invertibility of $\mathcal{L}_{11}(\omega)$, we have that

$$g_n(\omega) \leq \sum_{i=0}^{n-1} \frac{\max_{u\in U(\sigma^{-i-1}\omega), \|u\|=1} \|\mathcal{L}_{00}^{i}(\sigma^{-i}\omega)\mathcal{L}_{10}(\sigma^{-i-1}\omega)u\|}{\min_{u\in U(\sigma^{-i-1}\omega), \|u\|=1} \|\mathcal{L}_{11}^{i+1}(\sigma^{-i-1}\omega)u\|}.$$

Let $\epsilon < \frac{\lambda-\mu}{4}$. Using Sublemma 2.13, we have that for $\mathbb{P}$-almost every ω, there is a constant $C'(\omega) < \infty$ such that

$$g_n(\omega) \leq C'(\omega) \sum_{i=0}^{n-1} e^{(\mu-\lambda+3\epsilon)i}.$$

Hence, $M(\omega) := \sup_{n\in\mathbb{N}} g_n(\omega) < \infty$ for $\mathbb{P}$-almost every ω.

Next, we show that the sequence of subspaces $Y^{(n)}(\omega) = \mathcal{L}^{(n)}_{\sigma^{-n}\omega} U(\sigma^{-n}\omega)$ forms a Cauchy sequence in $\mathcal{G}_k(X)$.

Let $m > n$. By homogeneity of the norm, the expression

$$\max\{\sup_{x\in Y^{(n)}(\omega)\cap B} d(x, Y^{(m)}(\omega)\cap B), \sup_{x\in Y^{(m)}(\omega)\cap B} d(x, Y^{(n)}(\omega)\cap B)\}$$

coincides with

$$\max\{\sup_{x\in Y^{(n)}(\omega)\cap S(X)} d(x, Y^{(m)}(\omega)\cap B), \sup_{x\in Y^{(m)}(\omega)\cap S(X)} d(x, Y^{(n)}(\omega)\cap B)\},$$

where $S(X)$ is the unit sphere in X.

First, let $x \in Y^{(n)}(\omega)\cap S(X)$. Then $x = \mathcal{L}^{(n)}_{\sigma^{-n}\omega} u$, with $u \in U(\sigma^{-n}\omega)$. Since $\mathcal{L}^{(m-n)}_{11}(\sigma^{-m}\omega)$ is invertible for $\mathbb{P}$-almost every ω, there exists $u' \in U(\sigma^{-m}\omega)$ such that $\mathcal{L}^{(m-n)}_{\sigma^{-m}\omega} u' = u+v$, for some $v \in V_+(\sigma^{-n}\omega)$. Since $v = \mathcal{L}^{(m-n)}_{10}(\sigma^{-m}\omega)u'$ and $u = \mathcal{L}^{(m-n)}_{11}(\sigma^{-m}\omega)u'$, we have that $\|v\| \le M(\sigma^{-n}\omega)\|u\|$.

Let $y = \mathcal{L}^{(m)}_{\sigma^{-m}\omega} u' \in Y^{(m)}(\omega)$. Then, $y = x + \mathcal{L}^{(n)}_{\sigma^{-n}\omega}(v)$. Also, $\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega}(v)\| \le D_1(\omega)e^{n(\mu+\epsilon)}M(\sigma^{-n}\omega)\|u\|$, with $D_1(\omega)$ as in Sublemma 2.13(1). Using Sublemma 2.13(3), we also have that

$$1 = \|x\| = \|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} u\| \ge C(\omega)e^{n(\lambda-\epsilon)}\|u\|.$$

Letting $K(\omega) = \frac{D_1(\omega)}{C(\omega)}$ and $\alpha = \lambda-\mu-2\epsilon$, we get that $d(x, Y^{(m)}(\omega)) \le \|y-x\| = \|\mathcal{L}^{(n)}_{\sigma^{-n}\omega}(v)\| \le K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}$. By the triangle inequality, $d(x, Y^{(m)}(\omega)\cap B) \le 2d(x, Y^{(m)}(\omega))$. Therefore,

$$d(x, Y^{(m)}(\omega)\cap B) \le 2K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}. \tag{5}$$

Second, let $y \in Y^{(m)}(\omega)\cap S(X)$. Then, $y = \mathcal{L}^{(m)}_{\sigma^{-m}\omega} u'$ for some $u' \in U(\sigma^{-m}\omega)$. Let $\mathcal{L}^{(m-n)}_{\sigma^{-m}\omega} u' = u + v$, with $u \in U(\sigma^{-n}\omega)$ and $v \in V_+(\sigma^{-n}\omega)$.

Using once again the definition of $M(\omega)$ at the beginning of the proof, we have that $\|v\| \le M(\sigma^{-n}\omega)\|u\|$, combined with Sublemma 2.13 we obtain that

$$\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} v\| \le K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} u\|.$$

On the other hand, since $\mathcal{L}^{(n)}_{\sigma^{-n}\omega}(u+v) = y$, we have

$$\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} u\| \le \|y\| + \|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} v\| \le 1 + K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} u\|.$$

Therefore, whenever $K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n} < 1$, we have

$$\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega} u\| \le \frac{1}{1-K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}}.$$

Since $\mathcal{L}^{(n)}_{\sigma^{-n}\omega}u \in Y^{(n)}(\omega)$, we have

$$d(y, Y^{(n)}(\omega)) \leq \|y - \mathcal{L}^{(n)}_{\sigma^{-n}\omega}u\| = \|\mathcal{L}^{(n)}_{\sigma^{-n}\omega}v\| \leq \frac{K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}}{1 - K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}}.$$

As before, we use the triangle inequality to conclude that

$$d(y, Y^{(n)}(\omega) \cap B) \leq \frac{2K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}}{1 - K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}}. \tag{6}$$

Combining (5) and (6), we get that

$$d(Y^{(n)}(\omega), Y^{(m)}(\omega)) \leq \frac{2K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}}{1 - K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}}.$$

Therefore,

$$d(Y^{(m')}(\omega), Y^{(m)}(\omega)) \leq \frac{4K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}}{1 - K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n}} \text{ for every } m, m' > n,$$

provided $K(\omega)M(\sigma^{-n}\omega)e^{-\alpha n} < 1$.

Since $M(\omega) < \infty$ for $\mathbb{P}$-almost every ω, there exists an $A > 0$ such that $\mathbb{P}(M(\omega) < A) > 0$. By ergodicity of σ, for $\mathbb{P}$-almost every ω, there exist arbitrarily large values of n for which $M(\sigma^n\omega) < A$, proving that $Y^{(m)}(\omega)$ is a Cauchy sequence. Therefore, it is convergent. Let us call its limit $Y(\omega)$.

Measurability of $Y^{(n)}(\omega)$ comes from Corollary B.14. Hence, measurability of $Y(\omega)$ follows from the fact that the pointwise limit of Borel-measurable functions from a measurable space to a metric space is Borel-measurable.

Proof of (2)**.**
By closedness of $\mathcal{G}_k(X)$, we know that $Y(\omega) \in \mathcal{G}_k(X)$. Since $Y(\omega)$ is the limit of subspaces of $V(\omega)$ and $V(\omega)$ is closed, it follows that $Y(\omega) \subset V(\omega)$. We also have that $V_+(\omega)$ is a k-codimensional subspace of $V(\omega)$. Hence, to show that $Y(\omega) \oplus V_+(\omega) = V(\omega)$, we must show that $Y(\omega) \cap V_+(\omega) = \{0\}$. Let $x \in Y^{(n)}(\omega)$, with $\|x\| = 1$. Then, $x = \mathcal{L}^{(n)}_{\sigma^{-n}\omega}u'$ for some $u' \in U(\sigma^{-n}\omega)$. Writing $x = u + v$, with $u \in U(\omega)$, $v \in V_+(\omega)$, we have that $\|v\| \leq M(\omega)\|u\|$. Thus, $1 = \|x\| \leq \|u\|(1+M(\omega))$, which yields $\|u\| \geq \frac{1}{1+M(\omega)}$. Since this holds for every n, every $x \in Y(\omega)$ with $\|x\| = 1$ when decomposed as $x = u + v$ with $u \in U(\omega)$, $v \in V_+(\omega)$, also satisfies $\|u\| \geq \frac{1}{1+M(\omega)}$. Therefore $Y(\omega) \cap V_+(\omega) = \{0\}$.

The second statement follows directly from Theorem 2.8.

To show invariance, we observe that $\mathcal{L}_\omega Y^{(n)}(\omega) = Y^{(n+1)}(\sigma\omega)$. Also, by the previous argument combined with Sublemma 2.13 , we have that for every $n \geq 0$, $(\mathcal{L}_\omega, Y^{(n)}(\omega)) \in G(N, k, 0)$ for some $N > 0$ (see notation of Lemma B.13), and $(\mathcal{L}_\omega, Y(\omega)) \in G(N, k, 0)$ as well. Hence, by Lemma B.13, $\lim_{n\to\infty} \mathcal{L}_\omega Y^{(n)}(\omega) = \mathcal{L}_\omega Y(\omega)$. Thus, $\mathcal{L}_\omega(Y(\omega)) = Y(\sigma\omega)$.

Proof of (3)**.**
Let $Y_-(\omega) = \bigoplus_{i<j} Y_i(\omega)$. Assume $Y'(\omega)$ is a measurable equivariant complement of $V_+(\omega)$ in $V(\omega)$. We will show that $Y'(\omega) = Y(\omega)$ for $\mathbb{P}$-almost every

$\omega \in \Omega$. Let $R(\omega) = \Pi_{V_+ \| Y \oplus Y_-(\omega)}$. Remark B.19 and Lemma B.20 imply that R is $(\mathcal{F}, \mathcal{S})$ measurable. Let $h(\omega) = \|R(\omega)|_{Y'(\omega)}\|$. Then, h is non-negative and, in view of Lemma B.16, h is $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$ measurable.

We claim that for $\mathbb{P}$-almost every ω, $\lim_{n\to\infty} h(\sigma^n \omega) = 0$. We will show this in the next paragraph. Given this, we can finish the proof as follows. Let $E_i = \{\omega : h(\omega) \leq \frac{1}{i}\}$. The claim implies that $\mathbb{P}$-almost every ω, $\sigma^n \omega \in E_i$ for sufficiently large n. Hence, by the Poincaré recurrence theorem, $\mathbb{P}(\Omega \setminus E_i) = 0$ for all $i \in \mathbb{N}$, and therefore, since h is non-negative, $h(\omega) = 0$ for $\mathbb{P}$-almost every ω. This implies that $Y'(\omega) \subset Y(\omega) \oplus Y_-(\omega)$. On the other hand, $Y(\omega), Y'(\omega) \subset V(\omega)$, and by part (2), $V(\omega) \cap Y_-(\omega) = \{0\}$. So $Y'(\omega) \subset Y(\omega)$. Since $Y(\omega)$ and $Y'(\omega)$ have the same dimension, $Y'(\omega) = Y(\omega)$ as claimed.

The proof of $\lim_{n\to\infty} h(\sigma^n\omega) = 0$ proceeds as in [19, §3.2]. Let $y' \in Y'(\omega) \setminus \{0\}$. Since $R(\omega)y' \in V_+(\omega)$, Sublemma 2.13(1) implies that for every $\epsilon > 0$, there exists some $D'(\omega) < \infty$ such that $\|\mathcal{L}_\omega^{(n)} R(\omega) y'\| \leq D'(\omega) e^{n(\mu+\epsilon)} \|y'\|$. By Remark 2.14, for every $\epsilon > 0$, there exists some $C'(\omega) > 0$ such that $\|\mathcal{L}_\omega^{(n)} y'\| \geq C'(\omega) e^{n(\lambda-\epsilon)} \|y'\|$. Let $\epsilon < \frac{\lambda-\mu}{4}$. Then, $\frac{\|\mathcal{L}_\omega^{(n)} R(\omega) y'\|}{\|\mathcal{L}_\omega^{(n)} y'\|} \leq \frac{D'(\omega)}{C'(\omega)} e^{-n(\lambda-\mu-2\epsilon)}$ for every $n \geq 0$. Consider the closed sets

$$D_N = \{y' \in Y'(\omega) : \|\mathcal{L}_\omega^{(n)} R(\omega) y'\| \leq N e^{-n(\lambda-\mu-2\epsilon)} \|\mathcal{L}_\omega^{(n)} y'\| \text{ for all } n \in \mathbb{N}\}.$$

Since $\bigcup_{N\in\mathbb{N}} D_N = Y'(\omega)$, the Baire category principle implies that there exists $N \in \mathbb{N}$, $y' \in Y'(\omega)$ and $\delta > 0$ such that $\overline{B_\delta(y')} \cap Y'(\omega) \subset D_N$. By linearity of $\mathcal{L}_\omega$, $\overline{B_1(\frac{y'}{\delta})} \cap Y'(\omega) \subset D_N$.

Let $x \in Y'(\omega)$ with $\|x\| = 1$. Then, $\|\mathcal{L}_\omega^{(n)} R(\omega)(\frac{y'}{\delta}+x)\| \leq N e^{-n(\lambda-\mu-2\epsilon)} \|\mathcal{L}_\omega^{(n)}(\frac{y'}{\delta}+x)\|$ and $\|\mathcal{L}_\omega^{(n)} R(\omega)(\frac{y'}{\delta})\| \leq N e^{-n(\lambda-\mu-2\epsilon)} \|\mathcal{L}_\omega^{(n)}(\frac{y'}{\delta})\|$.

By invariance of $V_+(\omega), Y(\omega)$ and $Y_-(\omega)$, we see that $R(\sigma^n\omega)\mathcal{L}_\omega^{(n)} = \mathcal{L}_\omega^{(n)} R(\omega)$. Hence,

$$\begin{aligned}
\|R(\sigma^n\omega)\mathcal{L}_\omega^{(n)}(x)\| &\leq N e^{-n(\lambda-\mu-2\epsilon)} \Big(\|\mathcal{L}_\omega^{(n)}(\frac{y'}{\delta})\| + \|\mathcal{L}_\omega^{(n)}(\frac{y'}{\delta} + x)\| \Big) \\
&\leq 2N e^{-n(\lambda-\mu-2\epsilon)} C''(\omega) e^{n(\lambda+\epsilon)} \Big(\|\frac{y'}{\delta}\| + 1 \Big),
\end{aligned}$$

where the existence of such $C''(\omega) < \infty$ is guaranteed by Theorem 2.8. Furthermore, using invariance of $Y'(\omega)$, we get that $\mathcal{L}_\omega^{(n)}(Y'(\omega)) = Y'(\sigma^n\omega)$. Thus,

$$\begin{aligned}
h(\sigma^n\omega) &\leq \frac{\sup_{x\in Y'(\omega)\cap S^1(X)} \|R(\sigma^n\omega)\mathcal{L}_\omega^{(n)}(x)\|}{\inf_{x\in Y'(\omega)\cap S^1(X)} \|\mathcal{L}_\omega^{(n)}(x)\|} \\
&\leq \frac{2NC''(\omega)(\|\frac{y'}{\delta}\| + 1)}{C'(\omega)} e^{-n(\lambda-\mu-4\epsilon)},
\end{aligned}$$

where the last inequality follows from the existence of constants $C'(\omega)$ and $C''(\omega)$ as before. By the choice of ϵ, we get that $\lim_{n\to\infty} h(\sigma^n\omega) = 0$ as claimed.

Proof of (4)**.**
We want to show that for $i = 1, 2$ and $\mathbb{P}$-almost every $\omega \in \Omega$, the following holds

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Pi_i(\sigma^n\omega)\| = 0.$$

Since the maps $\Pi_i(\omega)$ are projections to non-trivial subspaces, it follows that all the norms involved are at least 1. We will show upper bounds. In view of Lemma C.2, it suffices to show

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_i(\sigma^{-n}\omega)\| = 0.$$

It suffices to show that for each j, $\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{Y_j||V_{j+1}(\sigma^{-n}\omega)}\| = 0$, where the map is defined on $V_j(\sigma^{-n}\omega)$.

Fix j, and suppose $1/n \log \|\Pi_{Y_j||V_{j+1}(\sigma^{-n}\omega)}\| \not\to 0$ as $n \to \infty$. Then there exists a sequence $n_1 < n_2 < \ldots$ such that $\log \|\Pi_{Y_j||V_{j+1}(\sigma^{-n_i}\omega)}\| > \epsilon n_i$.

Pick $\delta < \min(\epsilon/2, (\lambda_j - \lambda_{j+1})/2)$. Then, for $\mathbb{P}$-almost every ω, there exist $C(\omega)$, $D(\omega)$ and $E(\omega)$ such that

$$\begin{aligned}
\|\mathcal{L}^{(n)}(\sigma^{-n}\omega)|_{V_{j+1}}\| &< C(\omega)e^{(\lambda_{j+1}+\delta)n} \text{ for all } n;\\
\|\mathcal{L}^{(n)}(\sigma^{-n}\omega)(y)\| &> D(\omega)e^{(\lambda_j-\delta)n} \text{ for all } n \text{ and } y \in B \cap Y_j(\sigma^{-n}\omega);\\
\|\mathcal{L}^{(n)}(\sigma^{-n}\omega)|_{V_j}\| &< E(\omega)e^{(\lambda_j+\delta)n} \text{ for all } n.
\end{aligned}$$

Now by hypothesis there exists a sequence $y_{n_i} + v_{n_i} \in Y_j(\sigma^{-n_i}\omega) \oplus V_{j+1}(\sigma^{-n_i}\omega)$ with $\|y_{n_i}\| \approx 1$, $\|v_{n_i}\| \approx 1$, but $\|y_{n_i} + v_{n_i}\| < e^{-\epsilon n_i}$. Then, we have

$$\begin{aligned}
\|\mathcal{L}^{(n_i)}(\sigma^{-n_i}\omega)(y_{n_i})\| &> D(\omega)e^{(\lambda_j-\delta)n_i};\\
\|\mathcal{L}^{(n_i)}(\sigma^{-n_i}\omega)(v_{n_i})\| &< C(\omega)e^{(\lambda_{j+1}+\delta)n_i};\\
\|\mathcal{L}^{(n_i)}(\sigma^{-n_i}\omega)(y_{n_i} + v_{n_i})\| &< E(\omega)e^{(\lambda_j+\delta)n_i}\|y_{n_i} + v_{n_i}\|\\
&< E(\omega)e^{(\lambda_j+\delta-\epsilon)n_i}.
\end{aligned}$$

The triangle inequality gives

$$D(\omega)e^{(\lambda_j-\delta)n_i} < E(\omega)e^{(\lambda_j+\delta-\epsilon)n_i} + C(\omega)e^{(\lambda_{j+1}+\delta)n_i}.$$

Since $\lambda_j - \delta > \max(\lambda_j + \delta - \epsilon, \lambda_{j+1} + \delta)$ this gives a contradiction for sufficiently large n_i. Hence, $\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_2(\sigma^{-n}\omega)\| = 0$ for $\mathbb{P}$-almost every ω, as claimed.

Using that $\Pi_1 + \Pi_2 = \mathrm{Id}$, the corresponding statement for Π_1 follows immediately. □

3 Oseledets splittings for random Piecewise Expanding maps

In this section, we present an application of the semi-invertible operator Oseledets theorem, Theorem 2.10, to the setting of random piecewise expanding maps. It is worked out in considerable generality in the one-dimensional setting and in a special case in higher-dimensions, based on results of Cowieson. The main result of this section is Theorem 3.19.

We first discuss the class of piecewise expanding maps and random piecewise expanding dynamical systems in §3.1. Then, fractional Sobolev spaces and some of their relevant properties are briefly reviewed in §3.2. In §3.3, we recall the definition of the transfer operator of a piecewise expanding map acting on a fractional Sobolev space. Strong measurability is established in the one-dimensional case. Quasi-compactness is proved in §3.5.

Convention

Throughout this section, $C_\#$ will denote various constants that are allowed to depend only on parameters d, p, t and α, as well as on a C^∞ compactly supported function $\eta : \mathbb{R}^d \to [0, 1]$, that is chosen and fixed depending only on the dimension d, as appears in Thomine [43] and Baladi and Gouëzel [2].

3.1 Random Piecewise Expanding Dynamical Systems

Definition 3.1. *A map T is called a* piecewise expanding $C^{1+\alpha}$ map *of a compact region $X_0 \subset \mathbb{R}^d$ if:*

- *There is a finite (ordered) collection of disjoint subsets of X_0, $O^1, \ldots, O^I$, each connected and open in $\mathbb{R}^d$, whose boundaries are unions of finitely many compact C^1 hypersurfaces with boundary, and whose union agrees with X_0 up to a set of Lebesgue measure 0;*
- *For each $1 \le i \le I$, $T|_{O_i}$ agrees with a $C^{1+\alpha}$ map T_i defined on a neighbourhood of $\overline{O_i}$ such that T_i is a diffeomorphism onto its image.*
- *There exists $\mu > 1$ such that for all $x \in \overline{O_i}$, $\|DT_i(x)(v)\| \ge \mu\|v\|$ for all $v \in \mathbb{R}^d$.*

We define $b^T := I$ to be the number of branches of T. The collection $\{O^1, \ldots, O^I\}$ is called the *branch partition* of T. The collection of $C^{1+\alpha}$ expanding maps of X_0 will be denoted $\mathrm{PE}^{1+\alpha}(X_0)$. The collection of $C^{1+\alpha}$ expanding maps with a particular branch partition $\mathcal{P}$ will be denoted by $\mathrm{PE}^{1+\alpha}(X_0; \mathcal{P})$. In the special case where $d = 1$ and $X_0 = [0, 1]$, we denote the collection of maps satisfying the above conditions $\mathrm{LY}^{1+\alpha}$. In this case, the elements of $\mathcal{P}$ are intervals.

Finally, we define a metric d_{PE} on $\mathrm{PE}^{1+\alpha}(X_0)$ as follows. Let $S, T \in \mathrm{PE}^{1+\alpha}(X_0)$. Let the branches for S be $(O_i^S)_{i=1}^{b^S}$ and for T be $(O_i^T)_{i=1}^{b^T}$ (recall that a piecewise expanding map is assumed to consist of an *ordered* collection of domains and maps). If $b^T \neq b^S$, or $O_i^S \cap O_i^T = \emptyset$ for some i, we define $d_{\mathrm{PE}}(S, T) = 1$. Otherwise we define

$$d_{\mathrm{PE}}(S,T) = \max_i \|(S_i - T_i)|_{O_i^S \cap O_i^T}\|_{C^{1+\alpha}} + \max_i \Big| \|S_i\|_{C^{1+\alpha}} - \|T_i\|_{C^{1+\alpha}} \Big| + \max_i d_H(O_i^S, O_i^T),$$

where d_H denotes Hausdorff distance. In the one-dimensional case we call the metric d_{LY}. We endow $\mathrm{PE}^{1+\alpha}(X_0)$ with the Borel σ-algebra.

Remark 3.2. *There is a definition of distance for Lasota-Yorke maps, related to the Skorohod metric, that has been previously used in the literature; see for instance Keller and Liverani. [29]. That notion of distance is not adequate for our purposes, because it allows maps to behave badly in sets of small Lebesgue measure.*

Definition 3.3. *A* ***random*** $C^{1+\alpha}$ ***piecewise expanding dynamical system*** *on a domain* X_0 *is given by a tuple* $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, T)$ *where* $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ *is a probability preserving transformation and* $T : \Omega \to PE^{1+\alpha}(X_0)$ *satisfying*

R1. (Measurability) $T : \Omega \to PE^{1+\alpha}(X_0)$ *given by* $\omega \mapsto T_\omega$ *is a measurable function.*

R2. (Number of branches) The function $\omega \mapsto b^{T_\omega}$ *is* $\mathbb{P}$*-log-integrable,* b^{T_ω} *being the number of branches of* T_ω*.*

R3. (Distortion) There exists a constant $\mathcal{D}$ *such that* $\|DT_\omega\|_\alpha \leq \mathcal{D}$ *for* $\mathbb{P}$*-almost every* $\omega \in \Omega$*.*

R4. (Minimum expansion) There exists $a < 1$ *such that for* $\mathbb{P}$*-almost every* $\omega \in \Omega$*,* $\|\mu_{T_\omega}^{-1}\|_\infty \leq a$*, where* $\mu_{T_\omega}(x) := \inf_{\|v\|=1} \|DT_\omega(x)v\|$*.*

R5. (Branch geometry) There exists a constant L *such that for* $\mathbb{P}$*-almost every* $\omega \in \Omega$*, each branch domain* O_i *is bounded by at most* L C^1 *hypersurfaces.*

A random piecewise expanding dynamical system will denote a random $C^{1+\alpha}$ piecewise expanding dynamical system for some $0 < \alpha \leq 1$. In the case where $d = 1$ and $X_0 = [0, 1]$, we refer to these systems as **random Lasota-Yorke type dynamical systems**. We will also refer to random C^2 piecewise expanding dynamical systems (with the obvious definition).

3.2 Fractional Sobolev spaces

Here we introduce spaces of functions suitable for our purposes. Their choice is motivated by recent work of Baladi and Gouëzel [2]. Much of the development in this subsection parallels that done in [2] (see also Thomine's work [43], the specialization of [2] to the expanding case).

While the other works consider the case of a single map, we work with random dynamical systems. One new feature is that we need to ensure the strong measurability of the family of Perron-Frobenius operators. In the context of a single map, it is often sufficient to prove inequalities with constants depending on the map, showing only that the constants are finite. A second new feature in the random context is that one needs to maintain control of the quantities describing compositions of maps as the inequalities are iterated.

For this reason we give references to the earlier works where possible and emphasize those points where differences arise.

Let $t \geq 0$ and $1 < p < \infty$. Let $H_p^t(\mathbb{R}^d)$, or simply H_p^t, be the image of $L_p(\mathbb{R}^d)$ under the injective linear map $\mathcal{J}_t : L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d)$ given by

$$\mathcal{J}_t(g) = \mathcal{F}^{-1}(a_{-t}\mathcal{F}(g)),$$

where $\mathcal{F}$ denotes the Fourier transform, and $a_t(\zeta) := (1 + |\zeta|^2)^{\frac{t}{2}}$. H_p^t endowed with the norm

$$\|f\|_{H_p^t} := \|\mathcal{F}^{-1}(a_t\mathcal{F}(f))\|_{L_p(\mathbb{R}^d)}$$

is a Banach space known as (local) fractional Sobolev space. Thus, $\mathcal{J}_t$ is a surjective isometry from $L_p(\mathbb{R}^d)$ to H_p^t. Since $L_p(\mathbb{R}^d)$ is separable and reflexive, its isometric image H_p^t is also separable and reflexive. Also, the space of differentiable functions with compact support, $C_0^\infty(\mathbb{R}^d)$, is dense in H_p^t. See for example Strichartz [41] and references therein for these and other properties of H_p^t.

This subsection closely follows Baladi and Gouëzel. In what follows, we assume that $p > 1$ and $0 < t < \min\{\alpha, \frac{1}{p}\}$. The following properties will be used in the sequel. The first one is taken from Triebel [44, Corollary 4.2.2], and concerns multiplication by Hölder functions. The second one is a refinement of a result of Strichartz [41, Corollary II3.7], and deals with multiplication by characteristic functions of intervals. The third one deals with multiplication by characteristic functions of higher dimensional sets. The last one is related to a result of Baladi and Gouëzel [2, Lemma 25], about composition with smooth functions.

Lemma 3.4 (Multiplication by C^α functions)**.** *(Triebel [44, Corollary 4.2.2]) There exists a constant $C_\#$, depending only on t and α, such that for any $g \in C^\alpha(\mathbb{R}^d, \mathbb{R})$, and for any $f \in H_p^t$, we have that $fg \in H_p^t$, with*

$$\|gf\|_{H_p^t} \leq C_\# \|g\|_{C^\alpha} \|f\|_{H_p^t}.$$

Lemma 3.5 (Multiplication by characteristic functions in one dimension)**.**

(a) (Strichartz [41, Corollary II3.7]) There exists some constant $C_\#$ depending only on t and p such that for every $f \in H_p^t$, and interval $I' \subset \mathbb{R}$, $\|1_{I'}f\|_{H_p^t} \leq C_\# \|f\|_{H_p^t}$.

(b) Let $f \in H_p^t$. Then, for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $I' \subset \mathbb{R}$ is an interval of length at most δ, then $\|1_{I'}f\|_{H_p^t} \leq \epsilon$.

Proof of (b). Since $C_0^\infty(\mathbb{R})$ is dense in H_p^t, there exists (using part (a)) $g \in C^\alpha(\mathbb{R}) \cap H_p^t$ such that $\|1_{I'}(f - g)\|_{H_p^t} \leq \epsilon/2$ for all intervals I'. By Lemma 3.4 we have $\|1_{I'}g\|_{H_p^t} \leq C_\# \|g\|_{C^\alpha} \|1_{I'}\|_{H_p^t}$. If I' is of length at most δ, then $\|1_{I'}\|_{H_p^t} \leq C_\# \delta^{1/p-t}$. Hence if $\delta > 0$ is chosen sufficiently small we obtain that for all intervals I' of length at most δ, $\|1_{I'}g\|_{H_p^t} \leq \epsilon/2$ completing the proof. $\square$

Lemma 3.6 (Multiplication by characteristic functions of nice sets)**.**

(a) *(Strichartz [41, Corollaries II3.7 and II4.2]) There exists some constant $C_\#$ depending only on t and p such that for every set $O \subset \mathbb{R}^d$ intersecting almost every line parallel to some coordinate axis in at most L connected components, and for every $f \in H_p^t$, we have that $\|1_O f\|_{H_p^t} \le C_\# L \|f\|_{H_p^t}$.*

(b) *(Sickel [40, Proposition 4.8]) Let $\mathcal{P}$ be a branch partition as in Definition 3.1. Then, there exists a constant C depending on $\mathcal{P}$ such that for every $O \in \mathcal{P}$ and every f in H_p^t, $1_O f \in H_p^t$ with $\|1_O f\|_{H_p^t} \le C\|f\|_{H_p^t}$.*

Lemma 3.7 (Composition with C^1 diffeomorphisms)**.**

(a) *(Thomine [43, Lemma 4.3]) Let $A : \mathbb{R}^d \circlearrowleft$ be a linear map. Then, there exists $C_\#$ such that for any $f \in H_p^t$,*

$$\|f \circ A\|_{H_p^t} \le C_\# |\det A|^{-\frac{1}{p}} \|A\|^t \|f\|_{H_p^t} + C_\# |\det A|^{-\frac{1}{p}} \|f\|_p.$$

(b) *Let $F : \mathbb{R}^d \circlearrowleft$ be a C^1 diffeomorphism with $\|DF\|_\infty, \|DF^{-1}\|_\infty < \infty$. Then, there exists $C_\#$ such that for any $f \in H_p^t$,*

$$\|f \circ F\|_{H_p^t} \le C_\# \|\det(DF^{-1})\|_\infty^{\frac{1}{p}} \max\{1, \|DF\|_\infty^t\} \|f\|_{H_p^t}.$$

(c) *Let $f \in H_p^t$. Then, for every $\epsilon > 0$ there exists $\delta > 0$ such that for every diffeomorphism $F : \mathbb{R}^d \to \mathbb{R}^d$ with $\|F - Id\|_{C^1} \le \delta$, we have $\|f \circ F - f\|_{H_p^t} \le \epsilon$.*

Proof. Part (b) follows via interpolation; this result is related to Lemma 4.3 of Thomine [43].

Now we prove (c). Let $f \in H_p^t$. In view of part (b) and the density of $C_0^\infty(\mathbb{R}^d)$ in H_p^t, we can find a $g \in C_0^\infty$ such that $\|f - g\|_{H_p^t} + \|(f - g) \circ F\|_{H_p^t} < \epsilon/2$. This gives $\|f \circ F - f\|_{H_p^t} \le \epsilon/2 + \|g \circ F - g\|_{H_p^t}$.

We recall that for $t \le s$, $H_p^s \subseteq H_p^t$, and the inclusion is continuous (see for example Strichartz [41, Corollary I1.3]). In particular, for each $t \le 1$, there exists a constant $C_\#$ such that for every $g \in H_p^1$, $\|g\|_{H_p^t} \le C_\# \|g\|_{H_p^1}$.

Since $g \in C_0^\infty(\mathbb{R}^d)$, $g, g \circ F - g \in H_p^1$. Let $K = \big(1 + \|\det(DF^{-1})\|_\infty\big)\mathrm{Leb}(\mathrm{supp}(g)) < \infty$. Then

$$\begin{aligned}
&\|g \circ F - g\|_{H_p^t} \\
\le & C_\# \|g - g \circ F\|_{H_p^1} \le C_\# (\|g \circ F - g\|_{L_p} + \|D(g \circ F) - Dg\|_{L_p}) \\
\le & C_\# K^{1/p} \Big(\|Dg\|_\infty \|I - F\|_\infty + \|D^2 g\|_\infty \|I - F\|_\infty + \|Dg\|_\infty \|1 - DF\|_\infty \Big),
\end{aligned}$$

since $\mathrm{Leb}(\mathrm{supp} h) \le K$ for all $h \in \{g \circ F - g, Dg \circ F - Dg, Dg \circ F \cdot DF - Dg \circ F\}$. Choosing δ sufficiently small makes the last expression smaller than $\frac{\epsilon}{2}$ and hence, $\|f \circ F - f\|_{H_p^t} \le \epsilon$. $\square$

3.3 Transfer operators

Given a map $T \in \mathrm{PE}^{1+\alpha}(X_0)$ with branches $T_i\colon O_i \to X_0$, we let $Q_i = T_i(O_i)$ and ξ_i be the inverse branch $T_i^{-1}\colon Q_i \to O_i$. Assume $p > 1$ and $0 < t < \min(\alpha, \frac{1}{p})$.

We let $\mathcal{H}_p^t = \mathcal{H}_p^t(X_0) \subset H_p^t$ be the subspace of functions supported on the domain X_0, with the induced norm $\|f\|_{\mathcal{H}_p^t} := \|f\|_{H_p^t}$. In view of Lemma 3.6, $\mathcal{H}_p^t$ is complete, and thus a Banach space. Lemma 3.6 shows that $\mathcal{H}_p^t$ is also separable. We recall from Baladi [1, Lemma 2.2] that the inclusion $\mathcal{H}_p^t \hookrightarrow L_p$ is compact.

Remark 3.8. *We note that the space of functions of bounded variation, which has been the most widely used Banach space to study Lasota-Yorke type maps, is not separable. This is the reason to look for alternatives. In Baladi and Gouëzel [2], the authors show, in particular, that the fractional Sobolev spaces $\mathcal{H}_p^t$, are suitable to study transfer operators associated to piecewise expanding maps.*

Definition 3.9. *The transfer operator, $\mathcal{L}_T : \mathcal{H}_p^t \circlearrowleft$, associated to a map $T \in PE^{1+\alpha}(X_0)$ is defined for every $f \in \mathcal{H}_p^t$ by*

$$\mathcal{L}_T f = \sum_{i=1}^{b^T} \left(1_{O_i^T} \cdot f\right) \circ \xi_i^T \cdot |D\xi_i^T| = \sum_{i=1}^{b^T} 1_{Q_i^T} \cdot f \circ \xi_i^T \cdot |D\xi_i^T|,$$

where $|A|$ denotes the absolute value of the determinant of the linear map A.

Remark 3.10. *The results of §3.2 imply that the linear operators corresponding to composition with smooth functions, multiplication by characteristic functions of elements of the branch partition and multiplication by C^α functions are bounded in $\mathcal{H}_p^t$. Clearly, $\mathcal{H}_p^t$ is invariant under $\mathcal{L}$, and thus the transfer operator acts continuously on $\mathcal{H}_p^t$. Furthermore, if $T \in PE^{1+\alpha}$ is onto, then $\mathcal{L}_T : \mathcal{H}_p^t \to \mathcal{H}_p^t$ is onto. For example, given $f \in \mathcal{H}_p^t$ and letting $g = \frac{|DT| \cdot f \circ T}{\sum_{i=1}^{b^T} 1_{Q_i^T} \circ T}$ gives $\mathcal{L}_T g = f$. Again by the results above, $g \in \mathcal{H}_p^t$.*

The following lemma provides a weak continuity property of the transfer operator acting on a fractional Sobolev space for Lasota-Yorke maps.

Lemma 3.11. *Let $L(\mathcal{H}_p^t)$ be endowed with the strong operator topology and $LY^{1+\alpha}$ be endowed with the metric $d_{LY^{1+\alpha}}$. Then, the map $\mathcal{L}$ sending a Lasota-Yorke map to its transfer operator $\mathcal{L} : LY^{1+\alpha} \to L(\mathcal{H}_p^t)$ given by $T \mapsto \mathcal{L}_T$ is continuous.*

Proof. Let $f \in \mathcal{H}_p^t$ and $T \in \mathrm{LY}^{1+\alpha}$. We will prove that $\lim_{S\to T} \|\mathcal{L}_S f - \mathcal{L}_T f\|_{\mathcal{H}_p^t} = 0$. Let $b = b^T$. Assume $d_{\mathrm{LY}^{1+\alpha}}(S,T) < 1$. Then, by definition of $d_{\mathrm{LY}^{1+\alpha}}$, $b^S = b$. For each $1 \le i \le b$, let $Q_i^{T\cap S} = Q_i^T \cap Q_i^S$ and $Q_i^{T\setminus S} = Q_i^T \setminus Q_i^S$.

Then,

$$\left\|\mathcal{L}_T f - \mathcal{L}_S f\right\|_{\mathcal{H}_p^t} \leq \sum_{i=1}^{b} \left\|1_{Q_i^{T\cap S}}(f\circ\xi_i^T\cdot|D\xi_i^T| - f\circ\xi_i^S\cdot|D\xi_i^S|)\right\|_{\mathcal{H}_p^t}$$
$$+\sum_{i=1}^{b}\left\|1_{Q_i^{T\setminus S}}\cdot f\circ\xi_i^T\cdot|D\xi_i^T|\right\|_{\mathcal{H}_p^t} + \sum_{i=1}^{b}\left\|1_{Q_i^{S\setminus T}}\cdot f\circ\xi_i^S\cdot|D\xi_i^S|\right\|_{\mathcal{H}_p^t}.$$

We finish the proof by bounding the terms separately in the following lemma.

Sublemma 3.12.

(I) Bound on common branches. For every $1 \leq i \leq b$,

$$\lim_{S\to T}\left\|1_{Q_i^{T\cap S}}(f\circ\xi_i^T\cdot|D\xi_i^T| - f\circ\xi_i^S\cdot|D\xi_i^S|)\right\|_{\mathcal{H}_p^t} = 0.$$

(II) Bound on remaining terms. For every $1 \leq i \leq b$,

$$\lim_{S\to T}\left\|1_{Q_i^{T\setminus S}}\cdot f\circ\xi_i^T\cdot|D\xi_i^T|\right\|_{\mathcal{H}_p^t} = 0 \quad \textit{and} \quad \lim_{S\to T}\left\|1_{Q_i^{S\setminus T}}\cdot f\circ\xi_i^S\cdot|D\xi_i^S|\right\|_{\mathcal{H}_p^t} = 0.$$

□

Proof of Sublemma 3.12 (I). We start by noting that we can fix a way of choosing extensions of each T_i to a diffeomophism $\tilde{T}_i$ of $\mathbb{R}$, in such a way that $\|\tilde{S}_i - \tilde{T}_i\|_{C^{1+\alpha}} \leq 2\|(S_i - T_i)|_{Q_i^S\cap Q_i^T}\|_{C^{1+\alpha}}$. In what follows, we drop the tildes for convenience. Using Lemmas 3.4, 3.5 and 3.7 repeatedly, we have

$$\left\|1_{Q_i^{T\cap S}}(f\circ\xi_i^T\cdot|D\xi_i^T| - f\circ\xi_i^S\cdot|D\xi_i^S|)\right\|_{\mathcal{H}_p^t} \leq C_\#\left\|f\circ\xi_i^T\cdot|D\xi_i^T| - f\circ\xi_i^S\cdot|D\xi_i^S|\right\|_{H_p^t}$$
$$\leq C_\#\left\||D\xi_i^T| - |D\xi_i^S|\right\|_{C^\alpha}\left\|f\circ\xi_i^T\right\|_{H_p^t} + C_\#\left\|D\xi_i^S\right\|_{C^\alpha}\left\|f\circ\xi_i^T - f\circ\xi_i^S\right\|_{H_p^t}$$
$$\leq C_\#\Big(\left\||D\xi_i^T| - |D\xi_i^S|\right\|_{C^\alpha}\left\|f\right\|_{\mathcal{H}_p^t}\left\|DT_i\right\|_\infty^{\frac{1}{p}} + \Big(\left\|D\xi_i^T\right\|_{C^\alpha}+1\Big)\left\|f\circ\xi_i^T - f\circ\xi_i^S\right\|_{H_p^t}\Big),$$

where $T_i := (\xi_i^T)^{-1}$, and in the last inequality we use the fact that $\left\|D\xi_i^S\right\|_{C^\alpha} < \left\|D\xi_i^T\right\|_{C^\alpha} + 1$ whenever $d_{\mathrm{LY}^{1+\alpha}}(S,T) < 1$. The first term goes to 0 as $S \to T$ because $d_{\mathrm{LY}^{1+\alpha}}(S,T) \geq \frac{1}{2}\left\||D\xi_i^S| - |D\xi_i^T|\right\|_{C^\alpha}$. It remains to show that $\lim_{S\to T}\left\|f\circ\xi_i^T - f\circ\xi_i^S\right\|_{H_p^t} = 0$. By Lemma 3.7(b), showing the above is equivalent to proving that $\lim_{S\to T}\left\|f - f\circ\xi_i^S\circ T_i\right\|_{H_p^t} = 0$. This is a direct consequence of Lemma 3.7(c) and the observation that $\lim_{S\to T}\left\|\xi_i^S\circ T_i - Id\right\|_{C^1} = 0$. □

Proof of Sublemma 3.12 (II). Fix $1 \le i \le b$. First, we observe that $\mathrm{Leb}(Q_i^{S\setminus T}) \le 2d_H(Q_i^S, Q_i^T) \le 2(\|T\|_\infty + 1)d_{LY^{1+\alpha}}(S,T)$. Since ξ_i^S and ξ_i^T are contracting, then

$$\lim_{S\to T} \mathrm{Leb}\big(\xi_i^T(Q_i^{T\setminus S})\big) = 0 \quad \text{and} \quad \lim_{S\to T} \mathrm{Leb}\big(\xi_i^S(Q_i^{S\setminus T})\big) = 0. \tag{7}$$

We now show that

$$\lim_{S\to T} \left\| 1_{Q_i^{T\setminus S}} \cdot f\circ \xi_i^T \cdot |D\xi_i^T| \right\|_{\mathcal{H}_p^t} = 0.$$

Being the set difference of two intervals, the set $Q_i^{T\setminus S}$ is either empty, or an interval, or the union of two intervals. Thus, we let $Q_i^{T\setminus S} = \bigcup_{\gamma_i\in\Gamma_i} Q_{\gamma_i}^{T\setminus S}$ be the decomposition of $Q_i^{T\setminus S}$ into intervals, where $\#\Gamma_i \in \{0,1,2\}$.

Let $\gamma_i \in \Gamma_i$. Using Lemmas 3.4 and 3.7(b), respectively, we obtain

$$\begin{aligned}\left\| 1_{Q_{\gamma_i}^{T\setminus S}} \cdot f\circ \xi_i^T \cdot |D\xi_i^T| \right\|_{\mathcal{H}_p^t} &\le C_\# \left\| D\xi_i^T \right\|_{C^\alpha} \left\| 1_{Q_{\gamma_i}^{T\setminus S}} \cdot f\circ \xi_i^T \right\|_{\mathcal{H}_p^t} \\ &\le C_\# \left\| D\xi_i^T \right\|_{C^\alpha} \left\| DT_i \right\|_\infty^{\frac{1}{p}} \left\| 1_{\xi_i^T(Q_{\gamma_i}^{T\setminus S})} f \right\|_{\mathcal{H}_p^t}.\end{aligned}$$

Since $Q_{\gamma_i}^{T\setminus S} \subseteq Q_i^{T\setminus S}$, (7) implies that $\lim_{S\to T} \mathrm{Leb}(\xi_i^T(Q_{\gamma_i}^{T\setminus S})) = 0$. Lemma 3.5(b) yields $\lim_{S\to T} \left\| 1_{\xi_i^T(Q_{\gamma_i}^{T\setminus S})} f \right\|_{\mathcal{H}_p^t} = 0$. Therefore,

$$\lim_{S\to T} \left\| 1_{Q_{\gamma_i}^{T\setminus S}} \cdot f\circ \xi_i^T \cdot |D\xi_i^T| \right\|_{\mathcal{H}_p^t} = 0,$$

as claimed. Although the statements of Lemma 3.12(II) are not symmetric in T and S, interchanging the roles of T and S in the proof just presented, and recalling from (7) that $\lim_{S\to T} \mathrm{Leb}(\xi_i^S(Q_{\gamma_i}^{S\setminus T})) = 0$, we get that

$$\lim_{S\to T} \left\| 1_{Q^{S\setminus T}} \cdot f\circ \xi_i^S \cdot |D\xi_i^S| \right\|_{\mathcal{H}_p^t} = 0.$$

□

Let $\mathcal{T} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, T)$ be a random piecewise expanding $C^{1+\alpha}$ dynamical system. Suppose that the following conditions are satisfied.

S1. (Parameters) $\mathcal{H}_p^t$ is the fractional Sobolev space defined in §3.2, with $p > 1$ and $0 < t < \min\{\alpha, \frac{1}{p}\}$.

S2. (Strong measurability) The map $\mathcal{L}$ sending ω to the transfer operator of T_ω, $\mathcal{L}_{T_\omega}$, acting on $\mathcal{H}_p^t$, is strongly measurable.

Then we call the tuple $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{H}_p^t, \mathcal{L})$ the strongly measurable random linear system *associated* to $\mathcal{T}$. For brevity, we will use write $\mathcal{L}_\omega$ instead of $\mathcal{L}_{T_\omega}$. Also, the notation from §3.1 will be abbreviated. For example, instead of b^{T_ω}, we will write b^ω, and so on.

We note that Lemma 3.11 guarantees that for any random $C^{1+\alpha}$ Lasota-Yorke dynamical system, condition [S2] is automatically satisfied. A second situation that we consider is that of a random piecewise expanding dynamical system in higher dimensions with a fixed, suitably regular branch partition. In this situation one sees directly that [S2] holds also.

3.4 Random Lasota-Yorke Inequalities

Given a collection $\mathcal{C}$ of subsets of a set, we recall that its intersection multiplicity is given by $\max_{x\in\bigcup\mathcal{C}} \#\{C \in \mathcal{C} : x \in C\}$. Given $T \in \mathrm{PE}^{1+\alpha}(X_0)$, the *complexity of T at the end*, denoted by $C_e(T)$, is the intersection multiplicity of $\{\overline{T(O_i^T)}\}_{1\le i\le b^T}$. The *complexity of T at the beginning*, $C_b(T)$, is the intersection multiplicity of $\{\overline{O_i^T}\}_{1\le i\le b^T}$. We note that in the one-dimensional Lasota-Yorke case, $C_b(T)$ is always equal to 2 (even when compositions of maps are taken), whereas in higher dimensions the complexity at the beginning can grow without bound as maps are composed. Examples of Tsujii [45] and Buzzi [8] show that this can lead to singular ergodic properties of the map including non-existence of absolutely continuous invariant measures.

As is well-known, quasi-compactness can be derived from Lasota-Yorke type inequalities of the form $|\!|\!|\mathcal{L}f|\!|\!| \le A\|f\| + B|\!|\!|f|\!|\!|$, where $|\!|\!|\cdot|\!|\!|$ is a stronger norm than $\|\cdot\|$ and the inclusion $(Y, |\!|\!|\cdot|\!|\!|) \hookrightarrow (Y, \|\cdot\|)$ is compact. Hennion's theorem shows that the essential spectral radius is governed by B.

The following Lasota-Yorke type inequality is based on results of Thomine [43, Theorem 2.3]. In that work, rather than a random dynamical system, a single dynamical system is considered. Thomine (and the previous work of Baladi and Gouëzel) took a great deal of care to bound the 'B' term, but did not need to control the 'A' term other than to say that it is finite. In our context, we need the additional fact that A depends in a measurable way on our dynamical system. That this holds can be seen by a careful examination of the proofs of Thomine; and Baladi and Gouëzel. One feature of the proof that needs attention is that these papers replace the norm $\|\cdot\|_{\mathcal{H}_p^t}$ by an equivalent norm depending on properties of the map T^n. In our context, we would obtain results in different norms for different compositions $T_\omega^{(n)}$. We avoid this at the expense of increasing the A term. More specifically we make use of a bound of the form

$$\sum_{m\in\mathbb{Z}} \|\eta_{m,r}u\|_{\mathcal{H}_p^t}^p \le C_\# \left((1+r^{pt})\|u\|_p^p + \|u\|_{\mathcal{H}_p^t}^p\right),$$

where $(\eta_{m,r})_{m\in\mathbb{Z}^d}$ is a partition of unity of $\mathbb{R}^d$ obtained by scaling a fixed partition of unity by a factor r in the variable, and satisfies $\eta_{m,r}(x) = \eta_{0,r}(x+m/r)$.

Lemma 3.13 (Strong $L_p - \mathcal{H}_p^t$ Lasota-Yorke inequality)**.**
Suppose $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{H}_p^t, \mathcal{L})$ is a strongly measurable random linear system associated to a random $C^{1+\alpha}$ piecewise expanding dynamical system $\mathcal{T}$. Then there exists a constant $C_{\mathcal{R}}$, depending only on p, t, α, η and L (from Definition

3.3), and a measurable function $A_{\mathcal{R},n}(\omega)$ such that for every $\omega \in \Omega$, we have

$$\|\mathcal{L}_\omega^{(n)} f\|_{\mathcal{H}_p^t} \le A_{\mathcal{R},n}(\omega)\|f\|_p + B_{\mathcal{R},n}(\omega)\|f\|_{\mathcal{H}_p^t}, \tag{S-LY}$$

where

$$A_{\mathcal{R},n}(\omega) \text{ is a measurable function of } \omega; \text{and}$$

$$B_{\mathcal{R},n}(\omega) = C_{\mathcal{R}} n \left(C_b(T_\omega^{(n)})\right)^{\frac{1}{p}} \left(C_e(T_\omega^{(n)})\right)^{1-\frac{1}{p}} \||DT_\omega^{(n)}|^{\frac{1}{p}-1}\mu_{\omega,n}^{-t}\|_\infty,$$

where $\mu_{\omega,n}(x) := \inf_{\|v\|=1} \|DT_\omega^{(n)}(x)v\|$.

This inequality will prove sufficient to control the index of compactness, but does not give enough information to control the maximal Lyapunov exponent since we have no control of the $A_{\mathcal{R},n}(\omega)$ term. The following inequality remedies the situation by providing an inequality with no A term, at the expense of having a larger (but still log-integrable) B term. The availability of both inequalities will allow us to apply Lemma C.5. The proof of the weak Lasota-Yorke inequality is straightforward using some of the ingredients of the stronger version.

Lemma 3.14 (Weak $L_p - \mathcal{H}_p^t$ Lasota-Yorke inequality)**.**
Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{H}_p^t, \mathcal{L})$ be a random Lasota-Yorke type dynamical system. Then, for each $n \in \mathbb{N}$ there exists a $\mathbb{P}$-log-integrable function $\tilde{A}_{\mathcal{R},n} : \Omega \to \mathbb{R}$ such that for every $\omega \in \Omega$,

$$\|\mathcal{L}_\omega^{(n)} f\|_{\mathcal{H}_p^t} \le \tilde{A}_{\mathcal{R},n}(\omega)\|f\|_{\mathcal{H}_p^t}. \tag{W-LY}$$

3.5 Quasi-compactness

In this subsection we prove quasi-compactness by bootstrapping a strategy of Buzzi [7] based on multiple Lasota-Yorke inequalities as elaborated in Appendix C.2.

Lemma 3.15. *Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, T)$ be a random piecewise expanding dynamical system with ergodic base. Then the following hold:*

1. *There exist $C_e^* < \infty$ and $C_b^* < \infty$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$,*

$$\lim_{n\to\infty} (C_e(T_\omega^{(n)}))^{\frac{1}{n}} = C_e^*; \quad \text{and} \quad \lim_{n\to\infty} (C_b(T_\omega^{(n)}))^{\frac{1}{n}} = C_b^*;$$

2. *There exists $\chi < 1$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$, $\lim_{n\to\infty} \|\mu_{\omega,n}^{-1}\|_\infty^{\frac{1}{n}} = \chi$. Furthermore, $\lim_{n\to\infty} \||DT_\omega^{(n)}|^{-1}\|_\infty^{\frac{1}{n}} \le \chi^d$.*

Proof. The sequences $(C_e(T_\omega^{(n)}))_{n\in\mathbb{N}}$, $(C_b(T_\omega^{(n)}))_{n\in\mathbb{N}}$, $(\|\mu_{\omega,n}^{-1}\|_\infty)_{n\in\mathbb{N}}$ and $(\||DT_\omega^{(n)}|^{-1}\|_\infty)_{n\in\mathbb{N}}$, are submultiplicative. Log-integrability of C_e and C_b is assured by Definition 3.3 since we have $C_b(T_\omega), C_e(T_\omega) \le \log b^\omega$. Hence the existence of the limits follows from the Kingman subadditive ergodic theorem [33]. That $\chi < 1$ follows from condition R4. The last statement follows from $|DT_\omega^{(n)}(x)|^{-1} \le \mu_{\omega,n}(x)^{-d}$. ∎

Lemma 3.16 (Quasi-compactness: Lasota-Yorke case)**.** *Let $0 < \alpha < 1$ and let $\mathcal{T}$ be a $C^{1+\alpha}$ random Lasota-Yorke dynamical system satisfying the additional condition that the function $\omega \mapsto \log^+ \operatorname{var}(|DT_\omega|^{-1})$ is $\mathbb{P}$-integrable. Then there exist parameters $p > 1, 0 < t < \min(\alpha, \frac{1}{p})$ such that the associated random linear system, $\mathcal{R}$, is quasi-compact with*

$$\kappa^* \leq \left(1 - \frac{1}{p}\right)(\log C_e^* + \log \chi) + t \log \chi < \lambda^* = 0.$$

Proof of Lemma 3.16. Since we are in the Lasota-Yorke case, strong measurability of $\omega \mapsto \mathcal{L}_\omega$ follows from Lemma 3.11. We also have $C_b^* = 1$. By Lemma 3.15, $C_e^* < \infty$. By hypothesis, $\chi < 1$. Fix $0 < t < \alpha$. Now if p is sufficiently close to 1, t satisfies $t < \min(\alpha, 1/p)$ and the inequality $\left(1 - \frac{1}{p}\right)(\log C_e^* + \log \chi) + t \log \chi < 0$ holds. By Lemma C.5, we see $\kappa^* < 0$. On the other hand we have $\|\mathcal{L}_\omega^{(n)} 1\|_{H_p^t} \geq C_\# \|\mathcal{L}_\omega^{(n)} 1\|_p \geq C_\# \|\mathcal{L}_\omega^{(n)} 1\|_1 = C_\#$ so that $\lambda^* \geq 0$. Accordingly Theorem 2.10 applies. Suppose for a contradiction that $\lambda^* > 0$.

Now the following are full measure sets: the set where the results of Theorem 2.10 hold; the set where the top Lyapunov exponent is λ^*; the set where the results of Lemma C.5 hold using $\|\cdot\| = \|\cdot\|_1$ and $|\!|\!|\cdot|\!|\!| = \|\cdot\|_{\mathrm{BV}}$ (by Buzzi's argument [7]); and the set where the results of Lemma C.5 hold using $\|\cdot\| = \|\cdot\|_p$ and $|\!|\!|\cdot|\!|\!| = \|\cdot\|_{H_p^t}$, because of Lemmas 3.13 and 3.14. Let the full measure set obtained by intersecting these be denoted by Ω_1.

Suppose $\omega \in \Omega_1$ and let f be a non-zero element of $Y_1(\omega)$. By standard properties of H_p^t, f may be approximated arbitrary closely in $\|\cdot\|_{H_p^t}$ be a C^∞ function g. Applying Lemma C.5 with $\|\cdot\|_1$ and $\|\cdot\|_{\mathrm{BV}}$, (note that $\|\mathcal{L}_\omega^{(n)} g\|_1 \leq \|g\|_1$ for all n), we get $\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} g\|_{\mathrm{BV}} \leq 0$ and hence $\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} g\|_p \leq 0$. Applying Lemma C.5 a second time using the conclusion of the first application as hypothesis, we get $\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} g\|_{H_p^t} \leq 0$. Letting π_1 be the projection onto the top Lyapunov subspace, we have that $\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} (g - \pi_1(g))\|_{H_p^t} < \lambda^*$. Thus $\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} \pi_1(g)\|_{H_p^t} < \lambda^*$. By Theorem 2.10, this implies $\pi_1(g)$ is 0. Since g can be chosen arbitrarily close to f and π_1 is bounded, this is a contradiction. $\square$

We now show that results of Cowieson [9] may be exploited to give families of random dynamical systems in higher dimensions for which one can establish an Oseledets splitting for the Perron-Frobenius cocycle. The framework of [9] has a key simplifying feature, namely that there is a *fixed* partition $\mathcal{P}$ of the domain X_0 into disjoint open pieces on each of which the map is continuous and expanding. For this reason, the analogue of Lemma 3.12 is straightforward: there is no issue with accounting for differences between partitions. On the other hand a new difficulty appears in higher dimensions, namely that it is no longer true *a priori* that the complexity at the beginning, $C_b(T_\omega^{(n)})$, is bounded in n. The necessity of controlling C_b is demonstrated by results of Tsujii [45] and Buzzi [8] and indeed C_b appears in Baladi and Gouëzel [2].

Theorem 3.17 (Cowieson[9])**.** *Let $\mathcal{P}$ be a fixed branch partition of a compact region $X_0 \subset \mathbb{R}^d$. There is a quantity M and a dense G_δ subset, $Cow(X_0;\mathcal{P})$, of $PE^2(X_0;\mathcal{P})$ with the following property: For any $n > 0$ and $T \in Cow(X_0;\mathcal{P})$, there is a neighbourhood U of T such that for any $T_1, T_2, \dots, T_n \in U$, $C_b(T_n \circ \cdots \circ T_1) \le M$.*

Lemma 3.18 (Quasi-compactness: Cowieson case)**.** *Let $d > 1$, let X_0 be a compact region of $\mathbb{R}^d$ and let $\mathcal{P}$ be a branch partition of X_0. Let $T \in Cow(X_0;\mathcal{P})$. Then there exist parameters $p > 1$, $0 < t < 1/p$, a constant $\tau < 0$ and a neighbourhood N of T with the following property:*

Let $\mathcal{T}$ be a random C^2 piecewise expanding dynamical system with ergodic base. Suppose that for $\mathbb{P}$-almost every ω, T_ω has branch partition $\mathcal{P}$ and that $T_\omega \in N$. Then if $\mathcal{L}_\omega$ is the corresponding family of transfer operators acting on $\mathcal{H}_p^t(X_0)$, then $\mathcal{R}$, the random linear system associated to $\mathcal{T}$, is quasi-compact with

$$\kappa^* \le \tau < \lambda^* = 0.$$

Proof. Let $t = \frac{1}{2}$, let $T \in \mathrm{Cow}(X_0;\mathcal{P})$, and let k be the number of elements of $\mathcal{P}$. Let $a < 1$ be such that $\mu_T^{-1} < a$, where $\mu_T = \operatorname{essinf}_{x\in X_0, \|v\|=1} \|DT(x)v\|$, and let M be as in guaranteed by Theorem 3.17. Let $p > 1$ be such that $k^{1-\frac{1}{p}} a^{d(1-\frac{1}{p})+t} < 1$. We may further assume that $p < d/(d-1)$. This fixes all the data necessary to determine $C_{\mathcal{R}}$.

Let n_0 be such that $\beta := C_{\mathcal{R}} n_0 M^{1/p} k^{n_0(1-\frac{1}{p})} a^{n_0(d(1-\frac{1}{p})+t)} < 1$ and let $\tau = \log \beta / n_0$ so that $\tau < 0$. We now apply Theorem 3.17 to deduce that there is a neighbourhood N of T such that for all any n_0-fold composition of elements of N, each respecting the branch partition $\mathcal{P}$, the complexity at the beginning is bounded above by M. We further reduce N (to a smaller open neighbourhood of T) by requiring that $\mu_S^{-1} < a$ for all $S \in N$.

Now if $\mathcal{T}$ is a random C^2 piecewise expanding dynamical system where the maps all belong to N then we have ensured that the quantity $B_{\mathcal{R},n_0}$ appearing in Lemma 3.13 is at most $e^{n_0\tau}$. As in the proof of Lemma 3.16 we obtain $\kappa^* \le \tau$.

To see that $\lambda^* = 0$, we argue as in Lemma 3.16. We initially apply Lemma C.5 with $\|\cdot\| = \|\cdot\|_{L^1}$ and $|\!|\!|\cdot|\!|\!| = \|\cdot\|_{BV}$ to deduce for $f \in C^\infty$ (using results from Cowieson's paper [9]) that $\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} f\|_{\mathrm{BV}} \le 0$. Then, since there are constants $C_\#, C_\#(X_0)$ such that for any function g supported on X_0, $\|g\|_{L^{d/(d-1)}} \le C_\# \|g\|_{\mathrm{BV}}$ (see Giusti [22, Theorem 1.28]) and $\|g\|_{L^p} \le C_\#(X_0)\|g\|_{L^{d/(d-1)}}$, we obtain sufficient conditions for the second application of Lemma C.5, taking this time $\|\cdot\| = \|\cdot\|_{L^p}$ and $|\!|\!|\cdot|\!|\!| = \|\cdot\|_{\mathcal{H}_p^t}$. The remainder of the proof is exactly as in Lemma 3.16. ∎

In view of the quasi-compactness just obtained, we can apply Theorem 2.10 to get our main application theorem, ensuring the existence of an Oseledets splitting for random Lasota-Yorke dynamical systems or Cowieson-type random piecewise expanding dynamical systems.

Theorem 3.19. *Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, T)$ be a random $C^{1+\alpha}$ piecewise expanding dynamical system satisfying the hypotheses of Lemma 3.16 or Lemma 3.18* *with parameters p and t. Then, there exist $1 \leq l \leq \infty$, and exceptional Lyapunov exponents $0 = \lambda_1 > \lambda_2 > \cdots > \lambda_l > \kappa^*$ (in the case $l = \infty$, we have $\lim \lambda_n = \kappa^*$), measurable families of finite-dimensional equivariant spaces $Y_1(\omega), \ldots, Y_l(\omega) \subset X$ and a measurable equivariant family of closed subspaces $V(\omega) \subset X$ defined on a full $\mathbb{P}$ measure, σ-invariant subset of Ω so that $X = V(\omega) \oplus \bigoplus_{j=1}^{l} Y_j(\omega)$, for every $f \in V(\omega) \setminus \{0\}$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} f\|_{\mathcal{H}_p^t} \leq \kappa^*$, and for every $f \in Y_j(\omega) \setminus \{0\}$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} f\|_{\mathcal{H}_p^t} = \lambda_j$. Furthermore, the norms of the associated projections are tempered with respect to σ.*

Remark 3.20. *We remark that in both the scenarios that we consider, the existing proofs of Buzzi [7] and Cowieson [9] establish the existence of random absolutely continuous invariant measures. One can check, using techniques such as those in [7, Proposition 3.2] that their densities lie in the leading Oseledets subspace, $Y_1(\omega)$.*

4 Future work

Several interesting questions remain open for future research. In relation to previous works concerned with exponential decay of correlations, it is natural to look for conditions that provide further information about the structure of the Oseledets splitting, either in a general framework, or in the specific situation of random composition of piecewise expanding maps. Of particular interest would be to ensure simplicity of the leading Lyapunov exponent, and to obtain bounds on the number of exceptional Lyapunov exponents, including finiteness. Some progress has already been achieved in this direction in the settings of smooth expanding maps (see work of Baladi, Kondah and Schmitt [4]) subshifts of finite type (see work of Kifer [31, 32]) and piecewise smooth expanding maps of the interval (see work of Buzzi [6]).

In a different direction, it would be interesting to investigate applications of the abstract semi-invertible Oseledets theorem (Theorem 2.10) to a more general class of expanding maps, allowing for non-constant branch partitions and, more ambitiously, to piecewise hyperbolic maps.

Finally, we hope that the constructive approach to the identification of Oseledets spaces turns out to be useful for numerical studies of non-autonomous dynamical systems. A possible alternative would be to first attempt to identify the Oseledets filtration, perhaps using some existing numerical method. Then, one could inductively approximate Oseledets spaces by (1) fixing a sufficiently dense subset of a basis of a suitable Banach space, (2) pushing forward these elements under a numerical approximation of the transfer operator and (3) subtracting the projection along the corresponding level in the filtration to lower Oseledets spaces, previously obtained by this procedure.

A Strong measurability in separable Banach spaces

Let X be a separable Banach space and let $\mathcal{B}_X$ be the standard Borel σ-algebra generated by the open subsets of X. Fix a countable dense sequence $x_1, x_2, \ldots$ in X for the remainder of this appendix. It is well known that any open subset of X is a countable union of sets of the form $U_{i,j}$, where $U_{i,j} := \{x \in X : \|x - x_i\| < 1/j\}$. Hence, $\mathcal{B}_X$ is countably generated.

We denote by $L(X)$ the set of bounded linear operators from X to X. The strong operator topology, $\mathrm{SOT}(X)$, on $L(X)$ is the topology generated by the sub-base $\{V_{x,y,\epsilon} = \{T : \|T(x) - y\| < \epsilon\}\}$.

Definition A.1. *The* ***strong σ-algebra*** *on $L(X)$ is the σ-algebra $\mathcal{S}$ generated by sets of the form $W_{x,U} = \{T : T(x) \in U\}$, with $x \in X$ and $U \subset X$ open.*

For $r \in \mathbb{R}$, let $L_r(X)$ denote the linear maps from X to itself with norm at most r. That is, $L_r = \{T \in L(X) : \|T\| \leq r\}$.

Lemma A.2.

1. *For every $r \in \mathbb{R}$, $L_r \in \mathcal{S}$.*
2. *$\mathcal{S}$ is countably generated.*
3. *An open set in the strong operator topology lies in $\mathcal{S}$.*
4. *The strong σ-algebra is the Borel σ-algebra of the strong operator topology $SOT(X)$.*
5. *An open set in $L_n(X)$ (in the relative topology) is the union of countably many sets of the form $B_{i,j,m,n}$ (with terminology introduced in the proof).*

Proof. We first show that $L_r \in \mathcal{S}$. Let

$$\begin{aligned}\tilde{L}_r &= \bigcap_j \{T : \|T(x_j)\| \leq r\|x_j\|\} \\ &= \bigcap_j \bigcap_k \{T : \|T(x_j)\| < (r + 1/k)\|x_j\|\}.\end{aligned}$$

Then, $\tilde{L}_r$ is a countable intersection of sets in the sub-base and therefore $\tilde{L}_r \in \mathcal{S}$. We claim that $L_r = \tilde{L}_r$. Notice that if $\|T\| \leq r$, then $T \in \tilde{L}_r$. Conversely let $T \in \tilde{L}_r$ and $x \in X$, let $x_j \to x$. Since T is bounded we have $T(x_j) \to T(x)$. Since $\|T(x_j)\| \leq r\|x_j\|$ and $\|x_j\| \to \|x\|$ we see that $\|T(x)\| \leq r\|x\|$. Since this holds for all x, we see that $\|T\| \leq r$. Thus, $L_r = \tilde{L}_r$, as claimed.

Set $V_{i,j,m} = \{T : \|T(x_i) - x_j\| < 1/m\}$. This clearly belongs to $\mathcal{S}$. We claim that an open set $U \subset L(X)$ in the strong operator topology is the union of sets of the form $B_{i,j,m,n} = V_{i,j,m} \cap L_n$. Let U be open and let $T \in U$. Then U contains a basic open neighbourhood of T, that is, a set of the form $\{S : \|S(y_i) - T(y_i)\| < \epsilon_i \text{ for } i = 1, \ldots, s\}$ (where $y_1, \ldots, y_s$ are elements of X). Let $n > \|T\|$ and let $m > \max(3/\epsilon_i)$. Choose x_{k_i} such that $\|x_{k_i} - y_i\| < \min(1/(2mn), \epsilon_i/(3n))$ and

x_{ℓ_i} such that $\|x_{\ell_i} - T(y_i)\| < \min(\epsilon_i/3, 1/(2m))$. Let $C = \bigcap_{i=1}^{s} B_{k_i,\ell_i,m,n}$. By an application of the triangle inequality if $S \in C$, then we have

$$\begin{aligned}\|S(y_i) - T(y_i)\| &\leq \|S(y_i) - S(x_{k_i})\| + \|S(x_{k_i}) - x_{\ell_i}\| + \|x_{\ell_i} - T(y_i)\| \\ &\leq n\|y_i - x_{k_i}\| + 1/m + \epsilon_i/3 < \epsilon_i,\end{aligned}$$

so that we see $C \subseteq U$. We also have

$$\begin{aligned}\|T(x_{k_i}) - x_{\ell_i}\| &\leq \|T(x_{k_i}) - T(y_i)\| + \|T(y_i) - x_{\ell_i}\| \\ &\leq \|T\|\|x_{k_i} - y_i\| + 1/(2m) \leq 1/m,\end{aligned}$$

so that $T \in C$. It follows that any open set U may be expressed as a countable union of finite intersections of $\mathcal{S}$-measurable sets of the form $B_{i,j,m,n}$, so that open sets belong to $\mathcal{S}$, proving (3). (5) is proved similarly.

Since $\mathcal{S}$ is generated by sets that are open in the strong operator topology, it follows that $\mathcal{S}$ is also generated by the $(B_{i,j,m,n})$, proving (2).

We have shown that $\mathcal{S}$ contains all open sets in the strong operator topology. By definition it is generated by a collection of sets that are open in the strong operator topology. It follows that $\mathcal{S}$ is the Borel σ-algebra of $\mathrm{SOT}(X)$.

□

Definition A.3. *A map $T : \Omega \to L(X)$ is called **strongly measurable** if for every $x \in X$, the map $T(\cdot)(x) : \Omega \to X$ given by $\omega \mapsto T(\omega)(x)$ is $(\mathcal{F}, \mathcal{B}_X)$ measurable.*

Lemma A.4. *$T : \Omega \to L(X)$ is strongly measurable if and only if it is $(\mathcal{F}, \mathcal{S})$-measurable.*

Proof. Recall that since X is separable, both $\mathcal{B}_X$ and $\mathcal{S}$ are countably generated Borel σ-algebras. $\mathcal{B}_X$ is generated by $U_{i,j} := \{x \in X \colon \|x - x_i\| < 1/j\}$ (where $\{x_i\}_{i\in\mathbb{N}}$ is a dense set in X) and in view of Lemma A.2, $\mathcal{S}$ is generated by the sets $B_{i,j,m,n} = V_{i,j,m} \cap L_n$ (where $V_{i,j,m} = \{T \colon \|T(x_i) - x_j\| < 1/m\}$). Furthermore, it is straightforward to check that

$$T^{-1}(V_{x,y,\epsilon}) = \{\omega : |T(\omega)(x) - y| < \epsilon\} = T(\cdot)(x)^{-1}(B_\epsilon(y)), \tag{8}$$

where $V_{x,y,\epsilon} = \{T \colon \|T(x) - y\| < \epsilon\}$.

Assume that $T : \Omega \to L(X)$ is strongly measurable. To show it is $(\mathcal{F}, \mathcal{S})$ measurable, it suffices to show that $T^{-1}(B_{i,j,m,n}) \in \mathcal{F}$. This follows from (8) and the fact that $L_n = \bigcap_j \bigcap_k \{T \colon \|T(x_j)\| < (n + 1/k)\|x_j\|\}$, which was established in the proof of Lemma A.2(1).

For the converse, suppose that $T : \Omega \to L(X)$ is $(\mathcal{F}, \mathcal{S})$ measurable. To show it is strongly measurable, we have to show that for every $x \in X$ and $i, j \in \mathbb{N}$, $T(\cdot)(x)^{-1}(U_{i,j}) \in \mathcal{F}$. Equation (8) gives that $T(\cdot)(x)^{-1}(U_{i,j}) = T^{-1}(V_{x,x_i,1/j})$. Since $V_{x,x_i,1/j} \in \mathcal{S}$ by Lemma A.2(3), the result follows. □

Lemma A.5. *The composition of strongly measurable maps is strongly measurable.*

Proof. In view of Lemma A.4, it suffices to show that the composition map $\Psi : L(X) \times L(X) \to L(X)$ given by $\Psi(T,S) = T \circ S$ is $(\mathcal{S} \otimes \mathcal{S}, \mathcal{S})$ measurable. We claim that for every $n \in \mathbb{N}$, the restriction of Ψ to $L_n(X) \times L(X)$ is continuous with respect to $(\tau_n(X), \mathrm{SOT}(X))$, where $\tau_n(X)$ is the product topology on $L_n(X) \times L(X)$, and where $L_n(X)$ is endowed with the subspace topology of $\mathrm{SOT}(X)$. Since $L(X)$ is $\bigcup_{n\in\mathbb{N}} L_n(X)$, the result then follows from Lemma A.2.

By Lemma A.2(2), the claim will follow from showing that for every x, y, ϵ, the set $\Psi^{-1}(V_{x,y,\epsilon}) \cap (L_n(X) \times L(X))$ lies in $\tau_n(X)$. Let $(T_0, S_0) \in \Psi^{-1}(V_{x,y,\epsilon}) \cap (L_n(X) \times L(X))$. Then, $\|T_0 \circ S_0(x) - y\| < \epsilon$. Let $\delta < \epsilon - \|T_0 \circ S_0(x) - y\|$. Then, for $(T,S) \in \left(V_{S_0(x), T_0\circ S_0(x), \frac{\delta}{2}} \cap L_n(X)\right) \times V_{x, S_0(x), \frac{\delta}{2n}}$, we have

$$\begin{aligned}\|T \circ S(x) - y\| &\leq \|T \circ S(x) - T \circ S_0(x)\| + \|T \circ S_0(x) - T_0 \circ S_0(x)\| + \|T_0 \circ S_0(x) - y\| \\ &< n\frac{\delta}{2n} + \frac{\delta}{2} + \|T_0 \circ S_0(x) - y\| < \epsilon.\end{aligned}$$

Thus, $\Psi(T,S) \in V_{x,y,\epsilon}$ and $\Psi^{-1}(V_{x,y,\epsilon})$ is open in $\tau_n(X)$, as claimed. □

Let $\Phi \colon L(X) \times X \to X$ be given by $(T,x) \mapsto T(x)$.

Lemma A.6.

1. *The restriction of Φ to $L_n(X) \times X$ is continuous, where $L_n(X)$ is endowed with the subspace topology of SOT(X).*

2. *Φ is $\mathcal{S} \times \mathcal{B}_X$-measurable.*

3. *If $\tau : \Omega \to L(X)$, given by $\omega \mapsto T_\omega$, is strongly measurable and $f : \Omega \to X$, given by $\omega \mapsto x_\omega$, is measurable, then $\omega \mapsto T_\omega(x_\omega)$ is measurable.*

Proof. Let U be an open subset of X and let $A = \Phi^{-1}U \cap (L_n(X) \times X)$. Let $(T,x) \in A$, so that $T(x) \in U$ and $T \in L_n(X)$. Since U is open there exists an $\epsilon > 0$ such that $B_\epsilon(T(x)) \subset U$.

Now let $N = \{S \in L_n(X) \colon \|S(x) - T(x)\| \leq \epsilon/2\}$. If $(S,y) \in N \times B_{\epsilon/(2n)}(x)$ then we see

$$\begin{aligned}\|\Phi(S,y) - \Phi(T,x)\| = \|S(y) - T(x)\| &\leq \|S(y) - S(x)\| + \|S(x) - T(x)\| \\ &\leq n\|y - x\| + \epsilon/2 < \epsilon.\end{aligned}$$

It follows that $N \times B_{\epsilon/(2n)}(x)$ is a subset of A so that A is open in the relative topology on $L_n(X) \times X$, hence proving (1).

If U is open, then $\Phi^{-1}(U) = \bigcup_n \Phi^{-1}(U) \cap (L_n(X) \times X)$. By the above, the set $\Phi^{-1}(U) \cap (L_n(X) \times X)$ is open in the relative topology on $L_n(X) \times X$. Since $L_n(X) \times X$ has a countable neighbourhood basis with respect to the relative topology (see Corollary 5), $\Phi^{-1}(U)$ may be expressed as the countable union of products of $\mathcal{S}$-measurable sets with $\mathcal{B}_X$ measurable sets. Therefore it is $\mathcal{S} \times \mathcal{B}_X$-measurable, proving (2).

Let $\tau \colon \Omega \mapsto L(X)$ be strongly measurable and $f \colon \omega \mapsto x_\omega$ be measurable. Let $\theta(\omega) = (T_\omega, x_\omega)$. Then the map $\omega \mapsto T_\omega(x_\omega)$ may be factorized as $\Phi \circ \theta$. It is therefore sufficient to show that $\theta^{-1}\Phi^{-1}U$ is measurable for any open set U in X. We showed above that $\Phi^{-1}U$ is $\mathcal{S}\times\mathcal{B}_X$-measurable so it suffices to show that θ is measurable. By definition, $\mathcal{S}\times\mathcal{B}_X$ is generated by sets of the form $A\times B$ with $A\in\mathcal{S}$ and $B\in\mathcal{B}_X$. The preimage of $A\times B$ under θ is $\tau^{-1}(A)\cap f^{-1}B$ which is, by assumption, the intersection of two measurable sets. Hence $\mathcal{S}\times\mathcal{B}_X$ is generated by a collection of sets whose preimages under θ are measurable and hence θ is measurable.

□

B The Grassmannian of a Banach space

This appendix collects some results about Grassmannians that we need in Section 2.

Let X be a Banach space. A closed subspace Y of X is called *complemented* if there exists a closed subspace Z such that X is the topological direct sum of Y and Z, written $X=Y\oplus Z$. That is, for every $x\in X$, there exist $y\in Y$ and $z\in Z$ such that $x=y+z$, and this decomposition is unique. The Grassmannian $\mathcal{G}(X)$ is the set of closed complemented subspaces of X. We denote by $\mathcal{G}^k(X)$ the collection of closed k-codimensional subspaces of X (these are automatically complemented). We denote by $\mathcal{G}_k(X)$ the collection of k-dimensional subspaces of X (these are automatically closed and complemented). We equip $\mathcal{G}(X)$ with the metric $d(Y,Y')=d_H(Y\cap B,Y'\cap B)$ where d_H denotes the Hausdorff distance and B denotes the closed unit ball in X. We let B^* denote the closed unit ball in X^*. We denote by $\mathcal{B}_{\mathcal{G}}$ the Borel σ-algebra coming from d.

There is a natural map $\perp$ from $\mathcal{G}(X)$ to $\mathcal{G}(X^*)$, namely $Y^\perp=\{\theta\in X^*\colon \theta(y)=0 \text{ for all } y\in Y\}$. We use the same notation for the map from $\mathcal{G}(X^*)$ to $\mathcal{G}(X)$ given by $W^\perp=\{y\in Y\colon \theta(y)=0 \text{ for all } \theta\in W\}$. Notice that if X is a reflexive Banach space, then the two notions of $\perp$ on X^* agree. It is well known that for a closed subspace Y of X, $Y^{\perp\perp}=Y$ (that $Y\subseteq Y^{\perp\perp}$ follows from the definitions; that $Y^{\perp\perp}\subseteq Y$ follows from the Hahn-Banach theorem).

The following result may be found in Kato [27, IV §2].

Lemma B.1. *The maps $\perp$ from $\mathcal{G}(X)$ to $\mathcal{G}(X^*)$ and from $\mathcal{G}(X)$ to $\mathcal{G}(X^*)$ are homeomorphisms.*

Definition B.2. *If $Y\in\mathcal{G}_k(X)$, we will say a basis $\{y_1,\ldots,y_k\}$ for Y is a* nice basis *if $\|y_i\|=1$ and $d(y_i,span(y_1,\ldots,y_{i-1})=1$ for each i. A subset of size k will be called ϵ-*nice *if $1-\epsilon<\|y_i\|<1+\epsilon$ and $d((y_i,span(y_1,\ldots,y_{i-1}))>1-\epsilon$ for each $i>1$. Clearly if $\epsilon<1$ an ϵ-nice set is linearly independent. If a set is ϵ-nice and a basis, we call it an ϵ-*nice basis.

Lemma B.3. *Each element of $\mathcal{G}_k(X)$ has a nice basis.*

Proof. Let $Y\in\mathcal{G}_k(X)$. We make the inductive claim that for each $m\le k$ we can find a sequence of elements $y_1,\ldots,y_m$ of norm 1 satisfying the claim for $1\le i\le m$. The base case is easy: let y_1 be any vector in Y of length 1. Suppose we have vectors $y_1,\ldots,y_m$ satisfying the claim where $m<k$. Let W

be the subspace of Y spanned by $y_1, \dots, y_m$. Let $x \in Y \setminus W$ (such an x exists because W is an m-dimensional subspace of Y and hence a proper subspace of Y). By compactness there is a $w \in W$ minimizing $\|x - w\|$. Then let y_{m+1} be a normalized version of $x - w$. This completes the inductive step and hence the proof. □

Lemma B.4. *Let $\{y_1, \dots, y_k\}$ be ϵ-nice (with $\epsilon < 2^{-k-2}$). If $\|\sum a_i y_i\| \leq 1$ then $|a_i| \leq 2^{k+1-i}$ for each i.*

Proof. Suppose for a contradiction that $y = \sum a_i y_i$ satisfies $\|y\| = 1$ and $|a_i| > 2^{k+1-i}$ for some i. Let i be the largest such index. Set $z = \sum_{j \leq i} a_j y_j$. Then $\|z - y\| \leq \sum_{j>i} |a_j|(1+\epsilon) \leq (2^{k+1-i} - 2)(1+\epsilon)$. On the other hand by the defining property of y_i we have $\|z\| \geq |a_i|(1-\epsilon)^2 > 2^{k+1-i}(1-2\epsilon)$. The triangle inequality then shows that $\|y\| > 2 - 2^{k+3-i}\epsilon > 1$, which contradicts the assumption. □

Lemma B.5. *Let $y_1, \dots, y_k$ be an ϵ-nice basis for a k-dimensional space Y (with $\epsilon < 2^{-k-2}$), then if W is a space such that $d(y_i, W) < \delta/2^{k+2}$, then $\sup_{y \in Y \cap B} d(y, W \cap B) < \delta$.*

Proof. Let $d(y_i, w_i) < \delta/2^{k+2}$. Given $y \in Y \cap B$, y may be expressed as $\sum a_i y_i$ with $|a_i| \leq 2^{k+1-i}$, by Lemma B.4. Let $w = \sum a_i w_i$. Then $\|y - w\| \leq \sum 2^{k+1-i}\delta/2^{k+2} < \delta/2$. It follows that $\|w\| < 1 + \delta/2$ so letting $w' = w$ if $\|w\| \leq 1$ and $w/\|w\|$ otherwise, we have $\|w' - w\| < \delta/2$. Since $w' \in W \cap B$ we deduce $d(y, W \cap B) < \delta$ so that $\sup_{y \in Y \cap B} d(y, W \cap B) < \delta$ as required. □

For $\epsilon < 2^{-k-2}$, let $NB_k^\epsilon(X) \subset X^k$ be the set of $k-$dimensional $\epsilon-$nice subsets of X.

Corollary B.6. *If $\epsilon < 2^{-k-2}$, then the function from $NB_k^\epsilon(X)$ to $\mathcal{G}_k(X)$ given by $(y_1, \dots, y_k) \mapsto span(y_1, \dots, y_k)$ is continuous.*

Proof. By Lemma B.5, if $(y_1, \dots, y_k), (y_1', \dots, y_k') \in NB_k^\epsilon(X)$ are such that $\|y_i - y_i'\| < 2^{-(k+2)}\delta$, then $d(Y, Y') < \delta$, where $Y = \text{span}(y_1, \dots, y_k)$ and $Y' = \text{span}(y_1', \dots, y_k')$. □

Lemma B.7. *(Symmetry of closeness in $\mathcal{G}_k(X)$ and $\mathcal{G}^k(X)$). Let Y and W be elements of $\mathcal{G}_k(X)$. Suppose that $\max_{y \in Y \cap B} d(y, W \cap B) = r < 3^{-k}/4$. Then one obtains $\max_{w \in W \cap B} d(w, Y \cap B) < 4 \cdot 3^k r$ and hence $d(W, Y) < 4 \cdot 3^k r$.*

Let Y and W be elements of $\mathcal{G}^k(X)$. Suppose that $\max_{y \in Y \cap B} d(y, W \cap B) = r < 3^{-k}/8$. Then, $d(W, Y) < 8 \cdot 3^k r$.

Proof. Using Lemma B.3, let $y_1, \dots, y_k$ be a nice basis for Y. By assumption there exist elements $w_1, \dots, w_k$ of $W \cap B$ such that $\|w_i - y_i\| < r$.

We first give a lower bound for $\|\sum_{i=1}^k a_i w_i\|$. Let $M = \max_i 3^i|a_i|$ and assume that $M = 3^j|a_j|$. Then we have

$$\begin{aligned}\left\|\sum_{i=1}^k a_i w_i\right\| &\geq \left\|\sum_{i=1}^j a_i w_i\right\| - \left\|\sum_{i=j+1}^k a_i w_i\right\| \\ &\geq \left\|\sum_{i=1}^j a_i y_i\right\| - \left\|\sum_{i=1}^j a_i (w_i - y_i)\right\| - \sum_{i=j+1}^k 3^{-i}M \\ &\geq |a_j| - r\sum_{i=1}^j |a_i| - 3^{-j}M/2 \\ &\geq M(3^{-j}/2 - r/2) \geq (3^{-k}/4)M,\end{aligned}$$

where for the first term of the third inequality we used the definition of nice basis. In particular if $\sum a_i w_i \in B$, we have $|M| \leq 4 \cdot 3^k$ so that $|a_i| \leq 4 \cdot 3^{k-i}$. Now given $w \in W \cap B$, write $w = \sum a_i w_i$ and let $y = \sum a_i y_i$. Then $\|w - y\| \leq \sum |a_i|\|w_i - y_i\| \leq 2\cdot 3^k r$. Rescaling y to move it inside B if necessary we obtain a point $y' \in Y \cap B$ with $\|w - y'\| \leq 4\cdot 3^k r$. It follows that $\max_{w\in W\cap B} d(w, Y\cap B) \leq 4 \cdot 3^k r$ as required.

A similar argument using the proof of Lemma B.1 gives the result in $\mathcal{G}^k$. ☐

Lemma B.8. *If $\epsilon < 2^{-k-2}$, the set $NB_k^\epsilon(X) \subset X^k$ is open.*

Proof. Let $(y_1, \ldots, y_k) \in NB_k^\epsilon(X)$. Let $\epsilon' < \epsilon$ be such that for every $1 \leq j \leq k$, $1 - \epsilon' \leq \|y_j\| \leq 1 + \epsilon'$ and $d(y_j, \mathrm{span}(y_1, \ldots, y_{j-1}) \geq 1 - \epsilon'$.

Let $r > 0$ and $(w_1, \ldots, w_k) \in \Pi_{i=1}^k B_r(y_i)$. Then, for every $1 \leq j \leq k$ we have $1 - \epsilon' - r \leq \|w_j\| \leq 1 + \epsilon' + r$ and, if $r > 0$ is sufficiently small, Lemmas B.5 and B.7 imply that $d(\mathrm{span}(y_1, \ldots, y_j), \mathrm{span}(w_1, \ldots, w_j)) \leq 2^{j+4}3^j r$.

It follows from the triangle inequality that

$$\begin{aligned}d(w_j, \mathrm{span}(w_1, \ldots, w_{j-1})) \geq& d(y_j, \mathrm{span}(y_1, \ldots, y_{j-1})) - \|y_j - w_j\| \\ &- 2\|w_j\| d(\mathrm{span}(y_1, \ldots, y_j), \mathrm{span}(w_1, \ldots, w_j)).\end{aligned}$$

Hence, $d(w_j, \mathrm{span}(w_1, \ldots, w_{j-1})) \geq 1 - \epsilon' - r - (1 + \epsilon' + r)2^{j+5}3^j r$. Thus, choosing r sufficiently small, the above yields that $(w_1, \ldots, w_k)$ satisfies all the conditions of an ϵ-nice basis. ☐

Lemma B.9. *(Disconnectedness of $\mathcal{G}(X)$). If $Y \in \mathcal{G}_j(X)$ and $dim(Y') > j$ then $d(Y, Y') \geq 2^{-j}/8$. If $Y \in \mathcal{G}^j(X)$ and $codim(Y') > j$ then $d(Y, Y') \geq 2^{-j}/16$.*

Proof. We have

$$d(Y, Y') = \max(\max_{y\in Y\cap B} d(y, Y' \cap B), \max_{y'\in Y'\cap B} d(y', Y \cap B)).$$

Suppose that the first term is less than $2^{-j}/8$. Let $y_1, \ldots, y_j$ be a nice basis for Y (as provided by Lemma B.3). Let $\|w_i - y_i\| < 2^{-j}/8$ and let W be the space spanned by the w_i. Lemma B.5 guarantees that $\sup_{y\in Y\cap B} d(y, W\cap B) < 1/2$. Since W is at most j-dimensional whereas $\dim(Y') > j$, let $z \in Y'\cap B$ satisfy $d(z, W) = 1$. Now for $y \in Y\cap B$, $d(z, W) \leq d(z, y) + d(y, W)$, so using $d(y, W) \leq 1/2$ we see that $d(z, y) \geq 1/2$. We have therefore shown that $\max_{y'\in Y'\cap B} d(y', Y) \geq 1/2$ under the assumption that the first term of $d(Y, Y')$ is small. In either case we see that $d(Y, Y')$ is bounded below by $2^{-j}/8$. The second statement follows from Lemma B.1. $\square$

Corollary B.10. *Suppose that $Y \in \mathcal{G}_k(X)$, $Y' \in \mathcal{G}_{k'}(X)$ and $\sup_{y\in Y\cap B} d(y, Y'\cap B) < \epsilon$. Then, if ϵ is sufficiently small, we have that $k' \geq k$. Similarly, suppose that $Y \in \mathcal{G}^k(X)$, $Y' \in \mathcal{G}^{k'}(X)$ and $\sup_{y\in Y\cap B} d(y, Y'\cap B) < \epsilon$. Then, if ϵ is sufficiently small, we have that $k' \leq k$.*

Proof. We present the proof of the second statement, which is the one used. The proof of the first one is entirely analogous.

Let $\epsilon < 3^{-k}/8$, and assume the hypotheses hold. We want to show that $k' \leq k$. Assume on the contrary that $k' > k$, and pick $\tilde{Y} \subset Y$ such that $\tilde{Y} \in \mathcal{G}^{k'}(X)$. Then, $\sup_{y\in \tilde{Y}\cap B} d(y, Y'\cap B) \leq \sup_{y\in Y\cap B} d(y, Y'\cap B) < \epsilon$. In view of Lemma B.7, $d(\tilde{Y}, Y') < 3^{k'}8\epsilon$, so $\sup_{y'\in Y'\cap B} d(y', Y\cap B) \leq \sup_{y'\in Y'\cap B} d(y', \tilde{Y}\cap B) < 3^{k'}8\epsilon$. So $d(Y, Y') < 3^{k'}8\epsilon$. If ϵ is sufficiently small, this contradicts Lemma B.9. Thus, $k' \leq k$. $\square$

Lemma B.11. *If X is separable, then $\mathcal{G}_k(X)$ is separable.*

Proof. Let $x_1, x_2, \ldots$ be a dense sequence in B. Then the collection of linearly independent k-element subsets of $\{x_1, x_2, \ldots\}$ is also countable. Let $Y \in \mathcal{G}_k(X)$. Let $y_1, \ldots, y_k$ be a nice basis for Y (as in Lemma B.3). Then given $\epsilon > 0$, let $(x_{i_j})_{j=1}^k \in B$ be chosen so that $\|x_{i_j} - y_j\| \leq \epsilon/2^{k+2}$. Let $W = \operatorname{span}(x_{i_1}, \ldots, x_{i_k})$. Lemma B.5 implies that $\max_{y\in Y\cap B} d(y, W) < \epsilon$. Then Lemma B.7 implies that $d(Y, W) \leq 4\cdot 3^k\epsilon$, so the separability is established. $\square$

Lemma B.12. *If X^* is separable, then $\mathcal{G}^k(X)$ is separable.*

Proof. Lemma B.1 implies that $\mathcal{G}^k(X)$ is homeomorphic to $\mathcal{G}_k(X^*)$. The result follows from Lemma B.11. $\square$

Lemma B.13. *Let $\Phi : L(X) \times \mathcal{G}(X) \to \mathcal{G}(X)$ be given by $\Phi(T, W) = T(W)$. For $0 \leq l \leq k$, let*

$$\tilde{G}(n, k, l) = \{(T, W) \in L_n(X) \times \mathcal{G}_k(X) : \dim(Ker\, T \cap W) \geq l\}.$$

This is a closed subset of $L_n(X) \times \mathcal{G}_k(X)$, where $L_n(X)$ is endowed with the restriction of the strong operator topology on $L(X)$. Also, let

$$\begin{aligned} G(n, k, l) &= \tilde{G}(n, k, l) \setminus \tilde{G}(n, k, l+1) \\ &= \{(T, W) \in L_n(X) \times \mathcal{G}_k(X) : \dim(Ker\, T \cap W) = l\}. \end{aligned}$$

Then, $\Phi|_{G(n,k,l)} : G(n, k, l) \to \mathcal{G}_{k-l}(X)$ is continuous.

Proof. To see that $\tilde{G}(n,k,l)$ is a closed set, suppose that $T \in L_n(X)$, $W \in \mathcal{G}_k(X)$ and $\dim(\mathrm{Ker}T \cap W) = s < l$. Let $\{Tw_1, \dots, Tw_{k-s}\}$ be a nice basis for $T(W)$. Let $M = \max \|w_i\|$. Let $U_\delta = \{S \in L_n(X) \colon \|S(w_i) - T(w_i)\| < \delta\}$ (a relatively open subset of $L_n(X)$), and let $V \in \mathcal{G}_k(X)$ be such that $d(W,V) < \delta$. Then in particular V contains elements $v_1, \dots, v_{k-s}$ such that $\|v_i - w_i\| < M\delta$. Now we have $\|S(v_i) - T(w_i)\| \leq \|S\|\|v_i - w_i\| + \|S(w_i) - T(w_i)\| \leq (nM+1)\delta$. By Lemma B.8, if $\epsilon > 0$ and δ is small enough, then $\{Sv_1, \dots, Sv_{k-s}\}$ is an ϵ-nice basis of $\mathrm{span}(Sv_1, \dots, Sv_{k-s}) \subset S(V)$. In particular, $S(V)$ has dimension at least $k-s$. It follows that $\dim(\mathrm{Ker}S \cap V) \leq s$, so $\tilde{G}(n,k,l)^c \subset L_n(X) \times \mathcal{G}_k(X)$ is open.

Let $(T,W) \in G(n,k,l)$, and let $r > 0$. Let $\{Tw_1, \dots, Tw_{k-l}\}$ be a nice basis for $T(W)$, and let M and U_δ as in the previous paragraph. We claim that if $\delta > 0$ is sufficiently small and $(S,V) \in \Big(U_\delta \times B_{\mathcal{G}}(W,\delta)\Big) \cap G(n,k,l)$, then $d(T(W), S(V)) < r$. Indeed, let $\epsilon > 0$. The previous argument shows that if δ is sufficiently small and $\|v_i - w_i\| < M\delta$ for each $1 \leq i \leq k-l$, then $\{Sv_1, \dots, Sv_{k-l}\}$ is an ϵ-nice basis for $S(V)$ such that $\|S(v_i) - T(w_i)\| \leq (nM+1)\delta$. Hence, by the proof of Corollary B.6, $d(T(W), S(V)) < 2^{k+2}(nM+1)\delta$. Taking $\delta < \frac{r}{2^{k+2}(nM+1)}$ yields the claim. $\square$

Corollary B.14. *The map $\Phi_k : L(X) \times \mathcal{G}_k(X) \to \mathcal{G}_{\leq k}(X)$ given by $\Phi_k(T,W) = T(W)$ is $(\mathcal{S} \otimes \mathcal{B}_{\mathcal{G}}, \mathcal{B}_{\mathcal{G}})$-measurable.*

Proof. Let us note that $L(X) \times \mathcal{G}_k(X) = \bigcup_{n \in \mathbb{N}} \bigcup_{l=0}^{k} G(n,k,l)$. Also, $G(n,k,l)$ is $\mathcal{S} \otimes \mathcal{B}_{\mathcal{G}}$ measurable, as $L_n(X)$ is $\mathcal{S}$ measurable, and $G(n,k,l)$ is the difference of two closed sets in $L_n(X) \times \mathcal{G}_k(X)$, by Lemma B.13. Since $\Phi|_{G(n,k,l)}$ is continuous again by Lemma B.13, then Φ_k is $(\mathcal{S} \otimes \mathcal{B}_{\mathcal{G}}, \mathcal{B}_{\mathcal{G}})$-measurable. $\square$

Lemma B.15. *Let X be separable, $\Pi : X \to Y$ a surjective bounded linear map, and $k \geq 0$. Then, the restriction of the induced map $\Pi_k : \mathcal{G}^k(X) \to \mathcal{G}(Y)$ to $\Pi^{-1}(\mathcal{G}^j(Y))$ is continuous for every $j \geq 0$. Furthermore, Π_k is measurable.*

Proof. First, we note that since Π is surjective, Y is separable and for every $W \in \mathcal{G}^k(X)$, $\Pi_k(W) \in \mathcal{G}^{\leq k}(Y)$.

For every $0 \leq j$, the set $\mathcal{G}^k(X) \cap \Pi_k^{-1}(\mathcal{G}^{\leq j}(Y))$ is relatively open in $\mathcal{G}^k(X)$. To see this, let $W \in \mathcal{G}^k(X) \cap \Pi_k^{-1}(\mathcal{G}^j(Y))$. By the above, $j \leq k$. Let $W' \in \mathcal{G}^k(X)$ be such that $d(W,W') \leq \delta$. By the open mapping theorem, there exists $r > 0$ such that $rB_{\Pi_k(W)} \subset \Pi_k(W \cap B_X)$. Let $z \in \Pi_k(W) \cap B_Y$. Then, there exist $w \in W \cap \frac{1}{r}B_X$ and $w' \in W' \cap \frac{1}{r}B_X$ such that $\Pi w = z$ and $\|w - w'\| \leq \frac{\delta}{r}$. Hence, $\|\Pi w - \Pi w'\| \leq \frac{\|\Pi\|\delta}{r}$. Thus,

$$\sup_{z \in \Pi_k(W) \cap B_Y} d(z, \Pi_k(W') \cap B_Y) \leq 2 \sup_{z \in \Pi_k(W) \cap B_Y} d(z, \Pi_k(W')) < \frac{2\|\Pi\|\delta}{r}.$$

If δ is sufficiently small, Corollary B.10 implies that $W' \in \Pi_k^{-1}(\mathcal{G}^{\leq j}(X))$.

It follows from the proof in the previous paragraph and Lemma B.7 that the restriction of Π_k to $\Pi_k^{-1}(\mathcal{G}^j(Y))$ is continuous.

The fact that $\Pi_k : \mathcal{G}^k(X) \to \mathcal{G}(Y)$ is measurable follows from the previous two paragraphs. $\square$

Lemma B.16. *The function $\nu : L(X) \times \mathcal{G}_k(X) \to \mathbb{R}$ given by $\nu(R, Y) = \|R|_Y\|$ is measurable when $L(X)$ is endowed with the strong σ-algebra $\mathcal{S}$.*

Proof. By Lemma A.2, it suffices to show continuity of ν as restricted to $L_n(X) \times \mathcal{G}_k(X)$, where $L_n(X)$ is endowed with the restriction of the strong operator topology. Let $Y \in \mathcal{G}_k(X)$ and let $R \in L_n(X)$. Let $\epsilon > 0$ and let $\delta < \epsilon/(n + k2^k)$. Let $\{y_1, \ldots, y_k\}$ be a nice basis for Y. Let $N = \{S \in L_n(X) \colon \|S(y_i) - R(y_i)\| < \delta\}$ and let $W \in \mathcal{G}_k(X)$ satisfy $d(W, Y) < \delta$. Now given $w \in B \cap W$, there exists a $y \in B \cap Y$ such that $\|w - y\| < \delta$ (or conversely given $y \in B \cap Y$, there exists a $w \in B \cap W$ such that $\|w - y\| < \delta$). We then have

$$\|S(w)\| \le \|R(y)\| + \|S(y) - R(y)\| + \|S(w) - S(y)\| \text{ and}$$
$$\|S(w)\| \ge \|R(y)\| - \|S(y) - R(y)\| - \|S(w) - S(y)\|.$$

It follows that $|\|S(w)\| - \|R(y)\|| \le \|S(y) - R(y)\| + \|S(w) - S(y)\|$.

The second term of the right side is bounded above by $n\delta$. For the first term, notice that by Lemma B.4, y may be expressed as a linear combination of y_i's with coefficients bounded above by 2^k. Hence the first term is bounded above by $k2^k\delta$. It follows that $|\|S(w)\| - \|R(y)\|| \le (n + k2^k)\delta$.

By taking $y \in Y \cap B$ for which $\|R(y)\| = \|R|_Y\|$ it follows that $\|S|_W\| > \|R|_Y\| - \epsilon$. Similarly taking $w \in W \cap B$ for which $\|S(w)\| = \|S|_W\|$ we obtain $\|R|_Y\| > \|S|_W\| - \epsilon$ so that $|\|R|_Y\| - \|S|_W\|| < \epsilon$ as required. $\square$

A pair of closed complemented subspaces Y, Z of X is called *complementary* if $Y \cap Z = \{0\}$ and $Y \oplus Z = X$. By the closed graph theorem, any pair of complementary spaces (Y, Z) specifies a bounded linear map $\Pi_{Y\|Z}$, which is the projection onto Y along Z, having kernel Z and image Y. By symmetry, the map $\Pi_{Z\|Y}$ is also a linear and bounded projection.

For $k \ge 0$, let

$$\mathrm{Comp}_k(X) = \{(Y, Z) \in \mathcal{G}_k(X) \times \mathcal{G}^k(X) : Y \cap Z = \{0\}, Y \oplus Z = X\},$$

and let $\mathrm{Comp}(X) = \bigcup_{k \ge 0} \mathrm{Comp}_k(X)$ be the set of complementary subspace pairs of X of finite dimension/codimension.

Lemma B.17. *Let $(Y, Z) \in Comp(X)$ and $Y' \in \mathcal{G}(X)$. Then,*

$$\|\Pi_{Z\|Y}|_{Y'}\| \le 2\|\Pi_{Z\|Y}\| d(Y, Y').$$

Proof. Let $y' \in Y'$ and $\epsilon > 0$. Let $y \in Y$ be such that $\|y' - y\| \le d(y', Y) + \epsilon$. Then,

$$\|\Pi_{Z\|Y}(y')\| = \|\Pi_{Z\|Y}(y' - y)\| \le \|\Pi_{Z\|Y}\|(d(y', Y) + \epsilon) \le \|\Pi_{Z\|Y}\|\big(2\|y'\| d(Y', Y) + \epsilon\big).$$

Letting $\epsilon \to 0$, the result follows. $\square$

Lemma B.18. *The map $\Psi : Comp(X) \to L(X)$ given by $\Psi(Y,Z) = \Pi_{Z\|Y}$ is continuous, where $Comp(X)$ carries the product topology induced by the metric on $\mathcal{G}(X)$ and $L(X)$ is endowed with the norm topology.*

Remark B.19. *Since the norm topology is finer than the strong operator topology on $L(X)$, Lemma B.18 yields that Ψ is also continuous when $L(X)$ is endowed with the strong operator topology.*

Proof of Lemma B.18. Let $\epsilon > 0$. Let $(Y,Z) \in \mathrm{Comp}(X)$ and $x \in X$. Since $\dim Y < \infty$, then $Y \cap \partial B$ is compact. Hence, $\zeta := \inf_{y \in Y \cap \partial B} d(y,Z) > 0$.

Let $\delta < \min\{\frac{1}{3}, \frac{\zeta}{8+2\zeta}\}$ and $(Y',Z') \in \mathrm{Comp}(Z)$ such that $d(Y,Y'), d(Z,Z') < \delta$. Then, $\inf_{y' \in Y' \cap \partial B} d(y',Z') \geq \frac{\zeta}{2}$. Indeed, let $y' \in Y' \cap \partial B$, and let $y \in Y \cap B$ be such that $\|y' - y\| < \delta$. Then, $1 - \delta < \|y\| < 1 + \delta$. Let $z \in Z$ be such that $\|y - z\| < d(y,Z) + \delta$. Then, $\|z\| \leq 2\|y\| + \delta < 3$ and

$$d(y',Z') \geq d(y,Z) - \|y - y'\| - d(z,Z') \geq \zeta(1-\delta) - \delta - 3\delta \geq \frac{\zeta}{2}.$$

We claim that $\|\Pi_{Y'\|Z'}\| < \frac{2}{\zeta}$ and $\|\Pi_{Z'\|Y'}\| < \frac{2}{\zeta}+1$. Indeed, let $x \in X \cap \partial B$, and write $x = y' + z'$, with $y' \in Y'$ and $z' \in Z'$. Then, $1 = \|y' + z'\| \geq d(y',Z') \geq \frac{\zeta}{2}\|y'\|$, so that $\|y'\| \leq \frac{2}{\zeta}$ and the first claim follows. The second claim follows from the triangle inequality.

Let $M = \max\{\|\Pi_{Y\|Z}\|, \|\Pi_{Z\|Y}\|\}$. Assume also that $\delta < \frac{\epsilon}{4M}\big(\frac{2}{\zeta} + 1\big)^{-1}$. Then, if $(Y',Z') \in \mathrm{Comp}(Z)$ is such that $d(Y,Y'), d(Z,Z') < \delta$, we have that

$$\begin{aligned}
\big|\|\Pi_{Z\|Y}\| - \|\Pi_{Z'\|Y'}\|\big| &\leq \|\Pi_{Z\|Y} - \Pi_{Z'\|Y'}\| \\
&\leq \|\big(\Pi_{Z\|Y} - \Pi_{Z'\|Y'}\big)|_{Z'}\|\|\Pi_{Z'\|Y'}\| + \|\big(\Pi_{Z\|Y} - \Pi_{Z'\|Y'}\big)|_{Y'}\|\|\Pi_{Y'\|Z'}\| \\
&\leq \|\Pi_{Y\|Z}|_{Z'}\|\|\Pi_{Z'\|Y'}\| + \|\Pi_{Z\|Y}|_{Y'}\|\|\Pi_{Y'\|Z'}\| \\
&\leq 2M(d(Z,Z') + d(Y,Y'))\big(\frac{2}{\zeta} + 1\big) < \epsilon,
\end{aligned}$$

where the third inequality follows from the fact that $\Pi_{Y\|Z} + \Pi_{Z\|Y} = \mathrm{Id}$, and the fourth one follows from Lemma B.17 and the claim above. ∎

Lemma B.20. *Let*

$$\begin{aligned}
NI(X) = \bigcup_{k,k' \geq 0} \{(Y,Z) \in \big(&\mathcal{G}_k(X) \times \mathcal{G}^{k'}(X)\big) \\
&\cup \big(\mathcal{G}^k(X) \times \mathcal{G}_{k'}(X)\big) \\
&\cup \big(\mathcal{G}_k(X) \times \mathcal{G}_{k'}(X)\big) : Y \cap Z = \{0\}\}
\end{aligned}$$

be the set of pairs of subspaces of X of finite dimension/codimension with trivial intersection.

Then, the map $\Psi' : NI(X) \to \mathcal{G}(X)$ be given by $\Psi'(Y,Z) = Y \oplus Z$ is continuous.

Proof. Let $(Y, Z) \in NI(X)$. Let $W \in \mathcal{G}(X)$ be such that $(Y \oplus Z, W) \in \mathrm{Comp}(X)$, so that $Y \oplus Z \oplus W = X$. Also, let $M = \max(\|\Pi_{Y\|Z\oplus W}\|, \|\Pi_{Z\|Y\oplus W}\|)$.

Let $\delta > 0$ and $Y', Z' \in \mathcal{G}(X)$ be such that $d(Y, Y'), d(Z, Z') < \delta$. Let $y \in Y, z \in Z$ be such that $\|y + z\| \leq 1$. Then $\|y\|, \|z\| \leq M$. Therefore, there exist $y' \in Y'$ and $z' \in Z'$ such that $\|y - y'\|, \|z - z'\| < M\delta$. Hence, $\|(y+z)-(y'+z')\| \leq \|y-y'\| + \|z-z'\| \leq 2M\delta$. Therefore, $\sup_{x\in(Y\oplus Z)\cap B} d(x, Y'\oplus Z') \leq 2M\delta$, and by the triangle inequality, $\sup_{x\in(Y\oplus Z)\cap B} d(x, (Y' \oplus Z') \cap B) \leq 4M\delta$. If δ is sufficiently small, Lemma B.9 implies that $(\mathrm{co})\dim Y = (\mathrm{co})\dim Y'$ and $(\mathrm{co})\dim Z = (\mathrm{co})\dim Z'$. Hence, using Lemma B.7, we get that $d(Y \oplus Z, Y' \oplus Z') < \tilde{M}\delta$, where $\tilde{M}$ depends on M, k and k'. Hence, Ψ' is continuous. $\square$

C Some facts from ergodic theory

C.1 A characterization of tempered maps

This appendix provides a characterization of tempered maps, based on the following theorem.

Theorem C.1 (Tanny)**.** *Let T be an ergodic measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. Let $f: X \to \mathbb{R}$ be a non-negative measurable function. Then either $f(T^n x)/n \to 0$ for μ-almost every x; or $\limsup f(T^n x)/n = \infty$ for μ-almost every x.*

The proof of the following lemma is based on a very concise proof of Tanny's theorem, attributed to Feldman, that appears in a Lyons, Pemantle and Peres [35].

Lemma C.2. *Let T be an invertible ergodic measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. Let $f: X \to \mathbb{R}$ be a non-negative measurable function. Then $f(T^{-n}x)/n \to 0$ for μ-almost every x as $n \to \infty$ if and only if $f(T^n x)/n \to 0$ for μ-almost every x as $n \to \infty$.*

Proof. Suppose that $f(T^{-n}x)/n \to 0$. Let $\epsilon > 0$. There exists for μ-almost every x an L such that $n \geq L$ implies $f(T^{-n}x)/n < \epsilon$. Fixing a sufficiently large L, the set $A = \{x : f(T^{-n}x)/n < \epsilon \text{ for all } n \geq L\}$ has measure at least $1/2$. Now we apply the Birkhoff ergodic theorem to $\mathbf{1}_A$. For almost every x, there exists an n_0 such that for $n \geq n_0$ one has $(1/n)(\mathbf{1}_A(x) + \ldots + \mathbf{1}_A(T^{n-1}x)) \in [2/5, 3/5]$. Fix such an x and let n_0 be the corresponding quantity. Then let $N > \max(n_0, 5L)$. We then have

$$\#\{0 \leq i < N : T^i(x) \in A\} \leq 3N/5.$$

On the other hand we have

$$\#\{0 \leq i < 2N : T^i(x) \in A\} \geq 4N/5.$$

It follows that there exists $i \in [N+L, 2N)$ with $T^i(x) \in A$. The fact that $T^i(x) \in A$ tells us that $f(T^N x) < \epsilon(i - N) < \epsilon N$. It follows that $f(T^N x)/N < \epsilon$. Since this holds for all large N and ϵ was arbitrary we deduce that $f(T^n x)/n \to 0$.

The converse statement follows immediately. $\square$

Combining the proof of Lemma C.2 with Tanny's theorem, we get the following.

Theorem C.3. *Let T be an invertible ergodic measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. Let $f\colon X \to \mathbb{R}$ be a non-negative measurable function. Then one of the following holds:*

- $f(T^n x)/n \to 0$ *for μ-almost every x as $n \to \pm\infty$; or*
- $\limsup_{n\to\infty} f(T^n x)/n = \infty$ *and* $\limsup_{n\to\infty} f(T^{-n} x)/n = \infty$ *for μ-almost every x.*

Proof. In view of Tanny's theorem, it is sufficient to show that if $f(T^{-n}x)/n \to 0$ a.e. then $f(T^n x)/n \to 0$ a.e. This follows from Lemma C.2. $\square$

C.2 Random version of Hennion's theorem

In this appendix, we present a result that allows us to bound the index of compactness and maximal Lyapunov exponent of some random dynamical systems satisfying certain Lasota-Yorke type inequalities. We remark that many parts of this lemma essentially appear in Buzzi [7]. We have modified the conclusion and weakened the hypotheses in one place. This result is based on the following theorem of Hennion [24].

Theorem C.4 (Hennion)**.** *Let $(X, \|\cdot\|)$ be a Banach space and suppose that Y is a closed subspace of X. Let Y be equipped with a finer norm $⦀.⦀$ (such that $\|y\| \le ⦀y⦀$ for all $y \in Y$) such that the inclusion of $(Y, ⦀\cdot⦀) \hookrightarrow (Y, \|\cdot\|)$ is compact. Suppose that $\mathcal{L}$ is a linear operator such that $\mathcal{L}(X) \subset X$ and $\mathcal{L}(Y) \subset Y$. Suppose further that for all $y \in Y$, one has the inequality*

$$⦀\mathcal{L}(y)⦀ \le A\|y\| + B⦀y⦀.$$

Then the index of compactness of $\mathcal{L}$ is bounded above by $2B$.

Lemma C.5. *Let $(X, \|\cdot\|)$ be a Banach space and let Y be a closed subspace. Let $⦀\cdot⦀$ be a finer norm on Y such that the inclusion of $(Y, ⦀\cdot⦀) \hookrightarrow (Y, \|\cdot\|)$ is compact. Let $\sigma\colon (\Omega, \mu) \to (\Omega, \mu)$ be an invertible ergodic measure preserving dynamical system and let $(\mathcal{L}_\omega)_{\omega\in\Omega}$ be a family of linear maps, each mapping X to X and Y to Y continuously. As usual, let $\mathcal{L}_\omega^{(n)} = \mathcal{L}_{\sigma^{n-1}\omega} \circ \ldots \circ \mathcal{L}_\omega$.*

Suppose we have the following inequalities:

$$\begin{aligned} &\text{(Strong L-Y)} \qquad ⦀\mathcal{L}_\omega f⦀ \le A(\omega)\|f\| + B(\omega)⦀f⦀ \text{ for all } f \in Y; \\ &\text{(Weak L-Y)} \qquad ⦀\mathcal{L}_\omega⦀ \le C(\omega), \end{aligned}$$

where $A(\omega)$, $B(\omega)$ and $C(\omega)$ are measurable functions, $C(\omega)$ is log-integrable and $\int \log B(\omega)\, d\mu(\omega) < 0$.

Then there exists a full measure subset $\Omega_1 \subset \Omega$ with the following properties:

1. $\lim_{n\to\infty} (1/n) \log ⦀\mathcal{L}_\omega^{(n)}⦀_{ic} \le \int \log B(\omega)\, d\mu(\omega)$ *for all $\omega \in \Omega_1$;*

2. For $\omega \in \Omega_1$, suppose that $f \in Y$ satisfies

$$\limsup_{n\to\infty}(1/n)\log \|\mathcal{L}_\omega^{(n)} f\| \le 0. \tag{9}$$

Then $\limsup_{n\to\infty}(1/n)\log |||\mathcal{L}_\omega^{(n)} f||| \le 0$.

Proof. For the first statement, notice that applying the strong Lasota-Yorke inequality we obtain inductively $|||\mathcal{L}_\omega^{(n)} f||| \le B(\sigma^{n-1})\dots B(\omega)|||f||| + D\|f\|$ for a constant D depending on ω and n. From Hennion's theorem, we deduce $|||\mathcal{L}_\omega^{(n)}|||_{\mathrm{ic}} \le 2B(\sigma^{n-1}\omega)\dots B(\omega)$. Taking logarithms, the conclusion then follows from the ergodic theorem.

We now show the second statement. There exists a $\delta > 0$ such that for any set S of measure at most δ, one has $\int_S(\log C - \log B)\,d\mu < -\int \log B\,d\mu$. Now since A is measurable, there exists a $K > 0$ such that $\mu(\{\omega\colon A(\omega) \ge K\}) < \delta$.

Set $\tilde{B}(\omega) = B(\omega)$ if $A(\omega) \le K$ and $C(\omega)$ otherwise. Set $\tilde{A}(\omega) = \min(A(\omega), K)$. We see that we have a hybrid Lasota-Yorke inequality obtained by applying the strong Lasota-Yorke inequality for cases in which $A(\omega) \le K$ and the weak inequality otherwise:

$$|||\mathcal{L}_\omega f||| \le \tilde{A}(\omega)\|f\| + \tilde{B}(\omega)|||f|||. \tag{10}$$

The advantage of this is that we still have $\int \log \tilde{B}\,d\mu < 0$ and $\tilde{A}$ is now uniformly bounded by K.

Applying the ergodic theorem (with the transformation being σ^{-1}) we obtain a measurable function $F(\omega)$ such that for $\mathbb{P}$-almost every ω, we have

$$\tilde{B}(\sigma^{-1}\omega)\dots\tilde{B}(\sigma^{-k}\omega) \le F(\omega) \text{ for all } k \ge 0. \tag{11}$$

Let $\beta > \int \log C$. Applying the ergodic theorem once more, we obtain, for $\mathbb{P}$-almost every ω, the bound

$$C(\sigma^{n+k-1}\omega)\dots C(\sigma^n\omega) \le H(\sigma^n\omega)e^{\beta k} \text{ for all } n, k \ge 0. \tag{12}$$

There exists a $B > 0$ such that $H(\omega)F(\omega) < B$ on a set of positive measure. By the ergodic theorem, for all $\delta > 0$, for almost every ω, there exists an n_0 such that

$$\forall N > n_0,\ \exists n \in [N(1-\delta), N) \text{ with} H(\sigma^n\omega)F(\sigma^n\omega) < B. \tag{13}$$

Now let Ω_1 be the set of full measure on which the conditions above hold. Fix an $\omega \in \Omega_1$ and let $f \in Y$ satisfy (9). Let $\epsilon > 0$ be arbitrary. Then by the hypotheses, there exists a constant L such that

$$\|\mathcal{L}_\omega^{(n)} f\| \le Le^{\epsilon n/2} \text{ for all } n \ge 0. \tag{14}$$

Now by iterating (10), we obtain the bound (valid for all $f \in Y$)

$$\|\mathcal{L}_\omega^n f\| \leq \tilde{B}(\sigma^{n-1}\omega)\ldots\tilde{A}(\omega)\|f\| + \tilde{B}(\sigma^{n-1}\omega)\ldots\tilde{B}(\sigma\omega)\tilde{A}(\omega)\|f\| + \\ \ldots + \tilde{B}(\sigma^{n-1}\omega)\tilde{A}(\sigma^{n-2}\omega)\|\mathcal{L}_\omega^{(n-2)}f\| + \tilde{A}(\sigma^{n-1}\omega)\|\mathcal{L}_\omega^{(n-1)}f\|.$$

Using the inequalities $\tilde{B}(\sigma^{n-1}\omega)\ldots\tilde{B}(\sigma^{n-k}\omega) \leq F(\sigma^n\omega)$ (from (11)), the fact that $\tilde{A}(\omega) \leq K$, and (14), we obtain an upper bound of the form

$$\|\mathcal{L}_\omega^n f\| \leq MF(\sigma^n\omega)e^{\epsilon n/2}, \tag{15}$$

for a suitable constant M.

Combining this with (12) we obtain

$$\|\mathcal{L}_\omega^{n+k} f\| \leq MF(\sigma^n\omega)H(\sigma^n\omega)e^{\epsilon n/2}e^{\beta k} \tag{16}$$

We can therefore obtain a bound for $\|\mathcal{L}_\omega^m f\|$ by minimizing the above over possible decompositions $m = n + k$. Let n_0 be as in (13) where δ is taken to be $\epsilon/(2\beta)$ and suppose $m > n_0$ is given. Then there exists a $k < \epsilon/(2\beta)m$ such that $F(\sigma^{m-k}\omega)H(\sigma^{m-k}\omega) \leq B$ so that

$$\|\mathcal{L}_\omega^m f\| \leq MBe^{\epsilon m/2}e^{\beta k} < MBe^{\epsilon m}.$$

It follows that $\limsup_{N\to\infty}(1/N)\|\mathcal{L}_\omega^{(N)} f\| \leq \epsilon$. Since ϵ is arbitrary, the proof is complete. $\square$

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