Stability and approximation of invariant measures of Markov chains in random environments *

Gary Froyland, and Cecilia González-Tokman

School of Mathematics and Statistics University of New South Wales Sydney NSW 2052, Australia e-mail: g.froyland@unsw.edu.au

ceciliagt@unsw.edu.au

Abstract: We consider finite-state Markov chains driven by stationary ergodic invertible processes representing random environments. Our main result is that the invariant measures of Markov chains in random environments (MCREs) are stable under a wide variety of perturbations. We prove stability in the sense of convergence in probability of the invariant measure of the perturbed MCRE to the original invariant measure. Our approach makes no assumptions on the transition matrix functions representing the Markov chains except measurability with respect to the random environment. We also develop a new numerical scheme to construct rigorous approximations of the invariant measures, which converge in probability as the resolution of the scheme increases. This numerical approach is illustrated with an example of a random walk in a random environment.

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1. Introduction

1.1. Set-up

Let $\{P(\omega), \omega \in \Theta\}$ be a family of Markov transition probabilities acting on a finite state space $(\mathcal{X}, \mathcal{A})$, where \mathcal{A} is the discrete σ -algebra. We call (Θ, \mathcal{B}) the set of *environments* on \mathcal{X} and doubly-infinite stochastic sequences $\vec{\omega} = \{\omega_n : n \in \mathbb{Z}\}$, taking values in Θ , random environments. Corresponding to this we have a forward stochastic sequence $\vec{X} = \{X_n : n \in \mathbb{Z}\}$ in \mathcal{X} . We require that $P(\omega : x, E)$ is $\mathcal{A} \times \mathcal{B}$ -measurable as a function of (x, ω) ; this is satisfied provided $P(\omega : x, y)$ is \mathcal{B} -measurable in ω for each $x, y \in \mathcal{X}$. If

$$P(X_{n+1} \in E | X_0, \dots, X_n, \vec{\omega}) = P(X_{n+1} \in E | X_n, \omega_n),$$

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then $(\vec{X}, \vec{\omega})$ is a Markov chain in a random environment (MCRE). Given a realisation $\vec{\omega}$ of the environmental sequence, the sample paths of the $X_n s$ in \mathcal{X} evolve as time non-homogeneous Markov chains with one-step transition probabilities from time n to n + 1 given by $P(\omega_n)$.

Using the shorthand $\Omega = \Theta_{-\infty}^{\infty} = \prod_{i=-\infty}^{\infty} \Theta$ and $\mathcal{F} = \mathcal{B}_{-\infty}^{\infty} = \prod_{i=-\infty}^{\infty} \mathcal{B}$, we define $\Psi = \mathcal{X} \times \Omega$, endowed with the σ -algebra $\mathfrak{F} = \mathcal{A} \times \mathcal{F}$. We assume that $\vec{\omega}$ has a stationary ergodic distribution π on Ω . Let σ be the coordinate shift, $(\sigma^k \vec{\omega})_n = (\vec{\omega})_{n+k}$. A transition probability on (Ψ, \mathfrak{F}) is defined by

$$Q(x,\vec{\omega}:\{y\}\times B) = P(\omega_0:x,y)\mathbf{1}_B(\sigma\vec{\omega}). \tag{1.1}$$

Let $\mu = \kappa \times \pi$, where κ is a counting measure on \mathcal{X} . The process $(\Psi, \mathfrak{F}, \mu, Q)$ is a Markov process in the sense of Foguel (see [6]). The process Q acts on measures ν on Ψ in the usual way: $\nu Q(F) = \int Q(\psi, F) d\nu(\psi)$. We call a measure ν on Ψ invariant if $\nu Q = \nu$. Furthermore, if $\nu \ll \mu$ is invariant, then its density $v := d\nu/d\mu$ satisfies the fixed point equation $\mathcal{L}(v) = v$, where $\mathcal{L} : L^1(\mu) \circlearrowleft$ is the linear operator defined by

$$\mathcal{L}v = d((v\mu)Q)/d\mu. \tag{1.2}$$

That is, if $\lambda(F) = \int_F v(\vec{\omega})\mu(d\vec{\omega})$, then $\mathcal{L}v$ is the density of λQ with respect to μ .

Cogburn [6] has proven existence of invariant measures $\nu \ll \mu$ for Markov chains in random environments. Our concern here is with the *stability* of such invariant measures to perturbations, both in the random environment and the family of transition probabilities.

We denote the finite state space as $\mathcal{X} = \{1, \ldots, d\}$, and for convenience introduce stochastic matrices $A(\vec{\omega})$ to denote the corresponding transition probabilities, as follows.

$$[A(\vec{\omega})]_{xy} := P(\omega_0 : x, y), \text{ for each } 1 \le x, y \le d,$$

where $[A]_{xy}$ denotes the x, y entry of the matrix A. Thus,

$$P^{n}(\omega_{0}:x,y) = [A(\vec{\omega})A(\sigma\vec{\omega})\dots A(\sigma^{n-1}\vec{\omega})]_{xy}, \text{ for each } n > 1, 1 \le x, y \le d.$$

1.2. Alternative viewpoint of the random environment and applications

In the formulation presented above, the random environment process $\sigma : \Omega \circlearrowleft$ is controlled by the left shift acting on a space of bi-infinite sequences. While this point of view is very broad, it is sometimes possible and more convenient to regard the environmental process as taking place on a more general probability space. This is the case, for instance, when the stationary process generating the environmental sequence $\vec{\omega}$ comes from an invertible ergodic transformation Ton a probability space Θ , with T preserving a probability measure $\hat{\pi}$.

A simple example where the probability space also has smooth structure, and at the same time illustrates the fact that no mixing conditions are imposed on the random environment, is: T is an irrational circle rotation and $\hat{\pi}$ is Lebesgue measure. Indeed, in this case, invertibility of T allows one to identify sequences $\vec{\omega}$ with single points $\omega \in \Theta$ by projecting on the 0-th entry, and in the other direction, by considering the full trajectory of $\omega \in \Theta$ under T, $\omega \mapsto \vec{\omega} = \{T^n \omega :$ $n \in \mathbb{Z}\}$. This procedure also provides an identification between π and $\hat{\pi}$. This possibility will be considered in various parts of §3 where, with a slight abuse of notation that should not confuse the reader, Ω and π will denote Θ and $\hat{\pi}$, respectively.

With this alternative viewpoint, Markov chains in random environments arise in analyses of time-dependent dynamical systems, such as models of stirred fluids [13] and circulation models of the ocean and atmosphere [8, 2, 4]. In these settings, one can convert the typically low-dimensional nonlinear dynamics into infinite-dimensional linear dynamics by studying the dynamical action on functions on the low-dimensional space, (representing densities of invariant measures with respect to a suitable reference measure). The driven infinite-dimensional dynamics is governed by cocycles of Perron-Frobenius operators. In numerical computations, these operators are often estimated by large sparse stochastic matrices [13] that involve perturbations in the form of discretisations of both the low-dimensional space and the random environment, resulting in a finitestate Markov chain in a random environment. Thus, the stability and rigorous approximation of invariant measures of Markov chains in random environments are important questions for applications in the physical and biological sciences. Other application areas include multiple-timescale systems of skew-product type (see eg. [27]), where the "random environment" is an aperiodic fast dynamics that drives the slow dynamics. In computations, both the fast and slow dynamics are approximated by discretised linear operators, leading to a Markov chain in a random environment.

1.3. Related results

Markov chains in random environments were considered in the 80's and 90's in a series of papers by Nawrotzki [23], Cogburn [5, 6] and Orey [25]. Central limit theorems [7] and large deviation results [32] have also been proved in this setting.

When Ω consists of a single point, one returns to the setting of a homogeneous Markov chain. The question of stability of the stationary distribution of homogeneous Markov chains under perturbations of the transition matrix has been considered by many authors [28, 22, 15, 14, 30, 31, 21]. These papers developed upper bounds on the norm of the resulting stationary distribution perturbation, depending on various functions of the unperturbed transition matrix, the unperturbed stationary distribution, and the perturbation. Our present focus is somewhat different: for Markov chains in random environments we seek to work with minimal assumptions on both the random environment and stochastic matrix function, and our primary concern is whether one can expect stability of

the invariant measures at all, and if so, in what sense. However, by enforcing stronger assumptions or requiring more knowledge about the driving process and the matrix functions, it may be possible to obtain bounds analogous to the homogeneous Markov chain setting.

Invariant measures of finite-state MCREs may be studied via a very powerful and general framework of so-called multiplicative ergodic theorems. When the matrices $A(\vec{\omega})$ are invertible, the celebrated multiplicative ergodic theorem (MET) of Oseledets [26] guarantees the π -a.e. existence of a measurable splitting of \mathbb{R}^d into equivariant subspaces, within which vectors experience an identical asymptotic exponential growth rate, known as a Lyapunov exponent. A recent extension [12] of the Oseledets theorem yields the same conclusion even when the matrices are not invertible, a situation that is relevant for MCREs. In the present setting of Markov chains in random environments, the maximal growth rate is $\log 1 = 0$, and the associated fastest growing Oseledets space corresponds exactly to the density $v(\vec{\omega})$ of the invariant measure of the MCRE with respect to μ .

In related work, Ochs [24] has linked convergence of Oseledets spaces to convergence of Lyapunov exponents in a class of random perturbations of general matrix cocycles. One of his standing hypotheses was that the matrices $A(\vec{\omega})$ were invertible, which is not a natural condition for the stochastic matrices in MCREs. For products of stochastic matrices, the top Lyapunov exponent is always 0, thus [24] yields convergence of the random invariant measures in probability, provided the matrices are invertible. The type of perturbations that we investigate generalise Ochs' "deterministic" perturbations in the context of stochastic matrices, which require Ω to be a compact topological space and σ to be a homeomorphism. Moreover, the arguments of Ochs do not easily extend to the noninvertible matrix setting. Our approach also enables the construction of an efficient rigorous numerical method for approximating the random invariant measure.

Another related result, regarding stability of so-called Oseledets splittings for semi-invertible matrix cocycles under iid perturbations, was recently obtained in [11]. The main result of [11] implies stability of invariant measures for MCREs with iid environment. The present paper does not impose the iid condition, indeed any ergodic random environment can be treated, and therefore the range of environmental processes we can handle is considerably richer.

1.4. Summary of results and outline of the paper

We demonstrate stability of invariant measures for MCRE, in the sense of convergence in probability. We show that the invariant measures are stable to the following types of perturbations:

1. Perturbing the random environment:

The environmental process σ is perturbed.

2. Perturbing the transition matrix function: The matrix function A is perturbed to a nearby matrix function.

3. Stochastic perturbations:

The system is perturbed by convolving with a stochastic kernel close to the identity.

4. Numerical schemes:

The system is perturbed by a Galerkin-type approximation scheme to numerically compute an estimate of the invariant measure.

An outline of the paper is as follows. In Section 2 we provide natural conditions under which the random invariant measure is unique. Section 3 proceeds through the four main types of perturbations listed above, deriving and confirming the necessary boundedness and convergence conditions. Numerical examples are given in Section 4. In Appendix A we present an abstract perturbation lemma that forms the basis of our results, and verify the hypotheses of this lemma in the stochastic matrix setting for the unperturbed MCRE. Appendix B collects the longer technical proofs.

2. Uniqueness of invariant measures of MCREs

In this section, we derive an easily verifiable condition for Q to have a unique invariant measure. Seneta [29] studied the *coefficient of ergodicity* in the context of stochastic matrices. We refer the reader to references therein for earlier appearances of related concepts.

Definition 2.1. Let M be a $d \times d$ stochastic matrix. The coefficient of ergodicity of M is

$$\tau(M) := \sup_{\|v\|_1 = 1, \sum_{i=1}^d [v]_i = 0} \|vM\|_1$$

One feature of this coefficient is that $1 \ge \tau(M) \ge \mu_2$, where μ_2 is the second eigenvalue of M. In particular, when $\tau(M) < 1$, the eigenspace corresponding to $\mu_1 = 1$ is one-dimensional. In the random case, an analogous statement holds.

Lemma 2.2 (Uniqueness). Suppose $\tilde{\tau}(P) := \int \tau(P(\vec{\omega})) d\pi(\vec{\omega}) < 1$. Then Q has at most one invariant probability measure.

The proof of Lemma 2.2 is deferred until Appendix B.1. The following consequence is relevant in our setting.

Corollary 2.3. Suppose there exist $\tilde{\Omega} \subset \Omega$ with $\pi(\tilde{\Omega}) > 0$ and $n \in \mathbb{N}$, such that for every $\vec{\omega} \in \tilde{\Omega}$, $P^{(n)}(\vec{\omega} : x, y) := [P(\vec{\omega}) \cdots P(\sigma^{n-1}\vec{\omega})]_{x,y}$ are positive. Then Q has at most one invariant measure.

Proof. The hypotheses of Lemma 2.2 hold for Q^n , giving a unique invariant probability measure for Q^n . Since invariant probability measures for Q are also invariant for Q^n , the claim follows.

3. Perturbations

In this section, we consider a given MCRE, encoded by a tuple $\mathfrak{M} = (\Omega, \mathcal{F}, \pi, \sigma, A, \mathbb{R}^d)$, as described in the introduction. We will refer to the MCRE as \mathfrak{M} . (Such an

 \mathfrak{M} is sometimes referred to as a *random dynamical system* [1].) We study stability properties of invariant measures under a variety of perturbations. The results rely on a general perturbation result, Lemma A.1, which is presented in Appendix A.

Throughout this section, we let \mathcal{V} be the Banach space of *d*-dimensional bounded measurable vector fields $v : \Omega \to \mathbb{R}^d$, with norm $||v||_* = \left|\sum_{i=1}^d |v_i|\right|_{L^{\infty}(\pi)} =$ $||v(\vec{\omega})|_{\ell^1}|_{L^{\infty}(\pi)}$. A non-negative element $v \in \mathcal{V}$ may be regarded as the density of a measure $\lambda \ll \mu$. That is, $\lambda(\cdot) = \int v_{\vec{\omega}}(\cdot) d\pi(\vec{\omega})$, where $v_{\vec{\omega}} = \delta_{\vec{\omega}} \times v(\vec{\omega})$ is supported on $\{\vec{\omega}\} \times \{1, \ldots, d\}$. The pre-dual of \mathcal{V} will be denoted by \mathcal{W} ; see Appendix A for details.

As introduced in §1, the linear operator $\mathcal{L} : L^1(\mu) \circlearrowleft$ associated with the MCRE \mathfrak{M} is defined in such a way that $\mathcal{L}v$ is the density of λQ with respect to μ . Thus, $\lambda \ll \mu$ is an invariant measure for the MCRE if and only if $\mathcal{L}v = v$. A useful characterisation of \mathcal{L} is given in Lemma A.5.

3.1. Perturbations of the random environment

We consider a sequence of environmental sequences governed by σ_n , n = 1, ..., that are "nearby" σ .

Proposition 3.1. Let $\{\mathfrak{M}_n\}_{n\in\mathbb{N}} = \{(\Omega, \mathcal{F}, \pi_n, \sigma_n, A, \mathbb{R}^d)\}_{n\in\mathbb{N}}$ be a sequence of MCREs, with π_n equivalent to π for $n \geq 1$, that satisfies

(I)
$$\lim_{n \to \infty} \left\| \frac{d\pi_n}{d\pi} \right\|_{L^{\infty}(\pi)} = 1.$$

(II) $\lim_{n \to \infty} \left\| \frac{d(\sigma_n^{-1})_*\pi}{d\pi} \right\|_{L^{\infty}(\pi)} = 1.$
(III) $\lim_{n \to \infty} \|g \circ \sigma - g \circ \sigma_n\|_{L^1(\pi)} = 0$ for each $g \in L^1(\pi).$

Then, for each $n \geq 1$, the MCRE \mathfrak{M}_n has an invariant measure with density $v_n \in \mathcal{V}$. Furthermore, there exists a subsequence of $\{v_n\}_{n \in \mathbb{N}}$ converging in probability to an \mathcal{L} -invariant $\tilde{v} \in \mathcal{V}$. Therefore, the measure ν , characterised by $\tilde{v} = d\nu/d\mu$, is invariant for the initial MCRE.

The proof of Proposition 3.1 will be deferred until §B.2.

Remark 3.2. In the context of Proposition 3.1, let us suppose the initial MCRE \mathfrak{M} has a unique invariant measure $\nu \ll \mu$ with density v; see §2 for a verifiable criterion for uniqueness. Then, the sequence of $\{v_n\}_{n\in\mathbb{N}}$ converges in probability to v, and it is not necessary to restrict to subsequences.

Remark 3.3.

- (i) Conditions (I) and (II) are automatically satisfied whenever σ and σ_n preserve a common ergodic invariant measure π .
- (ii) The function $\frac{d(\sigma_n^{-1})_*\pi}{d\pi}$, appearing in condition (II), is related to the Perron-Frobenius operator in dynamical systems. Whenever τ is non-singular with respect to π (that is, $\pi(A) = 0$ implies $\pi(\tau A) = 0$), then $\frac{d(\tau)_*\pi}{d\pi} = \mathcal{P}_{\tau} 1$,

where \mathcal{P}_{τ} : $L^{1}(\pi)$ is the Perron-Frobenius operator associated to τ (with respect to the reference measure π).

- (iii) Let us consider the alternative viewpoint of random environments introduced in §1.2. If σ is an ergodic invertible π preserving transformation of a metric space (Ω, ρ) , then $\sup_{\omega \in \Omega} \rho(\omega, \sigma_n \circ \sigma^{-1}\omega) \to 0$ as $n \to \infty$, implies $\|g \circ \sigma - g \circ \sigma_n\|_{L^1(\pi)}$ for any $g \in L^1(\pi)$ (eg. Corollary 5.1.1 [19]).
- (iv) The result here is considerably more general than Ochs [24] applied to stochastic invertible matrices. Ochs considers Ω a topological space, σ a homeomorphism, and A a continuous matrix function. Moreover, Ochs has a further requirement that, in our language, σ and σ_n , $n \ge 0$, all preserve π . The convergence result in our setting is equivalent to Ochs (convergence in probability).

Example 3.4. The following example fits in the alternative viewpoint of random environments introduced in §1.2. Let $\Omega = \mathbb{T}^D = \mathbb{R}^D/\mathbb{Z}^D$, the D-dimensional torus and σ be rigid rotation by an irrational vector $\alpha \in \mathbb{R}^D$, which preserves Ddimensional volume π . Let $\sigma_n(\omega) = \omega + \alpha_n$, $n \ge 0$ where $\alpha_n \in \mathbb{R}^D$ is irrational, and $\alpha_n \to \alpha$ as $n \to \infty$. Then for any given stochastic matrix function $A : \Omega \to \mathcal{M}_{d \times d}(\mathbb{R})$, one has $v_n \to v$ in probability. If, for example, the $A(\omega)$ represent a random walk on the finite set of states $\{1, \ldots, d\}$ where there is a positive probability to remain in place and walk both left and right for each ω , then $A^{(d-1)}(\omega)$ is a positive matrix for all $\omega \in \Omega$ and by Corollary 2.3 there is a unique invariant probability measure for the MCRE. See Section 4 for numerical computations.

3.2. Perturbations of the transition matrix function

We consider a sequence of matrix functions A_n , $n \in \mathbb{N}$, that are nearby A.

Proposition 3.5. Let $A_n : \Omega \to \mathcal{M}_{d \times d}(\mathbb{R})$ be a sequence of measurable stochastic matrix-valued functions that converge in measure to A; that is

$$\lim_{n \to \infty} \pi(\{\vec{\omega} \in \Omega : |A_n(\vec{\omega}) - A(\vec{\omega})|_{\ell^{\infty}} > \epsilon\}) \to 0 \text{ for each } \epsilon > 0.$$

For each $n \geq 1$, the MCRE $\{\mathfrak{M}_n\}_{n\in\mathbb{N}} := \{(\Omega, \mathcal{F}, \pi, \sigma, A_n, \mathbb{R}^d)\}_{n\in\mathbb{N}}$ has an invariant measure with density $v_n \in \mathcal{V}$. Furthermore, there exists a subsequence of $\{v_n\}_{n\in\mathbb{N}}$ converging in probability to an \mathcal{L} -invariant $\tilde{v} \in \mathcal{V}$. Therefore, the measure ν , characterised by $\tilde{v} = d\nu/d\mu$, is invariant for the initial MCRE.

If \mathfrak{M} has a unique invariant measure $\nu \ll \mu$, then one does not require subsequences of $\{v_n\}_{n\in\mathbb{N}}$ in Proposition 3.5; see Remark 3.2.

Proof of Proposition 3.5. Existence of v_n for π a.e. $\vec{\omega}$ follows from the Multiplicative Ergodic Theorem [12]. As in the proof of Proposition 3.1, the linear operators associated to \mathfrak{M}_n , $\mathcal{L}'_n : \mathcal{W} \circlearrowright$ are defined by $\mathcal{L}'_n f(\vec{\omega}) = A_n(\vec{\omega}) f(\sigma \vec{\omega})$ and by Lemma A.7 these are bounded. We need to check that $\|(\mathcal{L} - \mathcal{L}'_n)f\| \to 0$

as $n \to \infty$.

$$\begin{aligned} \|(\mathcal{L} - \mathcal{L}'_n)f\| &= \int \max_{1 \le i \le d} |[(A(\vec{\omega}) - A_n(\vec{\omega}))f(\sigma\vec{\omega})]_i| \ d\pi(\vec{\omega}) \\ &\le \int |A(\vec{\omega}) - A_n(\vec{\omega})|_{\ell^{\infty}} |f(\sigma\vec{\omega})|_{\ell^{\infty}} \ d\pi(\vec{\omega}). \end{aligned}$$

Define $\mathcal{A}_{\epsilon,n} = \{ \vec{\omega} \in \Omega : |A(\vec{\omega}) - A_n(\vec{\omega})|_{\ell^{\infty}} < \epsilon \}.$ Then

$$\begin{split} \int |A(\vec{\omega}) - A_n(\vec{\omega})|_{\ell^{\infty}} |f(\sigma\vec{\omega})|_{\ell^{\infty}} d\pi(\vec{\omega}) \\ &\leq \epsilon \int_{\mathcal{A}_{\epsilon,n}} |f(\sigma\vec{\omega})|_{\ell^{\infty}} d\pi(\vec{\omega}) + \int_{\mathcal{A}_{\epsilon,n}^c} |A(\vec{\omega}) - A_n(\vec{\omega})|_{\ell^{\infty}} |f(\sigma\vec{\omega})|_{\ell^{\infty}} d\pi(\vec{\omega}) \\ &\leq \epsilon ||f|| + 2 \int_{\mathcal{A}_{\epsilon,n}^c} |f(\vec{\omega})|_{\ell^{\infty}} d\pi(\vec{\omega}). \end{split}$$

Without loss, let f have unit norm ||f|| = 1, and select some $\delta > 0$. Then choosing $\epsilon = \delta/2$, there is an N such that for $n \ge N$, $\int_{\mathcal{A}_{\epsilon,n}^c} |f(\sigma \vec{\omega})|_{\ell^{\infty}} d\pi(\vec{\omega}) < \delta/4$ and thus $||(\mathcal{L} - \mathcal{L}'_n)f|| < \delta$ for $n \ge N$.

Hence, Lemma A.1 yields a weak-* limit for $\{v_n\}_{n\in\mathbb{N}}$ in $(\mathcal{V}, \|\cdot\|_*)$, and Lemma A.8 shows that convergence takes place in probability.

Remark 3.6. In the context of stochastic matrices, Proposition 3.5 is analogous to Ochs [24], except that we can additionally handle non-invertible matrices.

3.3. Stochastic perturbations

We now consider the situation where the MCRE is subjected to an averaging process. In this section we assume that Ω is a compact metric space with metric ρ . For example, if $\Omega = \Theta^{\mathbb{Z}}$, where $\Theta = \{1, \ldots, k\}$ has the discrete metric, $\rho'(i, j) = 1$ for $i \neq j$. Then, Ω is a compact metric space with metric $\rho(\vec{\omega}, \vec{\zeta}) := 2^{-n}$, where $n = \min\{|j| : \omega_j \neq \zeta_j\}$ (see e.g. [3, §1.4]).

For each $n \ge 1$ let $k_n : \Omega \times \Omega \to \mathbb{R}$ be a non-negative measurable function satisfying $\int k_n(\vec{\omega}, \vec{\zeta}) d\pi(\vec{\zeta}) = 1$ for a.a. $\vec{\omega} \in \Omega$. Define $(\mathcal{L}_n v)(\vec{\omega}) = \int k_n(\vec{\omega}, \vec{\zeta}) v(\sigma^{-1}\vec{\zeta}) A(\sigma^{-1}\vec{\zeta}) d\pi(\vec{\zeta})$. We first require existence of fixed points of \mathcal{L}_n , which are invariant measures of the corresponding MCRE.

Proposition 3.7. If for every $\vec{\omega}_1, \vec{\omega}_2 \in \Omega$, $\int |k_n(\vec{\omega}_1, \vec{\zeta}) - k_n(\vec{\omega}_2, \vec{\zeta})| d\pi(\vec{\zeta}) \leq K_n(\varrho(\vec{\omega}_1, \vec{\omega}_2))$, where K_n is a function satisfying $\lim_{x\to 0} K_n(x) = 0$, then \mathcal{L}_n has a fixed point $v_n \in \mathcal{V}$.

The proof of Lemma 3.7 is deferred to §B.2.

Example 3.8. The assumptions of Proposition 3.7 hold for the following natural random perturbations:

(i) An important example is finite-memory approximations of the random environment. Assume $\pi = \prod_{i=-\infty}^{\infty} p$, where p is a probability measure on Θ

with full support, and let $B_r(\vec{\omega})$ be the closed ball of radius r centred at $\vec{\omega}$. Then, the stochastic kernel

$$k_n(\vec{\omega},\vec{\zeta}) = \pi(\bar{B}_{2^{-n}}(\vec{\omega}))^{-1} \mathbf{1}_{\bar{B}_{2^{-n}}(\vec{\omega})}(\vec{\zeta}),$$

satisfies the desired conditions. The corresponding MCRE \mathfrak{M}_n can only "see" the stochastic sequences in the random environment on a window [-n, n], and the matrices $A_n(\vec{\omega})$ are applied in an average sense, depending only $\omega_{-n}, \ldots, \omega_n$.

- (ii) k_n is uniformly continuous and π is arbitrary. In this case, the assumptions of Proposition 3.7 follow immediately from the definition of uniform continuity.
- (iii) The following example fits in the alternative random environment setting introduced in §1.2. Let $\Omega = \mathbb{T}^D$, π be equivalent to Lebesgue measure, $d\pi/d(Leb)$ uniformly bounded above, and $k_n \in L^1(Leb)$. In this case, the statement follows from continuity of translations in $L^1(Leb)$ (e.g. [16, Theorem 13.24]). This includes discontinuous k_n , for example, $k_n(\omega,\zeta) = \pi(B_{\epsilon_n})^{-1} \mathbf{1}_{B_{\epsilon_n}}(\omega-\zeta)$, where $\mathbf{1}_{B_{\epsilon_n}}$ is the characteristic function of an ϵ_n -ball centred at the origin. The linear operator \mathcal{L}_n associated to the MCRE \mathfrak{M}_n is performing a local averaging over an ϵ_n -neighbourhood. Note that there are no assumptions on continuity of σ .

Proposition 3.9. Let $v_n \in \mathcal{V}$ be a fixed point of \mathcal{L}_n for $n \ge 0$ as guaranteed by Lemma 3.7. If for each $f \in L^1(\pi)$,

$$\lim_{n \to \infty} \int \left| \int k_n(\vec{\omega}, \vec{\zeta}) f(\vec{\zeta}) \ d\pi(\vec{\zeta}) - f(\vec{\omega}) \right| \ d\pi(\vec{\omega}) = 0, \tag{3.1}$$

then one may select a subsequence of $\{v_n\}_{n\in\mathbb{N}}$ converging in probability to $\tilde{v} \in \mathcal{V}$. Further, \tilde{v} is \mathcal{L} -invariant, and therefore the measure ν , characterised by $\tilde{v} = d\nu/d\mu$, is invariant for the initial MCRE.

If \mathfrak{M} has a unique invariant measure $\nu \ll \mu$, then one does not require subsequences of $\{v_n\}_{n\in\mathbb{N}}$ in Proposition 3.9; see Remark 3.2.

Proof of Proposition 3.9. Firstly, one may check that

$$\mathcal{L}'_n f(\vec{\omega}) = \int k_n(\sigma\vec{\zeta}, \sigma\vec{\omega}) A(\vec{\omega}) f(\sigma\vec{\zeta}) \ d\pi(\vec{\zeta}).$$

Now, for $f \in \mathcal{W}$,

$$\begin{aligned} \|\mathcal{L}'_{n}f - \mathcal{L}'f\| &= \int \left| \int k_{n}(\sigma\vec{\zeta},\sigma\vec{\omega})A(\vec{\omega})f(\sigma\vec{\zeta}) \ d\pi(\vec{\zeta}) - A(\vec{\omega})f(\sigma\vec{\omega}) \right|_{\ell^{\infty}} \ d\pi(\vec{\omega}) \\ &= \int \left| A(\sigma^{-1}\vec{\omega}) \left(\int k_{n}(\vec{\zeta},\vec{\omega})f(\vec{\zeta}) \ d\pi(\vec{\zeta}) - f(\vec{\omega}) \right) \right|_{\ell^{\infty}} \ d\pi(\vec{\omega}) \\ &\leq \int \left| \int k_{n}(\vec{\zeta},\vec{\omega})f(\vec{\zeta}) \ d\pi(\vec{\zeta}) - f(\vec{\omega}) \right|_{\ell^{\infty}} \ d\pi(\vec{\omega}), \end{aligned}$$

since $|A(\sigma^{-1}\vec{\omega})|_{\ell^{\infty}} \equiv 1$.

Lemma A.1 then yields a weak-* limit for $\{v_n\}_{n\in\mathbb{N}}$ in $(\mathcal{V}, \|\cdot\|_*)$, and Lemma A.8 shows that convergence takes place in probability.

Remark 3.10. The condition (3.1) is reminiscent of what has been called a "small random perturbation" by Khas'minskii [17] and later Kifer [18], in the context of deterministic dynamical systems governed by a continuous map $T : X \to X$. In this setting, one asks about whether limits of invariant measures of stochastic processes formed by small random perturbations are invariant under the deterministic map T. A sufficient condition for this to be the case is: for each continuous $f : X \to \mathbb{R}$,

$$\lim_{n \to \infty} \sup_{x \in X} \left| \int_X P_n(x, dy) f(y) - f(Tx) \right| = 0,$$
(3.2)

where $P_n: X \times \mathfrak{B}(X) \to [0,1]$ is a transition function $(\mathfrak{B}(X))$ is the collection of Borel-measurable sets in X).

3.4. Perturbations arising from a numerical Galerkin scheme

When one knows the invariant measure of the environmental process *a priori*, one can in principle use finite-memory approximations, as in Example 3.8(i), to perform finite computations of the invariant measure for the MCRE. In many applications, however, the explicit knowledge of such a measure is not available. The goal of this section is to present a numerical approach that is useful in the context of random environments taking values on a manifold, as discussed in $\S1.2$.

In this setup, Lebesgue measure may be taken as a reference measure, even when it is not necessarily preserved by the environmental process. This consideration will allow us to prove convergence results for approximations that are numerically computable.

Throughout this section, let us assume Ω is a compact smooth Riemannian manifold, and let m be the natural volume measure, normalised on Ω . Suppose $\sigma : \Omega \circlearrowleft$ is invertible and it preserves an ergodic measure π . We assume that $\pi \equiv m$ and that $h := d\pi/dm$ is uniformly bounded above and below. For each n, let \mathcal{P}_n be a partition of $\Omega \pmod{m}$ into n non-empty, connected open sets $B_{1,n}, \ldots, B_{n,n}$. We require that $\lim_{n\to\infty} \max_{1\leq j\leq n} diam(B_{j,n}) = 0$.

For each n, we define a projection $\Pi_n : \mathcal{V} \to \overline{\mathcal{V}}$ by

$$\Pi_n(v) = \sum_{i=1}^n \left(\frac{1}{m(B_i)} \int_{B_i} v(\omega) \ dm(\omega) \right) \mathbf{1}_{B_i}.$$

Using the standard Galerkin procedure we define a finite-rank operator $\mathcal{L}_n := \prod_n \mathcal{L} \prod_n$.

This approach is related to Ulam's method [33], a common numerical procedure for estimating invariant measures of dynamical systems. We will introduce a new modification of the Ulam approach to numerically estimate the invariant measures of the MCRE, or more precisely its density $v(\omega)$, simultaneously for each $\omega \in \Omega$. We consider our numerical method to be a perturbation of the original MCRE and apply our abstract perturbation machinery.

It will be useful to consider the *m*-predual of \mathcal{L} , which we denote \mathcal{L}'_m : $\int \mathcal{L}v \cdot f \, dm = \int v \cdot \mathcal{L}'_m f \, dm$. It is easy to verify that $\mathcal{L}'_m f = A(\omega)f(\sigma\omega)h(\omega)/h(\sigma\omega)$. We first consider condition (c) of Lemma A.1.

Lemma 3.11. The following conditions are satisfied.

1. $\mathcal{L}'_n f = \prod_n (\mathcal{L}'_m(\prod_n (f \cdot h)))/h.$ 2. \mathcal{L}'_n is bounded. 3. $\int |\mathcal{L}'_n f - \mathcal{L}' f|_{\ell^{\infty}} d\pi \to 0$ as $n \to \infty$ for each $f \in \mathcal{W}.$

We defer the proof of Lemma 3.11 to $\S B.2$

3.4.1. Numerical considerations

We wish to construct a convenient matrix representation of \mathcal{L}_n .

Lemma 3.12. Let $\Pi_n(v) = \sum_{i=1}^n v^i \mathbf{1}_{B_i}$ where $v^i \in \mathbb{R}^d$. Then the action of \mathcal{L}_n can be written

$$\Pi_n \mathcal{L} \Pi_n(v) = \sum_{j=1}^n \left(\sum_{i=1}^n v^i L_{n,ij} \right) \mathbf{1}_{B_j},$$
(3.3)

where

$$L_{n,ij} = \frac{\int_{B_j \cap \sigma B_i} A(\sigma^{-1}\omega) \, dm(\omega)}{m(B_j)}.$$
(3.4)

The proof of Lemma 3.12 is deferred to §B.2.

We note that if $A(\omega) = A_i$ (a fixed matrix) for $\omega \in B_i \cap \sigma^{-1}B_j$, then the expression (3.4) simplifies:

$$L_{n,ij} = \frac{\int_{B_j \cap \sigma B_i} A(\sigma^{-1}\omega) \, dm(\omega)}{m(B_j)}$$

= $\frac{\int_{B_j \cap \sigma B_i} A(\sigma^{-1}\omega)/h(\omega) \, d\pi(\omega)}{m(B_j)}$
= $\frac{\int_{\sigma^{-1} B_j \cap B_i} A(\omega)/h(\sigma\omega) \, d\pi(\omega)}{m(B_j)}$
= $A_i \frac{\int_{B_j \cap \sigma B_i} 1/h(\omega) \, d\pi(\omega)}{m(B_j)}$
= $A_i \frac{m(B_j \cap \sigma B_i)}{m(B_j)}.$

One could for example for $\omega \in B_i$ replace $A(\omega)$ with $\bar{A}_i = 1/m(B_i) \int_{B_i} A(\omega) dm(\omega)$ for i = 1, ..., n. Such a replacement would create an additional triangle inequality term in the proof of Lemma 3.11(3) to handle the difference $\bar{A}(\omega) - A(\omega)$, but using the argument of the proof of Lemma 3.5 we see that any replacement that converges to A in probability, including the \bar{A} replacement will leave the conclusion of Lemma 3.11 (3) unchanged. Thus, supposing that we have made such a replacement, and denoting $P_{ij} = \frac{m(B_j \cap \sigma B_i)}{m(B_j)}$, we may write

$$L_{n} = \begin{pmatrix} \frac{P_{11}A_{1} & P_{12}A_{1} & \cdots & P_{1n}A_{1}}{P_{21}A_{2} & P_{22}A_{2} & \cdots & P_{2n}A_{2}}\\ \vdots & \vdots & \ddots & \vdots\\ \hline P_{n1}A_{n} & P_{n2}A_{n} & \cdots & P_{nn}A_{n} \end{pmatrix},$$
(3.5)

where each block is a $d \times d$ matrix. We now show that there is a fixed point of L_n .

Lemma 3.13. For each n, the matrix L_n has a fixed point v_n .

Proof. Let $S_n^d = \{x \in \mathbb{R}^{nd} : x \ge 0, \sum_{i=(k-1)d+1}^{kd} x_i = 1, k = 1, \dots, n\}$. A fixed point exists by Brouwer: the set S_n^d is convex and compact, and is preserved by L_n . To see the latter, the first block of length d of the image of $x = [x^1|\cdots|x^n]$ under L_n is given by $\sum_{i=1}^n P_{ij} x^i A_i$, where x^i is the i^{th} block of length d. As each A_i is row-stochastic, the sum of the entries of $x_i A_i$ remains 1; further note that $\sum_{i=1} P_{ij} = 1$ by the definition of P, so the summation is simply a convex combination of the $x_i A_i$.

Numerically, one seeks a fixed point $v_n = [v^1 | v^2 | \cdots | v^n] L_{ij} = [v^1 | v^2 | \cdots | v^n]$. One can for example initialise with

$$v_n^0 := [(1/d, \dots, 1/d) | (1/d, \dots, 1/d) | \cdots | (1/d, \dots, 1/d)]$$

and repeatedly multiply by the (sparse) matrix L_n .

Lemma A.1 and Lemma A.8 yield the following.

Proposition 3.14. Let $v_n \in \mathcal{V}$ be constructed as a fixed eigenvector of L_n in (3.5), and considered to be a piecewise constant vector field on Ω , constant on each partition element $B_i, i = 1, ..., n$. Then, there exists a subsequence of $\{v_n\}_{n\in\mathbb{N}}$ converging in probability to an \mathcal{L} -invariant $\tilde{v} \in \mathcal{V}$. Therefore, the measure ν , characterised by $\tilde{v} = d\nu/d\mu$, is invariant for the initial MCRE.

If \mathfrak{M} has a unique invariant measure $\nu \ll \mu$, then one does not require subsequences of $\{v_n\}_{n\in\mathbb{N}}$ in Proposition 3.14; see Remark 3.2.

Remark 3.15. The expression (3.5) is related to the constructions in [10] (Theorem 4.2) and [9] (Theorem 4.8). In [10], the focus was on estimating the top Lyapunov exponent of a random matrix product driven by a finite-state Markov chain, rather than approximating the top Oseledets space. In [9], the matrices A were finite-rank approximations of the Perron-Frobenius operator, mentioned in the introduction, and one sought an equivariant family of absolutely continuous invariant measures of random Lasota-Yorke maps (piecewise C^2 expanding interval maps) with a Markovian random environment. In the present paper,

we are able to handle non-Markovian random environments and require no assumptions on the transition matrix function beyond measurability. Propositions 3.14 and 3.5 have enabled a very efficient numerical approximation of the invariant measure for the MCRE by exploiting perturbations in both the random environment and the transition matrix function.

4. Numerical Examples

In this section we illustrate our results with numerical experiments. To emphasise the fact that no independence is assumed in the random environmental process, we explore a class of MCREs where the random environment is given by an irrational circle rotation.

More precisely, let $\Omega = S^1$, $\alpha \notin \mathbb{Q}$, and $\sigma(\omega) = \omega + \alpha \pmod{1}$ for $x \in \Omega$. We set π to Lebesgue on S^1 ; σ preserves π and is ergodic. For each $\omega \in \Omega$, the matrix $A(\omega)$ describes a nearest-neighbour random walk on states $\{1, \ldots, d\}$.

To give a specific example, for 1 < i < d, we allow possible transitions to states i - 1, i, i + 1 with conditional probabilities $A_{i,i-1}(\omega), A_{i,i}(\omega), A_{i,i+1}(\omega)$ given by

$$\left\{ \begin{array}{ll} 0.8 - 1.2\omega, 0.1 + \omega, 0.1 + 0.2\omega, & 0 \le \omega < 1/2; \\ 0.3 - 0.2\omega, 1.1 - \omega, -0.4 + 1.2\omega, & 1/2 \le \omega \le 1. \end{array} \right.$$

For i = 1, $A_{i,i-1}(\omega) = 0$ and $A_{i,i}(\omega)$, $A_{i,i+1}(\omega)$ are given by

$$\begin{cases} 0.9 - 0.2\omega, 0.1 + 0.2\omega, & 0 \le \omega < 1/2; \\ 1.4 - 1.2\omega, 1.1 - \omega, -0.4 + 1.2\omega, & 1/2 \le \omega \le 1. \end{cases}$$

For i = d, $A_{i,i+1}(\omega) = 0$ and $A_{i,i-1}(\omega)$, $A_{i,i}(\omega)$ are given by

$$\begin{cases} 0.8 - 1.2\omega, 0.2 + 1.2\omega, & 0 \le \omega < 1/2; \\ 0.3 - 0.2\omega, 0.7 + 0.2\omega, & 1/2 \le \omega \le 1. \end{cases}$$

Roughly speaking, the closer ω is to 0, the greater the tendency to walk left; the closer ω is to 1, the greater the tendency to walk right; and the closer ω is to 1/2, the greater the tendency to remain at the current state. The matrix A is a continuous function of ω , except at $\omega = 0$, however, our theoretical results only require A to be a measurable function of ω , so we can also handle very irregular A.

Using n = 5000 partition elements for Ω and d = 10, we form the (sparse) matrix L_n in (3.5), and compute the fixed left eigenvector; each of these operations takes less than 1 second in MATLAB. Figure 1 shows a numerical approximation of the random invariant measure using the approach of Section 3.4. The ω -coordinates are along the x-axis, and for a fixed vector $v(\omega) \in \mathbb{R}^{10}$, the 10 components are plotted as differently coloured vertical bars. The value of $v(\omega)_i$ is equal to the height of the i^{th} coloured vertical bar at x-coordinate ω ; note the total height is unity for all $\omega \in \Omega$.

Let us consider first Figure 1(a), where $\alpha = 1/(20\sqrt{2}) \approx 0.0354$. This value of α represents a relatively slow evolution of random environment coordinates. The

peak probabilities to be in state 1, the left-most state (dark blue), occur around $\omega = 0.5$, after the driven random walk has been governed by many matrices favouring walking to the left (from $\omega = 0$ up to $\omega = 0.5$). Once the driving rotation passes $\omega = 0.5$, the random walk matrices now favour movement to the right, and probability of being in state 1 (dark blue) decreases, while the probability of being in state 10, the right-most state, (dark red) increases, the latter finally reaching a peak around $\omega = 1$. This high probability of state 10 continues for one more iteration of σ , but once ω again passes α , the probability of being in state 10 quickly declines as the matrices again favour movement to the left.

Figure 1(b) reduces the resolution of the approximation from 5000 bins on Ω to 500. One sees that the result is still very accurate, with only the very fine irregularities beyond the resolution of the coarser grid unable to be captured.

Figures 1(c), (d) show approximations of the invariant measure with an identical setup to Figure 1(a), except that $\alpha = 1/\pi, 1/\sqrt{2}$, respectively. These rotations are relatively fast and so one does not see the unimodal "hump" shape in Figure 1(a); nevertheless, it is clear that there is a complicated interplay between the driving map σ and the resulting invariant measures.

Appendix A: A general perturbation lemma and MCREs

We begin with an abstract stability result for fixed points of linear operators.

Lemma A.1. Let $(B, \|\cdot\|)$ be a separable normed linear space with continuous dual $(B^*, \|\cdot\|_*)$. Let $L: (B^*, \|\cdot\|_*) \oslash$ and $L_n: (B^*, \|\cdot\|_*) \oslash$, $n = 1, \ldots$ be linear maps satisfying

- (a) there exists a bounded linear map $L' : (B, \|\cdot\|) \circlearrowleft$ such that $(L')^* = L$,
- (b) for each $n \in \mathbb{N}$ there is a $v_n \in (B^*, \|\cdot\|_*)$ such that $L_n v_n = v_n$, which we normalise so that $||v_n||_* = 1$ (where $||v_n||_* = \sup_{f \in B, ||f||=1} |v_n(f)|$), (c) for each n = 1, ..., there exists a bounded linear map $L'_n : B \bigcirc$ such that
- $(L'_n)^* = L_n$, satisfying $||(L' L'_n)f|| \to 0$ as $n \to \infty$ for each $f \in B$.

Then there is a subsequence $v_{n_i} \in B^*$ converging weak-* to some $\tilde{v} \in B^*$; that is, $v_{n_i}(f) \to \tilde{v}(f)$ as $j \to \infty$. Moreover, $L\tilde{v} = \tilde{v}$.

Proof. The existence of a weak-* convergent subsequence follows from (b) and the Banach-Alaoglu Theorem. To show that $L\tilde{v} = \tilde{v}$, for $f \in B$ we write

$$(L\tilde{v} - \tilde{v})(f) = (L\tilde{v} - Lv_n)(f) + (Lv_n - L_nv_n)(f) + (v_n - \tilde{v})(f).$$
(A.1)

Writing the first term of (A.1) as $|(\tilde{v} - v_n)(L'f)|$ this term goes to zero by boundedness of L' and weak-* convergence of v_n to \tilde{v} . Writing the second term as $|v_n(L'-L'_n)(f)|$ and applying (c), we see this term vanishes as $n \to \infty$. The third term goes to zero as $n \to \infty$ by weak-* convergence of v_n to \tilde{v} .

Remark A.2. Condition (a) is equivalent to L bounded and weak-* continuous; that is, $Lv_i \to Lv$ weak-* if $v_i \to v$ weak-* where $v, v_i \in B^*$ (see eq. Theorem 3.1.11 [20]).

Remark A.3. Condition (b) of Lemma A.1 may be replaced by

(b')
$$\lim_{n \to \infty} \frac{\|L_n v_n - v_n\|_*}{\|v_n\|_*} = 0.$$

Indeed, this condition would only add an extra term in Equation (A.1), that would also vanish as $n \to \infty$.

A.1. Application to MCREs

We now begin to define the objects $(B, \|\cdot\|)$ and its dual, and the operator Land its pre-dual L' in the setting of MCREs. Let us recall that Ω is the space of bi-infinite sequences $\vec{\omega} = \{\omega_n : n \in \mathbb{Z}\}$ with entries $\omega_n \in \Theta$, and the shift $\sigma : \Omega \circlearrowleft$ is an invertible, ergodic π -preserving transformation of Ω .

Recall that \mathcal{V} denotes the Banach space of *d*-dimensional bounded measurable vector fields $v: \Omega \to \mathbb{R}^d$, with norm $\|v\|_* = \left|\sum_{i=1}^d |v_i|\right|_{L^{\infty}(\pi)} = ||v(\vec{\omega})|_{\ell^1}|_{L^{\infty}(\pi)}$. Associated with \mathcal{V} is the Banach space \mathcal{W} of *d*-dimensional integrable functions $f: \Omega \to \mathbb{R}^d$, with norm $\|f\| = |\max_{1 \le i \le d} |f_i||_{L^1(\pi)} = ||f|_{\ell^{\infty}}|_{L^1(\pi)}$.

Lemma A.4. $(\mathcal{V}, \|\cdot\|_*)$ and $(\mathcal{W}, \|\cdot\|)$ are Banach spaces and $(\mathcal{V}, \|\cdot\|_*) = (\mathcal{W}, \|\cdot\|)^*$.

Proof. Given Banach spaces X_i , one can identify $(X_1 \oplus \cdots \oplus X_d)^*$ with $(X_1^* \oplus \cdots \oplus X_d^*)$, and with the right identification $x^*(y) = \sum_{i=1}^d x_i^*(y_i)$, [20, Theorem 1.10.13].

For the norms, note that using the usual formula for $\|\cdot\|_*$ in terms of $\|\cdot\|$ one has

$$\begin{split} \|v\|_{*} &= \sup_{\|f\|=1} \left| \int v(\vec{\omega}) \cdot f(\vec{\omega}) \ d\pi(\vec{\omega}) \right| = \sup_{f:\int \max_{i} |f_{i}| \ d\pi \leq 1} \left| \int \sum_{i=1}^{d} v_{i}(\vec{\omega}) f_{i}(\vec{\omega}) \ d\pi(\vec{\omega}) \right| \\ &\leq \sup_{f:\int \max_{i} |f_{i}| \ d\pi \leq 1} \int \sum_{i=1}^{d} |v_{i}(\vec{\omega})| \max_{1 \leq i \leq d} |f_{i}(\vec{\omega})| \ d\pi(\vec{\omega}) \\ &\leq \left| \sum_{i=1}^{d} |v_{i}(\vec{\omega})| \right|_{L^{\infty}(\pi)} \cdot \sup_{f:\int \max_{i} |f_{i}| \ d\pi \leq 1} \left| \max_{1 \leq i \leq d} |f_{i}| \right|_{L^{1}(\pi)} = \left| \sum_{i=1}^{d} |v_{i}| \right|_{L^{\infty}(\pi)} . \end{split}$$

The reverse inequality may be obtained as follows. For each $j \in \mathbb{N}$, let $f_{j,i} = \frac{sgn(v_i(\omega))}{\pi(\Omega_j)} \mathbbm{1}_{\Omega_j}$, where $\Omega_j = \left\{ \omega \in \Omega : \sum_{i=1}^d |v_i(\omega)| \ge (1 - 1/j) \left| \sum_{i=1}^d |v_i| \right|_{L^{\infty}(\pi)} \right\}$. Then, $\|f_j\| = 1$ and

$$\left| \int v(\omega) \cdot f_j(\vec{\omega}) \, d\pi(\vec{\omega}) \right| = \int \sum_{i=1}^d v_i(\vec{\omega}) f_{j,i}(\vec{\omega}) \, d\pi(\vec{\omega})$$
$$= \frac{1}{\pi(\Omega_j)} \int_{\Omega_j} \sum_{i=1}^d |v_i(\omega)| d\pi(\vec{\omega}) \ge (1 - 1/j) \left| \sum_{i=1}^d |v_i| \right|_{L^{\infty}(\pi)}.$$

Letting $j \to \infty$, we get that $||v||_* \ge \left|\sum_{i=1}^d |v_i|\right|_{L^{\infty}(\pi)}$.

Let $\mathcal{L} : \mathcal{V} \to \mathcal{V}$ be the linear operator defined in equation (1.2) by $\mathcal{L}v = d((v\mu)Q)/d\mu$, where Q is as defined in equation (1.1). The following characterisation of \mathcal{L} will be used repeatedly.

Lemma A.5. The action of the linear operator \mathcal{L} on $v \in L^1(\mu)$ may be expressed as follows:

$$(\mathcal{L}v)(\vec{\omega}) = v(\sigma^{-1}\vec{\omega})A(\sigma^{-1}\vec{\omega}).$$

Proof. Let $\lambda = v\mu$. Then, by definition, $\mathcal{L}v = d(\lambda Q)/d\mu$. Let us compute

$$\begin{split} \lambda Q(\{y\} \times B) &= \int Q((x,\vec{\omega}),\{y\} \times B)[v(\vec{\omega})]_x d\mu(x,\vec{\omega}) \\ &= \int [A(\vec{\omega})]_{x,y} \mathbf{1}_B(\sigma\vec{\omega})[v(\vec{\omega})]_x d\mu(x,\vec{\omega}) \\ &= \int [A(\sigma^{-1}\vec{\omega})]_{x,y} \mathbf{1}_B(\vec{\omega})[v(\sigma^{-1}\vec{\omega})]_x d\mu(x,\vec{\omega}) \\ &= \int [v(\sigma^{-1}\vec{\omega})A(\sigma^{-1}\vec{\omega})]_z \mathbf{1}_{\{y\} \times B}(z,\vec{\omega}) d\mu(z,\vec{\omega}) \end{split}$$

where the third equality follows from the fact that π is σ -invariant and $\mu = \kappa \times \pi$. Thus, $d(\lambda Q)/d\mu = v(\sigma^{-1}\vec{\omega})A(\sigma^{-1}\vec{\omega})$.

Lemma A.6. The operator \mathcal{L} has $\|\cdot\|_*$ -norm 1.

Proof. Using Lemma A.5, one has

$$\|\mathcal{L}v\|_{*} = \left|\sum_{i=1}^{d} |[\mathcal{L}v]_{i}|\right|_{\infty} = \left|\sum_{i=1}^{d} |[v \circ \sigma^{-1} \cdot A \circ \sigma^{-1}]_{i}|\right|_{\infty} \le \left|\sum_{i=1}^{d} |[v \circ \sigma^{-1}]_{i}|\right|_{\infty} = \|v\|_{*}$$

The inequality holds as $A(\vec{\omega})$ is row-stochastic for each $\vec{\omega}$, and the inequality is sharp if $v \ge 0$.

The following lemma shows that condition (a) of Lemma A.1 holds.

Lemma A.7. The operator $\mathcal{L}' : \mathcal{W} \to \mathcal{W}$ defined by $\mathcal{L}'f = A(\vec{\omega})f(\sigma\vec{\omega})$ satisfies $(\mathcal{L}')^* = \mathcal{L}$ and $\|\mathcal{L}'\| \leq 1$.

Proof. One has

$$(\mathcal{L}v)(f) = \int v(\sigma^{-1}\vec{\omega})A(\sigma^{-1}\vec{\omega})f(\vec{\omega}) \, d\pi(\vec{\omega}) = \int v(\vec{\omega})A(\vec{\omega})f(\sigma\vec{\omega}) \, d\pi(\vec{\omega}) = v(\mathcal{L}'f),$$

where $\mathcal{L}' f = A(\vec{\omega}) f(\sigma \vec{\omega})$. Moreover,

$$\|\mathcal{L}'f\| = \int \max_{1 \le i \le d} |[A(\vec{\omega})f(\sigma\vec{\omega})]_i| \ d\pi(\vec{\omega}) \le \int |A(\vec{\omega})|_{\ell^{\infty}} |f(\sigma\vec{\omega})|_{\ell^{\infty}} \ d\pi(\vec{\omega}) = \|f\|,$$

as each $A(\vec{\omega})$ is a row-stochastic matrix with $|A(\vec{\omega})|_{\ell^{\infty}} = 1$ ($|\cdot|_{\ell^{\infty}}$ is the maxrow-sum norm).

To conclude this section we have the following connection between weak-* convergence in L^{∞} and convergence in probability.

Lemma A.8. Let $g, g_n \in L^{\infty}(\pi)$ for $n \ge 0$. Then,

- 1. $g_n \to g \text{ weak-}^* \Rightarrow g_n \to g \text{ in probability.}$
- 2. If additionally, $|g_n|_{\infty}, n \ge 0$, are uniformly bounded then $g_n \to g$ weak-* $\Leftrightarrow g_n \to g$ in probability.

Remark A.9. In this paper our interest is in $v \in \mathcal{V}$ such that for each $\vec{\omega} \in \Omega$, $v(\vec{\omega})$ represents a $1 \times d$ probability vector (so $|v(\vec{\omega})|_{\ell^1} = 1$ for π -a.e. $\vec{\omega}$). Thus, $||v||_* = 1$ and we will always be in situation (2) of Lemma A.8 where statements of convergence may be regarded in either a weak-* sense or in probability. In the context of invariant measures for Markov chains in random environments, $v_n \to v$ in probability means $\lim_{n\to\infty} \pi(\{\vec{\omega} \in \Omega : |v_n(\vec{\omega}) - v(\vec{\omega})|_{\ell^1} > \epsilon\}) = 0$ for each $\epsilon > 0$.

Proof of Lemma A.8. Let us start with (1). After replacing g_n with $g_n - g$, it suffices to show that if $g_n \in L^{\infty}$ and $\int g_n f \ d\pi \to 0$ as $n \to \infty$ for each $f \in L^1$, then $|g_n| \to 0$ in probability. We show the contrapositive: Suppose $|g_n|$ does not converge to 0 in probability. We will show $\int |g_n| f \ d\pi$ does not converge to zero for every $f \in L^1$ and get a contradiction.

By assumption, there exists some $\epsilon > 0$ such that $\pi(\{\vec{\omega} : |g_n(\vec{\omega})| \ge \epsilon\}) \to 0$. So there exists a sequence of sets $E_n = \{|g_n| \ge \epsilon\}$ with $\pi(E_n) \to 0$ as $n \to \infty$. Thus, $\limsup_n \pi(E_n) > 0$. Note that $|g_n| \ge \epsilon \mathbf{1}_{E_n}$ for $n \ge 0$. Let $E = \limsup_n E_n$, note that Fatou's lemma yields $\pi(E) \ge \limsup_n \pi(E_n) > 0$, and set $f = \mathbf{1}_E$. Then $\int |g_n| f \, d\pi \ge \epsilon \int \mathbf{1}_{E_n} \mathbf{1}_E \, d\pi = \epsilon \pi(E_n \cap E)$ for all $n \ge 0$. To finish, we will show that $\limsup_n \pi(E_n \cap E) > 0$.

We proceed by contradiction. Let $0 < \delta$ be such that $\limsup_n \pi(E_n) > \delta$. Suppose $\limsup_n \pi(E_n \cap E) = 0$. Then, there exists $N_0 \in \mathbb{N}$ such that for every $n \geq N_0, \pi(E_n \cap E) < \delta/2$. From the definition of $\limsup_n x \in E$ if and only if there is an infinite sequence $\{n_j\}$ such that $x \in E_{n_j}$ for all $j \geq 1$. Thus, for every $x \notin E$ there exists $t_x \in \mathbb{N}$ such that $x \notin E_n$ for every $n > t_x$. Let $t: \Omega \to \mathbb{N} \cup \{\infty\}$ be the function $x \mapsto t_x$ if $x \notin E$ and $t(x) = \infty$ if $x \in E$. That is, t is the supremum of n such that $x \in E_n$. Since the sets E_n are measurable, so is t. Hence, there exists N_1 such that $\pi(x \in \Omega \setminus E : t_x > N_1) < \delta/2$.

Let $n > \max(N_0, N_1)$ be such that $\pi(E_n) > \delta$. On the one hand, we have $\pi(E_n \cap E) < \delta/2$. On the other hand, $\pi(E_n \setminus E) \le \pi(x \notin E : t_x \ge n) < \delta/2$. Thus, $\pi(E_n) < \delta/2 + \delta/2$, which yields a contradiction.

Now we show the remaining part of (2). Let $\epsilon > 0$. If $\pi(|g_n - g| > \epsilon) \to 0$, then $|\int f(g_n - g)d\pi| \leq ||f||_1 \epsilon + \delta_f(\epsilon) ||g_n - g||_{\infty}$, where $\delta_f(\epsilon) := \sup_{\{\Gamma:\pi(\Gamma) \leq \epsilon\}} \int_{\Gamma} |f| d\pi$. In particular, $\delta_f(\epsilon) \to 0$ as $\epsilon \to 0$, and the claim follows.

Appendix B: Uniqueness and stability proofs

B.1. Uniqueness proof

Proof of Lemma 2.2. We argue by contradiction. Suppose there exist $v_1, v_2 \in \mathcal{V}$ distinct fixed points of \mathcal{L} . The set of $\vec{\omega} \in \Omega$ such that $v_1(\vec{\omega})$ and $v_2(\vec{\omega})$ are linearly dependent is σ invariant, so by ergodicity of σ it has measure 0 or 1. The latter is ruled out because v_1 and v_2 are distinct normalised fixed points. Hence, $v_1(\vec{\omega}) - v_2(\vec{\omega}) \neq 0$ for π -a.e. $\vec{\omega}$. Thus,

$$\tilde{\tau}(A) \ge \int \frac{\|(v_1 - v_2)(\vec{\omega})A(\vec{\omega})\|_1}{\|(v_1 - v_2)(\vec{\omega})\|_1} d\pi$$
$$= \int \frac{\|(v_1 - v_2)(\sigma\vec{\omega})\|_1}{\|(v_1 - v_2)(\vec{\omega})\|_1} d\pi.$$

However, the last expression is bounded below by 1, as the following sublemma, applied to $f(\vec{\omega}) = ||(v_1 - v_2)(\vec{\omega})||_1$, shows.

Sublemma B.1. Let $f \ge 0$ be such that $\frac{f \circ \sigma}{f}$ is a π -integrable function. Then,

$$\int \frac{f(\sigma\vec{\omega})}{f(\vec{\omega})} d\pi \ge 1.$$

Proof. Jensen's inequality yields

$$\int \frac{f(\sigma \vec{\omega})}{f(\vec{\omega})} d\pi \ge \exp\left(\int \log f(\sigma \vec{\omega}) - \log f(\vec{\omega}) d\pi\right) = 1,$$

where the equality follows from σ -invariance of π .

B.2. Stability proofs

Proof of Proposition 3.1. The strategy of proof is to verify the hypotheses of Lemma A.1, including the slight variation of hypothesis (b) given in Remark A.3. This will yield a weak-* limit for $\{v_n\}_{n\in\mathbb{N}}$ in $(\mathcal{V}, \|\cdot\|_*)$, and Lemma A.8 then shows that in fact convergence takes place in probability.

In view of Lemma A.5, if $\mu_n = \kappa \times \pi_n$ and $\lambda = v\mu_n$, then the density of λQ_n with respect to μ_n is given by $(\tilde{\mathcal{L}}_n v)(\vec{\omega}) = v(\sigma_n^{-1}\vec{\omega})A(\sigma_n^{-1}\vec{\omega})$. For each *n*, the existence of a fixed point of $\tilde{\mathcal{L}}_n$ is guaranteed at π a.a. $\vec{\omega}$ by the Multiplicative Ergodic Theorem [12]. Let us call such a fixed point v_n .

Since π_n and π are equivalent, we can also study the evolution under \mathfrak{M}_n of densities with respect to μ . Let us call this operator \mathcal{L}_n . That is, $\mathcal{L}_n v = d(v\mu Q_n)/d\mu$. Let $\rho_n = d\pi_n/d\pi$. It is straightforward to check that

$$\mathcal{L}_n v(\vec{\omega}) = \rho_n(\vec{\omega}) \tilde{\mathcal{L}}_n(\rho_n^{-1}(\vec{\omega})v(\vec{\omega})) = \frac{\rho_n(\vec{\omega})}{\rho_n(\sigma_n^{-1}\vec{\omega})} v(\sigma_n^{-1}\vec{\omega}) A(\sigma_n^{-1}\vec{\omega}).$$

In particular,

$$|\mathcal{L}_n v_n(\vec{\omega}) - v_n(\vec{\omega})|_{\ell^1} \le \left| \frac{\rho_n(\vec{\omega})}{\rho_n(\sigma_n^{-1}\vec{\omega})} - 1 \right| |v_n(\vec{\omega})|_{\ell^1}.$$
(B.1)

Thus,

$$\|\mathcal{L}_{n}v_{n} - v_{n}\|_{*} \leq \left\|\frac{\rho_{n}(\vec{\omega})}{\rho_{n}(\sigma_{n}^{-1}\vec{\omega})} - 1\right\|_{L^{\infty}(\pi)} \|v_{n}\|_{*}.$$
 (B.2)

Hence, hypothesis (I) of Proposition 3.1 ensures that condition (b') of Remark A.3 is satisfied.

Now we verify condition (c) of Lemma A.1. Notice that

$$\int \mathcal{L}_n v(f) d\mu = \int \frac{\rho_n(\vec{\omega})}{\rho_n(\sigma_n^{-1}\vec{\omega})} v(\sigma_n^{-1}\vec{\omega}) A(\sigma_n^{-1}\vec{\omega}) f(\vec{\omega}) d\mu$$
$$= \int \frac{\rho_n(\sigma_n\vec{\omega})}{\rho_n(\vec{\omega})} v(\vec{\omega}) A(\vec{\omega}) f(\sigma_n\vec{\omega}) d(\mathrm{Id} \times \sigma_n^{-1})_* \mu$$

Thus, since $\frac{d(\mathrm{Id} \times \sigma_n^{-1})_* \mu}{d\mu}(j, \vec{\omega}) = \frac{d(\sigma_n^{-1})_* \pi}{d\pi}(\vec{\omega})$ for every $j \in \mathcal{X}$, we get that

$$\mathcal{L}'_n f(\vec{\omega}) = \frac{\rho_n(\sigma_n \vec{\omega})}{\rho_n(\vec{\omega})} A(\vec{\omega}) f(\sigma_n \vec{\omega}) \frac{d(\sigma_n^{-1})_* \pi}{d\pi}(\vec{\omega}).$$

Therefore,

$$\begin{aligned} \|(\mathcal{L}' - \mathcal{L}'_n)f\| &\leq \int \max_{1 \leq i \leq d} \left| [A(\vec{\omega})(f(\sigma\vec{\omega}) - f(\sigma_n\vec{\omega}))]_i \right| \, d\pi(\vec{\omega}) \\ &+ \left\| \frac{\rho_n(\sigma_n\vec{\omega})}{\rho_n(\vec{\omega})} \frac{d(\sigma_n^{-1})_*\pi}{d\pi}(\vec{\omega}) - 1 \right\|_{L^{\infty}(\pi)} \int \max_{1 \leq i \leq d} \left| [A(\vec{\omega})f(\sigma_n\vec{\omega})]_i \right| \, d\pi(\vec{\omega}) \end{aligned}$$

The first term goes to zero as $n \to \infty$ by (III), since the entries of $A(\vec{\omega})$ are bounded between 0 and 1. The second term also goes to zero by virtue of (I) and (II).

Proof of Lemma 3.7. Let $S_{+,1}^{d-1} = \{x \in \mathbb{R}^d : x \ge 0, \sum_{i=1}^d x_i = 1\}$, and $\mathcal{S}_{+,1}^{d-1} = \{v \in \mathcal{V} : v(\vec{\omega}) \in S_{+,1}^{d-1}, \vec{\omega} \in \Omega\}$. For $v \in \mathcal{S}_{+,1}^{d-1}$,

$$\sum_{i=1}^{d} [\mathcal{L}_{n}v(\vec{\omega})]_{i} = \int \sum_{i=1}^{d} [v(\sigma^{-1}\vec{\zeta})A(\sigma^{-1}\vec{\zeta})]_{i}k_{n}(\vec{\omega},\vec{\zeta}) \ d\pi(\vec{\zeta}) = \int k_{n}(\vec{\omega},\vec{\zeta}) \ d\pi(\vec{\zeta}) = 1,$$

for π -a.e. $\vec{\omega}$; thus \mathcal{L}_n preserves $\mathcal{S}_{+,1}^{d-1}$. For $v \in \mathcal{S}_{+,1}^{d-1}$ we have

$$\sum_{i=1}^{d} |[\mathcal{L}_n v(\vec{\omega}_1) - \mathcal{L}_n v(\vec{\omega}_2)]_i|$$

$$\leq \int \sum_{i=1}^{d} |[v(\sigma^{-1}\vec{\zeta})A(\sigma^{-1}\vec{\zeta})]_i(k_n(\vec{\omega}_1,\vec{\zeta}) - k_n(\vec{\omega}_2,\vec{\zeta}))| \ d\pi(\vec{\zeta})$$

$$\leq \int |k_n(\vec{\omega}_1,\vec{\zeta}) - k_n(\vec{\omega}_2,\vec{\zeta})| \ d\pi(\vec{\zeta})$$

$$\leq K_n(\varrho(\vec{\omega}_1,\vec{\omega}_2)).$$

Fixing n, let v_0 be an arbitrary element of $\mathcal{S}_{+,1}^{d-1}$ and define $v_m^n = (1/m) \sum_{l=0}^{m-1} \mathcal{L}_n^l v_0$. The sequence v_m^n of \mathbb{R}^d -valued functions is uniformly bounded coordinate-wise below by 0 and above by 1. Further,

$$\begin{split} \sum_{i=1}^{d} |[\mathcal{L}_{n}^{2}v(\vec{\omega}_{1}) - \mathcal{L}_{n}^{2}(\vec{\omega}_{2})]_{i}| \\ &\leq \int \int \sum_{i=1}^{d} |[v(\sigma^{-1}\vec{\rho})A(\sigma^{-1}\vec{\rho})A(\sigma^{-1}\vec{\zeta})]_{i} \\ &\cdot (k_{n}(\sigma^{-1}\vec{\zeta},\vec{\rho})(k_{n}(\vec{\omega}_{1},\vec{\zeta}) - k_{n}(\vec{\omega}_{2},\vec{\zeta})))| \ d\pi(\vec{\rho})d\pi(\vec{\zeta}) \\ &\leq \int \int k_{n}(\sigma^{-1}\vec{\zeta},\vec{\rho})|k_{n}(\vec{\omega}_{1},\vec{\zeta}) - k_{n}(\vec{\omega}_{2},\vec{\zeta})| \ d\pi(\vec{\rho})d\pi(\vec{\zeta}) \\ &\leq K_{n}(\varrho(\vec{\omega}_{1},\vec{\omega}_{2})). \end{split}$$

By induction, one has the same result for all powers of \mathcal{L}_n and so one has that the sequence v_m^n is equicontinuous coordinate-wise. By Arzela-Ascoli, we can extract a subsequence $v_{m_j}^n$ that converges uniformly to some \tilde{v}^n . We show that \tilde{v}^n is a fixed point of \mathcal{L}_n via a triangle inequality of the form $\|\tilde{v}^n - v_m^n\|_* + \|v_m^n - \mathcal{L}_n v_m^n\|_* + \|\mathcal{L}_n v_m^n - \mathcal{L}_n \tilde{v}^n\|_*$. The first term goes to zero by uniform convergence, the second by telescoping, and the last because $\|\mathcal{L}_n\| = 1$ and by uniform convergence of v_m^n to \tilde{v}^n .

Proof of Lemma 3.11 (1). We repeatedly use the fact that $\int \prod_n v \cdot f \, dm = \int v \cdot \prod_n f \, dm$.

$$\int \Pi_n \mathcal{L} \Pi_n v(\omega) \cdot f(\omega) \, d\pi(\omega) = \int \Pi_n \mathcal{L} \Pi_n v(\omega) \cdot f(\omega) h(\omega) \, dm(\omega)$$

$$= \int v(\omega) \cdot \Pi_n (\mathcal{L}'_m(\Pi_n(f(\omega)h(\omega)))) \, dm(\omega)$$

$$= \int v(\omega) \cdot \Pi_n (\mathcal{L}'_m(\Pi_n(f(\omega)h(\omega)))) / h(\omega) \, d\pi(\omega)$$

Proof of Lemma 3.11 (2).

$$\begin{aligned} \|\mathcal{L}'_{n}f\| &= \int \left| \Pi_{n} \left(\frac{A(\omega)\Pi_{n}(f(\sigma\omega)h(\sigma\omega))h(\omega)}{h(\sigma\omega)} \right) / h(\omega) \right|_{\ell^{\infty}} d\pi(\omega) \\ &= \int \left| \Pi_{n} \left(\frac{A(\omega)\Pi_{n}(f(\sigma\omega)h(\sigma\omega))h(\omega)}{h(\sigma\omega)} \right) \right|_{\ell^{\infty}} dm(\omega) \\ &\leq \frac{1}{\inf_{\omega \in \Omega} h(\omega)} \int \left| \frac{A(\omega)\Pi_{n}(f(\sigma\omega)h(\sigma\omega))h(\omega)}{h(\sigma\omega)} \right|_{\ell^{\infty}} dm(\omega), \quad \text{by Sublemma B.2} \\ &\leq \frac{1}{\inf_{\omega \in \Omega} h(\omega)} \int \left| \frac{\Pi_{n}(f(\sigma\omega)h(\sigma\omega))h(\omega)}{h(\sigma\omega)} \right|_{\ell^{\infty}} dm(\omega), \quad \text{since } |A(\omega)|_{\ell^{\infty}} \leq 1 \\ &\leq \frac{\sup_{\omega \in \Omega} h(\omega)}{\inf_{\omega \in \Omega} h(\omega)^{2}} \int |\Pi_{n}(f(\sigma\omega)h(\sigma\omega))|_{\ell^{\infty}} dm(\omega) \\ &\leq \frac{\sup_{\omega \in \Omega} h(\omega)}{\inf_{\omega \in \Omega} h(\omega)^{3}} \int |f(\sigma\omega)h(\sigma\omega)|_{\ell^{\infty}} dm(\omega), \quad \text{by Sublemma B.2} \\ &\leq \frac{\sup_{\omega \in \Omega} h(\omega)^{2}}{\inf_{\omega \in \Omega} h(\omega)^{4}} \int |f(\sigma\omega)|_{\ell^{\infty}} d\pi(\omega) \\ &\leq \frac{\sup_{\omega \in \Omega} h(\omega)^{2}}{\inf_{\omega \in \Omega} h(\omega)^{4}} \|f\|. \end{aligned}$$

Sublemma B.2. Let $f \in \mathcal{W}$. Then $\|\Pi_n f\| \leq (\inf_{\omega \in \Omega} h(\omega))^{-1} \|f\|$.

Proof. Without loss, we consider the situation where \mathcal{P}_n consists of a single element, namely all of Ω . The argument extends identically to multiple-element partitions.

$$\begin{aligned} \|\Pi_n f\| &= \int \max_{1 \le i \le d} \left| \int f_i \, dm \right| \, d\pi \\ &\le \int \max_{1 \le i \le d} \left(\int |f_i| \, dm \right) \, d\pi \\ &= \max_{1 \le i \le d} \left(\int |f_i| \, dm \right) \\ &\le \left(\inf_{\omega \in \Omega} h(\omega) \right)^{-1} \int \max_{1 \le i \le d} |f_i| \, d\pi \\ &= \left(\inf_{\omega \in \Omega} h(\omega) \right)^{-1} \|f\|. \end{aligned}$$

Proof of Lemma 3.11 (3). We will use the facts that $\|\Pi_n\| \leq (\inf_{\omega \in \Omega} h(\omega))^{-1}$

(Sublemma B.2) and $|A(\omega)|_{\ell^{\infty}} \leq 1$ for each $\omega \in \Omega$.

$$\begin{split} &\int \left| \Pi_n \left(\frac{A(\omega) \Pi_n(f(\sigma\omega)h(\sigma\omega))h(\omega)}{h(\sigma\omega)} \right) / h(\omega) - A(\omega)f(\sigma\omega) \right|_{\ell^{\infty}} d\pi(\omega) \\ &= \int \left| \Pi_n \left(\frac{A(\omega) \Pi_n(f(\sigma\omega)h(\sigma\omega))h(\omega)}{h(\sigma\omega)} \right) - A(\omega)f(\sigma\omega)h(\omega) \right|_{\ell^{\infty}} dm(\omega) \\ &\leq \int \left| \Pi_n \left(\frac{A(\omega) \Pi_n(f(\sigma\omega)h(\sigma\omega))h(\omega)}{h(\sigma\omega)} \right) - \Pi_n \left(A(\omega)f(\sigma\omega)h(\omega) \right) \right|_{\ell^{\infty}} dm(\omega) \\ &+ \int \left| \Pi_n \left(A(\omega)f(\sigma\omega)h(\omega) \right) - A(\omega)f(\sigma\omega)h(\omega) \right|_{\ell^{\infty}} dm(\omega) \\ &\leq \frac{1}{\inf_{\omega \in \Omega} h(\omega)} \int \left| \frac{A(\omega) \Pi_n(f(\sigma\omega)h(\sigma\omega))h(\omega)}{h(\sigma\omega)} - A(\omega)f(\sigma\omega)h(\omega) \right|_{\ell^{\infty}} dm \\ &+ \int \left| (\Pi_n(f(\sigma\omega)h(\omega)) - f(\sigma\omega)h(\omega)) \right|_{\ell^{\infty}} dm(\omega). \end{split}$$

The second term goes to zero as $n \to \infty$ as $|\Pi_n g - g|_{L^1(m)} \to 0$ for any $g \in L^1(m)$. Continuing with the first term,

$$(\mathbf{B.3}) \le \left(\inf_{\omega \in \Omega} h(\omega)\right)^{-1} \int \left| \Pi_n(f(\sigma\omega)h(\sigma\omega))h(\omega) - f(\sigma\omega)h(\sigma\omega)h(\omega) \right|_{\ell^{\infty}} dm(\omega),$$

which also goes to zero as $n \to \infty$ as above.

Proof of Lemma 3.12.

$$\Pi_{n}\mathcal{L}\Pi_{n}(v) = \sum_{j=1}^{n} \left(\frac{1}{m(B_{j})} \int_{B_{j}} \left(\sum_{i=1}^{n} v^{i} \mathbf{1}_{B_{i}}(\sigma^{-1}\omega) \right) A(\sigma^{-1}\omega) dm(\omega) \right) \mathbf{1}_{B_{j}}$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} v^{i} \frac{1}{m(B_{j})} \int_{B_{j}} \mathbf{1}_{\sigma B_{i}}(\omega) A(\sigma^{-1}\omega) dm(\omega) \right) \mathbf{1}_{B_{j}}$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} v^{i} \frac{\int_{B_{j} \cap \sigma B_{i}} A(\sigma^{-1}\omega) dm(\omega)}{M(B_{j})} \right) \mathbf{1}_{B_{j}}$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} v^{i} L_{n,ij} \right) \mathbf{1}_{B_{j}}.$$

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FIG 1. Numerical approximations of the invariant measure $\nu = \int v_{\omega} d\pi(\omega)$, where $\pi = Leb$ and $v_{\omega} = \delta_{\omega} \times v(\omega)$, for the Markov chain in a random environment described in Section 5.1. Shown are cumulative distributions of $v_n(\omega) \in \mathbb{R}^{10}$ vs. ω for different random environments (different α) and different numerical resolutions (different n). As $n \to \infty$, the $v_n(\omega)$ converge in probability to the π -a.e. unique collection $\{v(\omega)\}_{\omega \in \Omega}$, which is equivariant: $v(\omega)A(\omega) =$ $v(\sigma\omega) \pi$ -a.e.