Scaling laws for bubbling bifurcations

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Abstract

We establish rigorous scaling laws for the average bursting time for bubbling bifurcations of an invariant manifold, assuming the dynamics within the manifold to be uniformly hyperbolic. This type of global bifurcation appears in nearly synchronized systems, and is conjectured to be typical among those breaking the invariance of an asymptotically stable hyperbolic invariant manifold. We consider bubbling precipitated by generic bifurcations of a fixed point in both symmetric and non-symmetric systems with a codimension one invariant manifold, and discuss their extension to bifurcations of periodic points. We also discuss generalizations to invariant manifolds with higher codimension, and to systems with random noise.

1 Introduction

The goal of this article is to quantify how quickly an attracting invariant manifold with internally chaotic dynamics loses stability through a bubbling bifurcation in a certain class of systems. This type of bifurcation occurs when the invariant manifold ceases to be a asymptotically stable due to one of its embedded orbits becoming unstable in a direction transverse to the manifold [ABS96]. Under this circumstance, the invariant manifold can still attract a set of positive Lebesgue measure (and thus, support a physical measure). However, this attractor is extremely sensitive to small perturbations that make the manifold non-invariant. This scenario arises, for example, in physical systems with approximate (but not exact) symmetry, and can give rise to intermittent dynamics called *bubbling*, where a trajectory spends most of its time near the manifold but occasionally bursts away.

There are experimental results and formal calculations for particular models that predict scaling laws for the average time between bursts and the size of the perturbed attractor as a function of bifurcation parameters in generic bifurcation scenarios, see [VHO+96] and references therein. Our results make rigorous the theoretical predictions presented in [ZHO03], concerning scaling laws for the average interburst time in terms of parameters and positive Lyapunov exponents of the bifurcating orbit. We prove the validity of similar scaling laws for more general dynamical systems displaying bubbling bifurcations. These results are applicable to generic systems, as well as to systems that have inherent symmetries.

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The scaling laws we describe involve two parameters. One is a normal parameter as defined in [ABS96]. The normal parameter does not affect the invariant manifold nor the dynamics within it; but does affect the dynamics transverse to the manifold. The other parameter is a *symmetry-breaking* parameter that we call q, which when nonzero, perturbs trajectories from the manifold that is invariant for $q = 0$. An example with two such parameters is as follows.

$$(u_n, v_n) \mapsto (u_{n+1}, v_{n+1}) = (G(u_n, 0) + k(v_n - u_n), G(v_n, q) + k(u_n - v_n)). \quad (1)$$

Here, the invariant manifold for $q = 0$ is the synchronization manifold $u = v$. The coupling strength k is a normal parameter.

In this article, we consider a model of such systems in the form of skew-product as follows:

$$(x_n, y_n) \mapsto (x_{n+1}, y_{n+1}) = (T(x_n), F(x_n, y_n, p, q)), \quad (2)$$

where p and q are the parameters of the model. We consider x and y to be coordinates along and transverse to the invariant manifold, respectively, where we have made the simplifying assumption that the dynamics of x are independent of y and the parameters. We study the case of a uniformly hyperbolic base map T with an invariant SRB (or *physical*) measure μ.

For $q = 0$, we assume that the system has an invariant manifold $\{(x, y)|y = 0\}$, that the corresponding invariant measure $\mu \times \delta_0$ undergoes a bubbling bifurcation at a fixed point $(x^*, 0)$ at $p = 0$ and that p is a *normal parameter* in the sense of [ABS96]. More generally, our results apply to systems that can be written in the form (2) by a choice of coordinates in a neighborhood of an invariant manifold. We discuss the scope of this model in §3.2. In more general cases, the orbit losing stability could be a periodic orbit. At various points in this paper we discuss how to extend our results to this case.

For $q = 0$, trajectories in the basin of $\mu \times \delta_0$ visit every neighborhood of $(x^*, 0)$. For definiteness, we assume that $(x^*, 0)$ is stable to perturbations of y for $p < 0$ and unstable for $p > 0$. The results of [ABS96] imply that generically, when $p > 0$ is sufficiently small the invariant measure $\mu \times \delta_0$ still has a basin of attraction with positive Lebesgue measure. For $p > 0$ and $q \neq 0$, trajectories that come close to $(x^*, 0)$ can *burst* away from $y = 0$. Depending on the y dynamics, trajectories that burst may come back close $y = 0$ and repeat the bursting behavior, or they may remain away from $y = 0$. The former type of dynamics is called bubbling and the latter, transient dynamics. The existence of a physical or SRB measure for $p > 0$ and $q \neq 0$ and its dependence on parameters is a difficult question. Some results in this direction, in the context of partially hyperbolic diffeomorphisms, can be found in [Dol04] and references therein. Our results do not distinguish between bubbling and transient phenomena, and estimate the average time it takes for a trajectory initialized near $y = 0$ to burst for the first time. In the case of bubbling, we expect this average bursting time also to be representative of the average time between bursts. For the sake of exposition, we refer to the bifurcation that leads to bursts as a bubbling bifurcation, whether or not bubbling actually occurs.

The scaling of the average bursting time for small p and q depends on the type of bifurcation the fixed point $(x^*, 0)$ undergoes when $q = 0$ and p passes through 0. As in [ZHO03], we consider both the case of a generic transcritical bifurcation (Theorem 1) and of a generic pitchfork bifurcation (Theorem 2).

Our main results are stated in §2.2. We study two qualitatively different forms of bursting: *multiplicative* and *additive*. In the former case, bursts are driven by the dominant effect of the expansion parameter p. In the latter case, bursts occur due to the accumulation of perturbations to the system, quantified by q. (In [ZHO03], these were called drift-dominated and noise-dominated, respectively. We have adopted the new terminology to avoid possible confusion with other common uses of the former terms.) We also distinguish between hard and soft bifurcations. A hard bifurcation occurs when the maximum burst size changes suddenly as p increases, while in a soft bifurcation the maximum burst size increases gradually with p.

Besides providing a proof for the results predicted in [ZHO03], we extend the range of parameters over which the scaling law is valid, obtaining uniform bounds for the logarithm of the average bursting time, proportional to the sum of positive Lyapunov exponents of the bifurcating fixed point in the invariant manifold. Furthermore, we extend those results to more general dynamics: the scaling law is valid for skew-product systems with uniformly hyperbolic maps in the base variables (x) and for fiber (y) dynamics displaying a *generic* type of bifurcation explained in §2.1 (conditions (i)-(v)) and generalized in §3.2. These bifurcations include generic transcritical and pitchfork bifurcations of fixed points. Period-doubling bifurcations can also be treated with our tools, since the second power of a map with a period-doubling bifurcation gives a map with a pitchfork bifurcation. Notice that a saddle-node bifurcation is not possible because the normal parameter assumption ensures that the fixed point persists on both sides of the bifurcation. Transcritical bifurcations can occur when the system is not symmetric with respect to reflection about the invariant manifold, while systems with reflectional symmetry will commonly have pitchfork bifurcations. As an example, the coupled system (1) is symmetric for $q = 0$, but if one of the coupling terms is eliminated it becomes asymmetric.

The structure of the paper is as follows. In Section 2, we first set up a model system in §2.1 and derive a Taylor approximation to F that we use in the subsequent sections. We state the main results in §2.2, and discuss three generalizations in §2.3; the case of a periodic bifurcating orbit in 2.3.1, the case of multiple transverse directions in 2.3.2, and a case of systems with random perturbations in 2.3.3. In Section 3 we analyze the dynamics and bifurcation of the invariant manifold. In §3.1, we prove some quantitative results about recurrence in hyperbolic systems that are relevant for our tasks. In §3.2, we discuss the mechanism of bubbling bifurcations and a generalization of the model presented in §2.1 to which our results apply. In Section 4 we prove the main results. In §4.1 we establish upper and lower bounds for the average bursting time in the linear regime (where the nonlinear terms in the Taylor approximation are small). In §4.2 we complete the proofs, extending those results to the nonlinear setting.

2 Statement of results

2.1 The model

Throughout this paper, we assume that we have a dynamical system with a hyperbolic invariant manifold X that undergoes a bubbling bifurcation. In our context, a bubbling bifurcation will be understood as the one that occurs when a parameter crosses a value at which the invariant manifold loses asymptotic stability. This loss of stability is due to one embedded orbit becoming unstable. In the terminology of [ABS96], at the bifurcation, the normal Lyapunov exponent to the invariant manifold X becomes 0 on one orbit but remains negative on other orbits.

To separate the dynamics on X from the transverse dynamics, we will work with skew-product systems: we assume that (near X) the dynamical system can be written in the form (2), where $T : X \circlearrowleft$ is a transitive C^2 Anosov diffeomorphism or a uniformly expanding map, and F is C^{1+1} in x and is C^3 as a function of y, p and q. We let μ be the SRB measure for T (see 3.1.1).

For $q = 0$ we impose the following conditions:

(i) $F(x, 0, p, 0) = 0$ for all x and p, so that $X = \{(x, y)|y = 0\}$ is an invariant manifold.

(ii) $x^* \in X$ is a fixed point. We let Λ be the sum of positive Lyapunov exponents of x^* associated to T.

(iii) X is asymptotically stable for $p < 0$, $\frac{\partial F}{\partial y}(x, 0, 0, 0) > 0$ for all x, $\frac{\partial F}{\partial y}(x^*, 0, 0, 0) = 1$ and $\frac{\partial^2 F}{\partial p \partial y}(x^*, 0, 0, 0) > 0$, so that $p = 0$ is a bifurcation value corresponding to the loss of asymptotic stability of X.

We remark that the assumption that $\frac{\partial F}{\partial y}(x, 0, 0, 0) > 0$ always holds if the map (2) is a diffeomorphism, because then $\frac{\partial F}{\partial y}(x, 0, 0, 0)$ must be nonzero for all x, and if it is negative we consider the second iterate of (2). The following assumption related to (iii) is generalized to the non-degeneracy condition (iii") in §3.2.

(iii') The global maximum of $\frac{\partial F}{\partial y}(\cdot, 0, 0, 0)$ is unique and occurs at x^*. This implies that the orbit $(x^*, 0, 0, 0)$ is the only orbit becoming unstable as p passes through 0.

(iv) The bifurcation of the fixed point x^* as p goes through 0 is either a generic transcritical bifurcation (in the asymmetric case) or a generic pitchfork bifurcation (in the symmetric case, where $F(x, y, p, 0) = -F(x, -y, p, 0)$).

We also assume the non-degeneracy condition:

(v) $\frac{\partial F}{\partial q}(x^*, 0, 0, 0) \neq 0$, so that varying q from 0 breaks the invariance of X near x^*.

With these requirements in mind, our model takes the following form:

$$\begin{cases} x_{n+1} = T(x_n) \\ y_{n+1} = F(x_n, y_n, p, q) \\ \qquad = (f(x_n) + h(x_n)p)y_n + qg(x_n) + \mathcal{O}(qy + p^2 y + pq + q^2 + y^2), \end{cases} \tag{3}$$

where $f(x) = \frac{\partial F}{\partial y}(x, 0, 0, 0)$, $g(x) = \frac{\partial F}{\partial q}(x, 0, 0, 0)$, $h(x) = \frac{\partial^2 F}{\partial p \partial y}(x, 0, 0, 0)$. Notice that $\frac{\partial^k F}{\partial p^k}(x, 0, 0, 0) = 0$ for all $k \geq 1$ by condition (i) above.

For definiteness, we assume $q \geq 0$, and we think of q as the strength of the asymmetry in the system; we also refer to the term $qg(x)$ as the *kick*. By (iii) above, $f(x^*) = 1$ and $h(x^*) > 0$, and by the non-degeneracy condition (v), we have $g(x^*) \neq 0$. In fact, without loss of generality, we assume $g(x^*) = 1 = h(x^*)$. This amounts to possibly rescaling q and p, and possibly changing the sign of y.

For $p > 0$ and $q = 0$, the invariant manifold X is no longer asymptotically stable due to the fixed point x^* becoming unstable in a direction transverse to the manifold. However, since $0 < f(x) < 1$ for $x \neq x^*$, then most orbits close to X continue to be attracted to X. This is due to the fact that when p is small, the transverse dynamics is contracting outside a neighborhood of $x = x^*$.

Let $a(x) = \frac{1}{\rho!}\frac{\partial^\rho F}{\partial y^\rho}(x, 0, 0, 0)$, where $\rho \in \{2, 3\}$ corresponds to the most significant non-linearity of the dynamics of x^* for $q = 0$, that is, $\rho = 2$ for a transcritical bifurcation and $\rho = 3$ for a pitchfork bifurcation. Then $a(x^*) \neq 0$, and without loss of generality, we can rescale y to assume $a(x^*) = \pm 1$. The remaining higher order terms involve only higher powers of y, p and q.

If the system does not have inherent symmetry constraints, we have generically that $\rho = 2$, and the bifurcation that x^* goes through as p crosses 0 is a transcritical bifurcation. In this case, we can write:

$$F(x, y, p, q) = (f(x)+h(x)p)y+qg(x)+a(x)y^2+\mathcal{O}(qy+p^2y+pq+q^2+y^3), \quad (4)$$

with $a(x^*) \neq 0$.

On the other hand, if the system is symmetric with respect to changing the sign of y, or if x^* undergoes a period-doubling bifurcation and we consider the second iterate of the map, the generic value is $\rho = 3$ and the corresponding generic bifurcation for x^* is a pitchfork bifurcation. In this case, we can write:

$$F(x, y, p, q) = (f(x)+h(x)p)y+qg(x)+a(x)y^3+b(x)y^2+\mathcal{O}(qy+p^2y+pq+q^2+y^4), \quad (5)$$

with $a(x^*) \neq 0$ and $b(x^*) = 0$ (of course, $b(x) = 0$ for all x if $F(x, y, p, 0)$ is an odd function of y).

In both scenarios, the size of the bursts may be small and determined by the size of the perturbation parameters. We call this case a soft transition; it happens if $qa(x^*)g(x^*) < 0$ in the asymmetric case, and if $a(x^*) < 0$ in the symmetric case. If $qa(x^*)g(x^*) > 0$ in the asymmetric case or $a(x^*) > 0$ in the symmetric case, the size of bursts is not so limited; we call this case a hard transition.

2.2 Main results

In order to state the results, we introduce some notation. For a fixed threshold $Y > 0$ and $\{(x_n, y_n)\}_{n\in\mathbb{Z}_+}$ trajectory of (2), we define its bursting time as:

$$\tau(Y, x_0, y_0) = \min_{n\geq 0}\{|y_n| > Y\}.$$

Recall that μ is the SRB measure for $T : X \circlearrowleft$. For y_0 fixed, we define the average bursting time as:

$$\tau(Y, y_0) = \frac{1}{2y_0}\int_{X\times[-y_0,y_0]} \tau(Y, x, y)d\mu(x)dy.$$

Since perturbations from the invariant manifold $y = 0$ are proportional to q, we will generally consider y_0 to be of order q and set $\tau(Y) := \tau(Y, q)$. We will simply write τ when the threshold is clear from the context.

Remark 2.1. Our proofs also apply to the case where T is a nontransitive Anosov diffeomorphism or, more generally, an Axiom A diffeomorphism, with x^* belonging to a hyperbolic attractor $\mathcal{A}$. In this case, the basin of the SRB measure μ supported in $\mathcal{A}$ may no longer have full Lebesgue measure, and there may be other SRB measures for T, supported away from $\mathcal{A}$.

Our main result in the case of generic transcritical bifurcations ($\rho = 2$) is the following.

Theorem 1. Consider a family of skew product systems as in (3), with F as in (4) satisfying all conditions in § 2.1 above (3). Assume that $p, q > 0$. Then, there is a constant $\tilde{C} > 1$ and a threshold Y independent of p and q in the hard transition case ($qa(x^*)g(x^*) > 0$), and proportional to $\max(p, \sqrt{q})$ in the soft transition case ($qa(x^*)g(x^*) < 0$), such that the scaling of the bursting time satisfies:

- (Multiplicative case). For each $\epsilon > 0$, if $(p, \frac{q}{p^2})$ is sufficiently close to $(0,0)$ and $q \geq p^2 e^{-p\tilde{C}^{\frac{1}{p}}}$, then
$$(1-\epsilon)\Lambda < \frac{\log \tau(Y)}{\frac{1}{p}\log \frac{p^2}{q}} < (1+\epsilon)\Lambda.$$
- (Additive case). There exists $C > 0$ independent of p, q and the map T on X such that for $(q, \frac{p^2}{q})$ sufficiently close to $(0,0)$,
$$C^{-1}\Lambda \leq \frac{\log \tau(Y)}{\frac{1}{q^{\frac{1}{2}}}} \leq C\Lambda.$$

(Recall that Λ is the sum of positive Lyapunov exponents of the fixed point x^*.) This result is proved in §4.2.1.

In the case of pitchfork bifurcations, which are generic for symmetric systems ($\rho = 3$), the main result is:

Theorem 2. Consider a family of skew product systems as in (3), with F as in (5) satisfying all conditions in Section 2.1 above (3). Assume that $p, q > 0$. Then, there is a constant $\tilde{C} > 1$ and a threshold Y independent of p and q in the hard transition case ($a(x^*) > 0$), and proportional to $\max(\sqrt{p}, \sqrt[3]{q})$ in the soft transition case ($a(x^*) < 0$), such that the scaling of the bursting time satisfies:

- (Multiplicative case). For each $\epsilon > 0$, if $(p, \frac{q}{p^{\frac{3}{2}}})$ is sufficiently close to $(0,0)$ and $q \geq p^{\frac{3}{2}} e^{-p\tilde{C}^{\frac{1}{p}}}$, then
$$(1-\epsilon)\Lambda \leq \frac{\log \tau(Y)}{\frac{1}{p}\log \frac{p^{\frac{3}{2}}}{q}} \leq (1+\epsilon)\Lambda,$$
- (Additive case). There exists $C > 0$ independent of p, q and the map T on X such that for $(q, \frac{p^{\frac{3}{2}}}{q})$ sufficiently close to $(0,0)$,
$$C^{-1}\Lambda \leq \frac{\log \tau(Y)}{\frac{1}{q^{\frac{2}{3}}}} \leq C\Lambda.$$

This result is proved in §4.2.2.

The results predicted in [ZHO03], with the additional hypothesis that q is not exponentially small compared to p, are consequences of Theorems 1 and 2. These results are:

Corollary 3. Consider the following model systems of bubbling bifurcations:

$$\begin{cases} x_{n+1} = 2x_n \pmod 1 \\ y_{n+1} = (f(x_n) + p)y_n + ay_n^\rho + q \quad \text{ for } |y| < 1 \text{ and } \rho \in \{2, 3\}, \end{cases} \tag{6}$$

where $f(0) = 1$, $0 < f(x) < 1$ for $x \neq 0$, $a \neq 0$, and parameters $p, q > 0$ are sufficiently small. In the multiplicative regime ($p^{\frac{\rho}{\rho-1}} \gg q > p^{\frac{\rho}{\rho-1}} e^{-p\tilde{C}^{\frac{1}{p}}}$, for some $\tilde{C} > 1$), for a threshold Y chosen as in the theorems above, the average bursting time obeys the following scaling laws:

$$\lim_{(p, \frac{q}{p^2}) \to (0,0)} \frac{\log \tau(Y)}{\frac{1}{p} \log \frac{p^2}{|a|q}} = \log 2, \quad \text{when the coupling is asymmetric } (\rho = 2) \text{ and}$$

$$\lim_{(p, \frac{q}{p^{\frac{3}{2}}}) \to (0,0)} \frac{\log \tau(Y)}{\frac{1}{p} \log \frac{p^{3/2}}{|a|^{1/2} q}} = \log 2, \quad \text{when the coupling is symmetric } (\rho = 3).$$

We have included in the conclusion of Corollary 3 terms from [ZHO03] involving a; while these terms do not affect the limits, they may make the limits converge faster.

Remark 2.2. The function $f(x) = \cos(2\pi x)$ considered in [ZHO03] does not meet our hypotheses because for technical reasons we have assumed f to be positive. However, our proofs can be adapted to such an f.

2.3 Generalizations

Here, we discuss three generalizations of our results. The first one concerns the replacement of the bifurcating fixed point by a periodic orbit. The second one is about the case of multidimensional transverse direction, that is, when the invariant manifold has codimension greater that one. The last one is to the case of random additive noise.

2.3.1 Periodic bifurcating orbit.

In case the bifurcating orbit is periodic of period d instead of a fixed point, after imposing non-degeneracy conditions, we could set up a model for bubbling bifurcations similar to that of 2.1. In this situation, when x gets near the periodic orbit, we would study the d-th power of T. The main difference with the fixed point case is that instead of having just one fixed point to keep track of, we would have d of them, and this introduces some technical difficulties. Although we do not carry out in detail all the calculations needed for this generalization, we do discuss the differences with the fixed point situation and provide ideas of how to extend the theorems in this case; see Remarks 3.7, 3.9 and 4.8.

2.3.2 Multidimensional transverse direction.

In this section we discuss a generalization of our analysis to the case when the codimension of the bifurcating invariant manifold X is larger than 1. As in hypotheses (i), (ii) and (iii) in §2.1, we assume that the attracting chaotic invariant manifold disappears when a direction transverse to X becomes unstable, that the orbit that first becomes unstable is a fixed point x^*, and that the bifurcation occurs at the parameter value $p = 0$.

Let $\vec{y}$ represent the multidimensional directions complementary to x. Our model system (3) then becomes:

$$\begin{cases} x_{n+1} = T(x_n) \\ \vec{y}_{n+1} = \vec{F}(x_n, \vec{y}_n, p, q). \end{cases}$$

If $\vec{F}$ is sufficiently smooth with respect to $\vec{y}$, p and q, we can bound the higher order terms as before. Following [ABS96], we impose the non-degeneracy condition that for $p = q = 0$, the fixed point x^* has a unique neutral direction transverse to X with eigenvalue 1, and that all other eigenvalues of $\frac{\partial \vec{F}}{\partial \vec{y}}(x^*, 0, 0, 0)$ have magnitude less than 1. We choose a norm defined by an inner product for $\vec{y}$, such that the corresponding norm of $\frac{\partial \vec{F}}{\partial \vec{y}}(x^*, 0, 0, 0)$ is equal to 1. Corresponding to (iii'), we assume that there are functions f and h on X, with f having a unique maximum of 1 at $x = x^*$, such that $\|\frac{\partial \vec{F}}{\partial \vec{y}}(x, 0, p, 0)\| = f(x) + h(x)p + \mathcal{O}(p^2)$. Then, one can show that the largest eigenvalue of $\frac{\partial \vec{F}}{\partial \vec{y}}(x^*, 0, p, 0)$ is $1 + h(x^*)p + \mathcal{O}(p^2)$. We call $\vec{v}(p)$ the corresponding eigenvector for the adjoint to $\frac{\partial \vec{F}}{\partial \vec{y}}(x^*, 0, p, 0)$, and let $\vec{g}(x) = \frac{\partial \vec{F}}{\partial q}(x, 0, 0, 0)$. Then, we can bound the growth of the norm of $\vec{y}$ as in the one-dimensional case,

$$|\vec{y}_{n+1}| \le (f(x_n) + h(x_n)p)|\vec{y}_n| + q\|\vec{g}(x_n)\| + \mathcal{O}(q|\vec{y}_n| + p^2|\vec{y}_n| + pq + q^2 + |\vec{y}_n|^2).$$

Thus, the analysis of §4.1.2 remains applicable. We can bound the growth of the norm of $\vec{y}$ from below in a similar way, with an additional error term of order $|x_n - x^*||\vec{y}_n|$.

$$\begin{aligned} \vec{y}_{n+1} \cdot \vec{v}(p) \ge \quad & (f(x_n) + h(x_n)p)\vec{y}_n \cdot \vec{v}(p) + q\vec{g}(x_n) \cdot \vec{v}(p) \\ & + \mathcal{O}(q|\vec{y}_n| + p^2|\vec{y}_n| + pq + q^2 + |\vec{y}_n|^2 + |x_n - x^*||\vec{y}_n|). \end{aligned}$$

(If $\vec{g}(x^*) \cdot \vec{v}(0) < 0$, we change the sign of $\vec{v}$.) In order for the analysis in §4.1.1 to be applicable, the conditions that need to be satisfied are that $\vec{g}(x^*) \cdot \vec{v}(0) \ne 0$, corresponding to (v), and, additionally, non-degeneracy conditions analogous to (iv) making x^* undergo a transcritical or pitchfork bifurcation. The main remaining complication to adapting the arguments in § 4 is that there is no analogue of Lemma 4.1 in this case, because the direction of $\vec{y}$ can rotate while x is away from x^*.

2.3.3 Random perturbations.

The results of this paper can be generalized to some random dynamical systems, such as the case where the deterministic mismatch $g(x_n)$ is replaced by a stationary sequence of independent random variables g_n [Gon]. We could also treat the case of combined random noise and deterministic mismatch with our methods.

3 Invariant manifold: dynamics and bifurcation

3.1 Dynamics on the invariant manifold

In this section, we present results we need for the base dynamics given by T, a transitive C^2 Anosov diffeomorphism. We assume T has a fixed point x^*, and derive quantitative dynamical properties that are used in our estimates in §4, following references [Bow75], [Che02] and [Aba04].

We note that all results of this section also apply to expanding maps. In particular, the model system with base dynamics given by $T(x) = mx \pmod 1$ is rich enough to give a good understanding of most of these properties. For general T, the analysis we present is somewhat more involved.

3.1.1 Existence of Markov partitions and SRB measures.

Classical works of Sinai [Sin68] and Bowen [Bow70] show that uniformly hyperbolic dynamical systems have Markov partitions of arbitrarily small diameter. Such partitions allow one to study the dynamics in symbolic terms, since all invariant measures of hyperbolic systems are projections of invariant measures on symbolic systems that are semi-conjugate to T.

Moreover, given any point $x \in X$, Pesin and Weiss show [PW97, Thm. 3] that, after possibly passing to a power of T, a Markov partition $\mathcal{R}$ can be chosen in such a way that x is in the interior of a Markov rectangle. We will choose such a *special* Markov partition with the hyperbolic fixed point x^* in the interior of a rectangle we call R_0.

Because of our hypotheses on T, there is always an invariant measure that is physically relevant: the SRB or physical measure [Sin68, Bow70]. We will call it μ. This measure is the one of interest for us, since its basin contains a full Lebesgue measure set of trajectories. Another relevant property of μ is exploited in §3.1.3, namely, that the μ measures of cylinders around a point (see definition below) are asymptotically determined by the sum of positive Lyapunov exponents.

For a fixed Markov partition $\mathcal{R} = \{R_0, \ldots, R_{D-1}\}$ of (X, T), we denote by $\omega_i(x)$ the index of the partition set to which $T^i(x)$ belongs, provided that $T^i(x)$ belongs to only one partition set. Note that this is undefined on the set for which $T^i(x)$ belongs to the boundary of a partition set, which has μ measure zero [Che02, Prop. 3.1]. We denote by $\Omega_{T,\mathcal{R}}$ the set of sequences $(\omega_i)_{i\in\mathbb{Z}} \subset \Sigma_D$ allowed by the dynamics of T. Cylinder sets are nonempty sets $S \subset X$ of the form $S = \{x \in X | \omega_i(x) = b_i, k \le i \le k + l\}$, up to a set of μ measure zero, for some $k \in \mathbb{Z}$, $l \ge 0$ and $b_i \in \{0, 1, \ldots, D-1\}$. Such a cylinder set S has length $l+1$ and is based at k. We write $\mathcal{C}(k, k+l)$ to denote the collection of all cylinders of length $l+1$ and base k. We say that two cylinders $S_i \in \mathcal{C}(k_i, k_i')$, $i = 1, 2$, are determined by non-overlapping words if either $k_1' < k_2$ or $k_2' < k_1$.

Below, we use $\mathbb{P}_\mu$ to denote the probability of an event with respect to μ. For example,

$$\mathbb{P}_\mu(\omega_k = b) = \mu(\{x \in X | \omega_k(x) = b\}).$$

We also use $\mathbb{E}_\mu$ for the expectation with respect to μ.

3.1.2 Expected hitting time.

For a μ measurable set S with $\mu(S) > 0$, let $\tau_S(x)$ be the first time the orbit of x visits (or hits) S, that is, $\tau_S(x) = \min\{k \geq 0 | T^k(x) \in S\}$. By ergodicity, the *hitting time* $\tau_S(x)$ is finite for μ almost every $x \in X$ and defines a μ measurable function on X. The following lemmas relate the expected hitting time with $\mu(S)$. The first one, which follows from [Aba04, §5], gives an upper bound and holds for cylinder sets. The second one gives a lower bound and is valid for all measurable sets S of sufficiently small measure.

Lemma 3.1. There exists a constant $\tilde{U} = \tilde{U}(T, \mathcal{R}) \geq 1$ such that for every cylinder set S,

$$\mathbb{E}_\mu(\tau_S) := \int_X \tau_S(x) d\mu(x) \leq \frac{\tilde{U}}{\mu(S)}.$$

Lemma 3.2. If $\mu(S) < \frac{1}{4}$ then

$$\mathbb{E}_\mu(\tau_S) := \int_X \tau_S(x) d\mu(x) \geq \frac{1}{4\mu(S)}.$$

Proof. Let $S_k := \{x \in X | \tau_S(x) = k\}$. Hence $S_k \subseteq T^{-k}(S)$ and therefore $\mu(S_k) \leq \mu(S)$, which gives the lower bound

$$\begin{aligned}
\mathbb{E}_\mu(\tau_S) = \sum_{k=0}^{\infty} k\mu(S_k) &= \sum_{k=1}^{\infty}\sum_{j=1}^{k} \mu(S_k) = \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \mu(S_k) = \sum_{j=1}^{\infty}\left(1 - \sum_{k=0}^{j-1}\mu(S_k)\right) \\
&\geq \sum_{j=1}^{\lfloor \frac{1}{\mu(S)} \rfloor} (1 - j\mu(S)) = \lfloor \frac{1}{\mu(S)} \rfloor - \frac{\lfloor \frac{1}{\mu(S)} \rfloor (\lfloor \frac{1}{\mu(S)} \rfloor + 1)}{2}\mu(S) \\
= \lfloor \frac{1}{\mu(S)} \rfloor \left(1 - \frac{\mu(S)(\lfloor \frac{1}{\mu(S)} \rfloor + 1)}{2}\right) &\geq \left(\frac{1}{\mu(S)} - 1\right)\frac{1 - \mu(S)}{2} \geq \frac{1}{4\mu(S)},
\end{aligned}$$

where the last inequality follows from the fact that $\mu(S) < \frac{1}{4}$. □

3.1.3 Consecutive number of iterates near a fixed point.

It is necessary for our purposes to understand the distribution of the number of consecutive iterates a trajectory spends in a neighborhood of the fixed point x^*. Following traditional notation, we let

$$B_x(n, \epsilon) := \{z | \operatorname{dist}(T^i x, T^i z) < \epsilon \ \forall i = 0, 1, \ldots, n\}.$$

A trajectory x stays within ϵ of x^* for n iterates if $x \in B_{x^*}(n, \epsilon)$. Let $\Xi := \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\dim X}\}$ be the Lyapunov spectrum of T at x^*. Let

$$\Lambda := \sum_{i=1}^{\dim X} (\lambda_i)_+, \text{ where } (\lambda)_+ := \max(\lambda, 0), \text{ and } \chi := e^\Lambda.$$

A lower bound on the number of iterates close to x^* is given by:

Lemma 3.3. There exist constants $C = C(T)$ and $\varphi = \varphi(\Xi)$ such that for all $\delta > 0$ sufficiently small,

$$\mu(B_{x^*}(n, \delta)) \geq C\delta^\varphi \chi^{-n}.$$

Proof. It is shown in [Bow75, § 4.4] that for $\epsilon > 0$ sufficiently small there is some constant $\tilde{C} = \tilde{C}(T, \epsilon)$

$$\mu(B_{x^*}(n, \epsilon)) \geq \tilde{C}\chi^{-n}.$$

Also, by Corollary 6.4.17 in [KH95], if $\epsilon > 0$ is sufficiently small, there is a constant $A = A(T)$ such that if $x \in B_{x^*}(n, \epsilon)$ then for $0 \leq i \leq n$, $\mathrm{dist}(T^i x, x^*) \leq A\epsilon e^{-\tilde{\lambda}\min(i,n-i)}$, where $\tilde{\lambda} := \min_{i=1,\dots,\dim X}(|\lambda_i|)$, the distance of Ξ to 0. Fix $\epsilon > 0$ and let $\varphi = 2\log\chi/\tilde{\lambda}$ and $k = \lceil \frac{\log\delta - \log A\epsilon}{\tilde{\lambda}} \rceil$. Then $A\epsilon e^{-\tilde{\lambda}k} \leq \delta$ and if $k \geq 0$ we have

$$\mu(B_{x^*}(n, \delta)) \geq \mu(B_{x^*}(n + 2k, \epsilon)) \geq \tilde{C}\chi^{-(n+2k)} \geq C\delta^{\varphi}\chi^{-n}. \qquad \square$$

Obtaining upper bounds for the time spent close to x^* requires a better understanding of the dynamical properties of T. Let $\xi_0(x), \xi_1(x), \dots$ be the number of consecutive iterates the trajectory of x spends in successive visits to a Markov rectangle R_0 containing x^* in its interior.

The following lemmas will be useful in §4.1.

Lemma 3.4. There is a constant $A = A(T, \mathcal{R}) > 0$ such that for every $k \in \mathbb{N}$ and $t > 0$ we have:

$$\mathbb{P}_\mu(\xi_k \geq t) \leq A\chi^{-t}.$$

Proof. This is a consequence of the so-called exponential cluster property for uniformly hyperbolic systems (see e.g. [Bow75, Che02]): there are constants $\tilde{C}$ and $\theta < 1$ such that given any two cylinders $S \in \mathcal{C}(0, a)$ and $S' \in \mathcal{C}(0, b)$,

$$|\mathbb{P}_\mu(S \cap T^{-n}(S')) - \mathbb{P}_\mu(S)\mathbb{P}_\mu(S')| \leq \tilde{C}\mathbb{P}_\mu(S)\mathbb{P}_\mu(S')\theta^{n-a}.$$

(Notice that $T^{-n}(S') \in \mathcal{C}(n, n + b)$, so $n - a$ represents the gap between the symbols determined by membership in S and those determined by membership in $T^{-n}(S')$.) In particular, there is a constant C such that for any two cylinders S and S' determined by non-overlapping allowed words, we have that S and S' are *independent up to a multiplicative factor* C in the following sense:

$$\mathbb{P}_\mu(S \cap S') \leq C\mathbb{P}_\mu(S)\mathbb{P}_\mu(S') \quad \text{and hence} \quad \mathbb{P}_\mu(S|S') \leq C\mathbb{P}_\mu(S),$$

where by $\mathbb{P}_\mu(S|S')$ we mean the conditional probability $\frac{\mathbb{P}_\mu(S \cap S')}{\mathbb{P}_\mu(S')}$.

To prove the lemma, we fix k and consider the following countable partition $\mathcal{Z}$ (modulo sets of μ measure 0) of $\Omega_{T,\mathcal{R}}$ as follows. Each element of $\mathcal{Z}$ consists of the cylinder set of sequences ω that share all symbols up to τ_k, where τ_k is the start of the k-th sequence of 0's. For example, for $k = 1$, sequences of the form $\mathbf{101}\dots$ and $\mathbf{1001}\dots$ would belong to the same element of $\mathcal{Z}$, but sequences of the form $\mathbf{110}\dots$ would be in a different element of the partition. This is a partition modulo sets of μ measure 0 since with probability 1, $\tau_k < \infty$.

By the exponential cluster property, we know that for any $Z \in \mathcal{Z}$,

$$\mathbb{P}_\mu(\xi_k \geq t|Z) \leq C\mathbb{P}_\mu(0_t),$$

where 0_t is the sequence consisting of t zeros. Therefore,

$$\mathbb{P}_\mu(\xi_k \geq t) = \sum_{Z \in \mathcal{Z}} \mathbb{P}_\mu(\xi_k \geq t|Z)\mathbb{P}_\mu(Z) \leq C\mathbb{P}_\mu(0_t) \sum_{Z \in \mathcal{Z}} \mathbb{P}_\mu(Z) = C\mathbb{P}_\mu(0_t).$$

The lemma follows from the fact that for rapidly mixing systems, there exist positive constants $\lambda, \tilde{A}$ such that:

$$\mathbb{P}_\mu(0_t) < \tilde{A}e^{-\lambda t}.$$

Furthermore, since μ is an SRB measure, we can take $\lambda = \log \chi = \Lambda$ [Bow75]. $\square$

Given a constant $c < 1$, let $\eta_k := \sum_{j=0}^{k} c^{k-j}\xi_j$.

Lemma 3.5. For $c < 1$ fixed, there are constants B and $0 < \theta < 1$, such that whenever $B \le t_* \le t$,

$$\mu_k := \mathbb{P}_\mu(\eta_k \ge t,\ \eta_j \ge t_* \text{ for } j = 1, \dots, k-1) \le A\theta^k \chi^{-t}.$$

Proof. First, we show that for any values of t, t_j, the events $\{\xi_k = t\}$ and $\{\eta_j \ge t_j \text{ for } 0 \le j < k\}$ are independent up to a multiplicative factor C given by the exponential cluster property. Let us make use of the partition $\mathcal{Z}$ from the proof of Lemma 3.4 and let $\mathcal{Z}_0 = \{Z \in \mathcal{Z} : \eta_j \ge t_j \text{ for } 0 \le j < k\}$. We remark that $\mathcal{Z}_0$ is well defined, since for $0 \le j < k$, η_j is constant μ-almost everywhere in each $Z \in \mathcal{Z}$. Thus, we have:

$$\begin{aligned}\mathbb{P}_\mu(\xi_k = t, \eta_j \ge t_j \text{ for } 0 \le j < k) &= \sum_{Z \in \mathcal{Z}} \mathbb{P}_\mu(\xi_k = t, \eta_j \ge t_j \text{ for } 0 \le j < k | Z)\mathbb{P}_\mu(Z) \\ &= \sum_{Z \in \mathcal{Z}_0} \mathbb{P}_\mu(\xi_k = t | Z)\mathbb{P}_\mu(Z) \le C\mathbb{P}_\mu(0_t)\mathbb{P}_\mu(\eta_j \ge t_j \text{ for } 0 \le j < k) \\ &\le CA\chi^{-t}\mathbb{P}_\mu(\eta_j \ge t_j \text{ for } 0 \le j < k).\end{aligned}$$

Now, using Lemma 3.4 as the base step, valid for all t_*, t, we will prove our result by induction. Assume we know that for some k, t_* and t we have that $\mu_k \le A\theta^k\chi^{-t}$. Then, for $k+1$ we have:

$$\begin{aligned}\mu_{k+1} \quad &\le \textstyle\sum_{s=0}^{t-ct_*} \mathbb{P}_\mu(\xi_{k+1} = s, \eta_k \ge \frac{t-s}{c}, \eta_j \ge t_* \text{ for } j = 0, \dots, k) \\ &\quad + \mathbb{P}_\mu(\xi_{k+1} \ge t - ct_*, \eta_j \ge t_* \text{ for } j = 0, \dots, k) \\ &\le \textstyle\sum_{s=0}^{t-ct_*} C(A\chi^{-s})(A\theta^k\chi^{-(\frac{t-s}{c})}) + C(A\chi^{-(t-ct_*)})(A\theta^k\chi^{-t*}) \\ &\le A\theta^k AC\left(\frac{\chi^{(\frac{1}{c}-1)}}{\chi^{(\frac{1}{c}-1)}-1} + 1\right)\chi^{-t}\chi^{-(1-c)t_*}.\end{aligned}$$

This establishes the inductive step and the result provided

$$\theta = AC\left(\frac{\chi^{(\frac{1}{c}-1)}}{\chi^{(\frac{1}{c}-1)} - 1} + 1\right)\chi^{-(1-c)B} \text{ and } B \text{ is large enough that } \theta < 1.$$

The proof of the second statement is similar and we omit it. $\square$

Let $N_k = \xi_0 + \xi_1 + \cdots + \xi_k$.

Lemma 3.6. For β sufficiently large and $0 < t \le \chi^{\beta/2}$, there is a constant $0 < \tilde{\theta} < 1$ such that

$$\tilde{\mu}_k(t) := \mathbb{P}_\mu(N_k \ge t + k\beta, N_j \ge j\beta \text{ for } j = 0, \dots, k-1) \le A\tilde{\theta}^k\chi^{-t}.$$

Proof. By an argument analogous to the proof of Lemma 3.5, the events $\{\xi_k \geq t\}$ and $\{N_j \geq j\beta \text{ for } 0 \leq j < k\}$ are independent up to a multiplicative factor C. Using Lemma 3.4 as the base step, valid for all t, we will proceed by induction. The base step follows from Lemma 3.4. Assume we know that for some k and t $\tilde{\mu}_k(t) \leq A\tilde{\theta}^k \chi^{-t}$. For $k+1$ we have:

$$\begin{aligned}\tilde{\mu}_{k+1}(t) \quad &\leq \textstyle\sum_{s=1}^{\lfloor t+\beta \rfloor} C\mathbb{P}_\mu(\xi_{k+1}=s)\tilde{\mu}_k(t+\beta-s) + C\mathbb{P}_\mu(\xi_{k+1} > \lfloor t+\beta \rfloor)\tilde{\mu}_k(0)\\ &\leq A\tilde{\theta}^k CA\chi^{-(t+\beta)}\big(t+\beta+1\big) \leq A\tilde{\theta}^{k+1}\chi^{-t},\end{aligned}$$

provided $\tilde{\theta} := AC\big(t+\beta+1\big)\chi^{-\beta}$. For any choice of β, this last inequality gives explicit restrictions on the allowed size of t in order for $\tilde{\theta} < 1$. In particular, for sufficiently large β, the argument is valid for any $t \leq \chi^{\beta/2}$. □

Remark 3.7. The analysis of this section dealt with the dynamics close to a fixed point x^* of T. Results of this section can be adapted to study the dynamics close to a periodic orbit of period d after taking the d-th power of T. This extension is not completely trivial, since in this case we would have d fixed points $x_1^*, \ldots, x_d^*$ to simultaneously keep track of. However, straightforward extensions of the arguments in §3.1.3, yield similar bounds for the analogue of ξ_k, η_k, N_k in this setting. In this case, Λ would be replaced by the sum of positive Lyapunov exponents of the periodic orbit and $\chi = e^\Lambda$ would change accordingly.

3.2 Bifurcation of the invariant manifold

In this section, we discuss the genericity of conditions (iii) and (iii') of §2.1 about the bifurcation of the invariant manifold when $q = 0$. While condition (iii') is non-generic, we will weaken it to a condition (iii") that we characterize as a non-degeneracy assumption.

The assumption in condition (iii) that X is asymptotically stable for $p < 0$ implies that the Lyapunov exponent, equal to the average of $\log \frac{\partial F}{\partial y}$, is nonpositive for all invariant measures of T when $p < 0$. Recall that $f(x) = \frac{\partial F}{\partial y}(x, 0, 0, 0) > 0$, and observe that f is Lipschitz by our smoothness hypothesis for F. Then by continuity, the average of $\log f(x)$ is nonpositive for all invariant measures of T, and further (by condition (iii) again) the average is zero for the delta measure at x^*. Thus, among all invariant measures of T, the average of $\log f(x)$ is maximized at the bifurcating orbit. It has been conjectured [YH99] and numerically supported [HO96] that generically, maximizing (optimal) invariant measures occur at measures with periodic support. In this respect, we expect in general the loss of stability of X to occur at a periodic orbit, and for simplicity we consider the case of a fixed point x^*. Furthermore, it is a topologically generic property of Lipschitz and smooth functions [Jen06] to have a unique maximizing invariant measure. Condition (iii') makes the stronger assumption that pointwise the maximum of $\log f(x)$ occurs at x^*. We can easily weaken this assumption by requiring that it be true for some change of coordinates. Specifically, we consider

$$\tilde{y} = \eta(x)y, \quad \text{with } \eta(x) > 0 \text{ for all } x \in X \text{ and } \eta(x^*) = 1. \tag{7}$$

Under such coordinate change, the evolution equations for system (3) and parameters $p = q = 0$ become:

$$\begin{cases} x_{n+1} = T(x_n) \\ \tilde{y}_{n+1} = \tilde{F}(x_n, \tilde{y}_n, 0, 0), \end{cases} \tag{8}$$

with $\tilde{F}(x, \tilde{y}, 0, 0) = \eta(T(x))F(x, 0, 0, 0) + \tilde{y}\frac{\eta(T(x))}{\eta(x)}f(x) + \mathcal{O}(\tilde{y}^2)$. The corresponding coefficient of the linear term in the Taylor expansion of $\tilde{F}$ with respect to $\tilde{y}$ becomes:

$$\tilde{f}(x) = \frac{\eta(T(x))}{\eta(x)}f(x).$$

Thus, condition (iii') can be replaced by

(iii") There exists a change of coordinates of the form (7) for which $\tilde{f}$ has a unique global maximum at x^*.

The following lemma suggests that it is plausible to expect such a change of coordinates.

Lemma 3.8. Let $f : X \to \mathbb{R}$ be a positive Lipschitz function. Suppose that among all T invariant measures, the average of $\log f$ is maximized at (the measure supported on) a fixed point x^*. Then, there exists a change of coordinates of the form (7) for which the global maximum of $\tilde{f}$ occurs at x^*.

Proof. Let $\phi(x) = \log f(x)$. This is well defined and Lipschitz, since f is positive and Lipschitz. Existence of a change of coordinates $\tilde{y} = \eta(x)y$ changing f into $\tilde{f}$ is equivalent to having a solution to the following co-homological equation:

$$\tilde{\phi}(x) = \phi(x) + \psi(T(x)) - \psi(x), \tag{9}$$

where $\tilde{\phi}(x) = \log \tilde{f}(x)$ and $\psi(x) = \log \eta(x)$.

When T is uniformly hyperbolic, the normal form theorem [Jen06, 4.7] ensures the existence of a Lipschitz solution ψ to (9) with the following property.

$$\phi(x^*) \geq \phi(x) + \psi(T(x)) - \psi(x) =: \tilde{\phi}(x).$$

Therefore, the change of coordinates from f to $\tilde{f}$ given by $\tilde{f}(x) = e^{\psi(x)}f(x)$ has a global maximum at x^*. ☐

With this result in mind, condition (iii") is similar to the assumption that the average of $\log f(x)$ over invariant measures of T has a unique maximum at x^*.

Remark 3.9. Lemma 3.8 extends to the case when the average of $\log f$ over the space of T invariant measures is maximized at a periodic orbit $x_1^*, \ldots, x_d^*$; after a coordinate change, the global maximum of f occurs at all d points of the orbit. In this case, our non-degeneracy assumption is that f is maximized only at these d points.

4 Proof of main results

All results in this section refer to dynamical systems of the form (3), satisfying assumptions (i)-(v) in §2.1 as well as either (iii') in §2.1 or (iii") in §3.2.

4.1 Average bursting time in the linear regime

The goal of this section is to derive a scaling law for the logarithm of the average bursting time τ, valid for burst amplitudes small enough that we can use a linear approximation to the y dynamics. We consider the effect of nonlinear terms in the following section. We set a threshold y value Y, and investigate the average time it takes for an initial condition starting close to X to burst (or *escape*) to the threshold.

When p is small, Lebesgue almost all orbits of T will spend most of their time in the region in which $f(x) + h(x)p < 1$, so that the y dynamics are contracting near $y = 0$. However, since x^* is in the support of μ, the x trajectory of Lebesgue almost every orbit will visit arbitrarily small neighborhoods of x^* and thus remain close to x^* for arbitrarily long period of time, eventually resulting in a burst.

A quantitative understanding of this statement allows us to find a cylinder set $S \subset X$ such that whenever the x trajectory enters it, the trajectory is guaranteed to reach the threshold Y. From this, we obtain an upper bound for the average bursting time in terms of $\mu(S)$, since once in S, the time it takes to burst is relatively negligible.

The lower bound needs further work, since in order to establish it, an understanding of all possible escape routes to the threshold Y is needed. In this part, we will identify a set $S' \subset X$ (not necessarily a cylinder but a union of cylinders) such that the x coordinate of any trajectory that escapes must visit S' before escaping. The definition of the set S' depends on the fact that trajectories may escape not only through one long sequence of expansive iterates, but instead could follow a sequence of alternating expanding and contracting periods. We note that our results will show that the former is asymptotically the most likely escape route, provided q is bounded below as in the multiplicative cases of Theorems 1 and 2. The set S' also depends on an intermediate y threshold that is presented in §4.1.2.

In order to establish upper and lower bounds on the average bursting time, we restrict ourselves to finding lower and upper bounds on the measure of trajectories that initiate a burst, $\mu(S)$ and $\mu(S')$. This is enough for our purposes, in view of Lemmas 3.1 and 3.2.

We introduce two parameters for the threshold size: $\alpha = \frac{Y}{q}$ quantifies the number of iterates to reach the threshold for $x = x^*$ and $p = 0$, ignoring higher order terms. The non-linearity parameter $s = \frac{Y^\rho}{\max(pY,q)}$ measures the size of the dominant non-linear term Y^ρ, relative to the largest term in the linearization of $y_{n+1} - y_n$ at $x_n = x^*$. Note that p, q and s determine Y and hence α.

Next, we bound the higher order terms in (3) by $\sigma p|y| + \zeta q$, where σ and ζ can be made arbitrarily close to 0 by making p, q and s small. In particular, we assume $\zeta, \sigma < 1$.

Throughout this section, we write $\tau = \tau(Y) = \tau(\alpha q)$, and recall that $\Lambda = \sum_{i=1}^{\dim X} (\lambda_i)_+ = \log \chi$ is the sum of positive Lyapunov exponents of x^* for T. We also recall that $q\Delta$ is upper bound on the *kick* $q(g(x) + \zeta)$. Let $0 < c < 1$ be an upper bound on $f(x) + h(x)p$ for $x \notin R_0$ and define $\tilde{l}(z) = \frac{1+z}{z\log(1+z)}$ and $K(p, \alpha) := \frac{1}{p\alpha\tilde{l}(p\alpha)} e^{\frac{p}{2}} \chi^{\frac{\log\frac{1}{c}}{4p(1+\tilde{l}(p\alpha))} - \frac{\Delta\tilde{l}(p\alpha)}{4c(1-c)(1+\tilde{l}(p\alpha))}}$. We say that parameters p, α satisfy condition $(\star)$ if α is sufficiently large and either $p\alpha < \frac{1}{2}\log\frac{1}{c}$ or

$p\alpha \geq \frac{1}{2}\log\frac{1}{c}$ and $K(p,\alpha) > 1$. We remark that when $p\alpha$ is sufficiently large and p, q and s are sufficiently small, $K(p,\alpha) > 1$ if $k(p) := \frac{1}{p}e^{p\chi^{\frac{\log\frac{1}{c}}{4p}}} > \alpha$. The main result of this section is the following.

Theorem 4. For sufficiently small parameters p, q and s such that the threshold size α satisfies condition $(\star)$, there is a constant $C > 1$ independent of p, q and the map T on X such that

$$C^{-1}\Lambda \leq \frac{\log\tau(q\alpha)}{\frac{1}{p}\log(1+p\alpha)} \leq C\Lambda.$$

Moreover, in the limit that $\alpha \to \infty$ and either $p\alpha \to 0$ or $p\alpha \to \infty$, C can be taken arbitrarily close to 1.

In order to prove the theorem, we first show that there is a manifold $X_{-\tilde{\kappa}q} := X \times \{-\tilde{\kappa}q\}$, with $\tilde{\kappa} = \mathcal{O}(1)$ that gets mapped above itself (in the y direction), therefore preventing all initial conditions starting above it to escape the strip $X \times [-\alpha q, \alpha q]$ from below. Then, we establish the upper and lower bounds in §4.1.1 and §4.1.2, respectively.

Lemma 4.1. There is a constant $\tilde{\kappa}$ independent of p and q such that the image of any initial condition (x_0, y_0) of (3) with $y_0 \geq -\tilde{\kappa}q$ satisfies $y_1 \geq -\tilde{\kappa}q$ for all sufficiently small p and q. Moreover, in this case, there exists $\tilde{x} > 0$ independent of p and q such that every trajectory for which $y_0 > -\tilde{\kappa}q$ that remains in the set $|x - x^*| < \tilde{x}$ for a sufficiently long number of iterates n_0, independent of p and q, reaches a positive y value, that is $y_{n_0} > 0$.

Proof. Consider $\tilde{x}$ sufficiently small so that when $|x_0 - x^*| < \tilde{x}$ we have that $g(x_0) > \frac{1}{2}$, $h(x_0) < \frac{3}{2}$, and such that there is $0 < r < 1$ depending only on f such that for $|x_0 - x^*| \geq \tilde{x}$, $f(x_0) \leq 1 - 2r$. When p and q are sufficiently small, $1 + 2p \geq f(x_0) + (h(x_0) + \sigma)p$ for $|x_0 - x^*| < \tilde{x}$, and $0 \leq f(x_0) + (h(x_0) - \sigma)p \leq 1 - r$ for $|x_0 - x^*| \geq \tilde{x}$. Let $\tilde{\kappa} > \frac{1 - \min_{x\in X}\{g(x)\}}{r}$. For $|x_0 - x^*| < \tilde{x}$ and $y_0 \geq -\tilde{\kappa}q$, $y_1 \geq -q\tilde{\kappa}(1 + 2p) + (\frac{1}{2} - \zeta)q$. Hence, if p and q are sufficiently small, $\zeta < \frac{1}{4}$ and $y_1 \geq -\tilde{\kappa}q$. For $|x_0 - x^*| \geq \tilde{x}$ and $y_0 \geq -\tilde{\kappa}q$, we have $y_1 \geq -\tilde{\kappa}q(1 - r) + q(\min_{x\in X}\{g(x)\} - \zeta)$. If p and q are sufficiently small, $\zeta < 1$ and by the choice of $\tilde{\kappa}$, $y_1 \geq -\tilde{\kappa}q$.

For the second statement, we know that if $|x_0 - x^*| < \tilde{x}$ and $y_0 \geq -\tilde{\kappa}q$, then $y_1 - y_0 \geq (\frac{1}{2} - \zeta - 2p\tilde{\kappa})q$. The result follows from the fact that we can apply the estimate repeatedly, as long as $|x_i - x^*| < \tilde{x}$. □

4.1.1 Upper bound for the bursting time.

In this section, we take an initial condition (x_0, y_0) starting above the manifold $X \times \{-\tilde{\kappa}q\}$ and find a neighborhood of $x = x^*$ so that whenever the trajectory remains in it for a sufficiently long number of iterates, it is guaranteed to escape.

First, we present a simple upper bound useful in the additive case. Another upper bound will be obtained in Proposition 4.3 by taking into account the expansiveness close to x^*.

Proposition 4.2. For any $\epsilon > 0$, if p, q and s are sufficiently small, and α is sufficiently large, we have:

$$\frac{\log\tau}{\alpha} < (1+\epsilon)\Lambda.$$

Proof. Assume $L-1$ is a Lipschitz constant for $|f|+|g|+|h|$. Let $0 < \tilde{\tilde{\delta}} < 1$, $\epsilon' > 0$ sufficiently small and $\tilde{\tilde{x}} = \min\{\frac{1-\tilde{\tilde{\delta}}}{L}, \frac{\tilde{\tilde{\delta}}\epsilon'}{L\alpha}\}$. Then, for p, q and s sufficiently small, if $|x-x^*| < \tilde{\tilde{x}}$ we have $g(x)-\zeta > \tilde{\tilde{\delta}} > 0$ and $f(x)+(h(x)-\sigma)p > 1-L\tilde{\tilde{x}} \geq 1-\frac{\tilde{\tilde{\delta}}\epsilon'}{\alpha}$. In this situation, for $|x_n - x^*| \leq \tilde{\tilde{x}}$ and $y_n \leq \alpha q$ we have:

$$y_{n+1} \geq \big(f(x_n) + h(x_n)p\big)y_n - \sigma p|y_n| + q\big(g(x_n) - \zeta\big) \geq y_n + (1-\epsilon')\tilde{\tilde{\delta}}q.$$

Therefore, a trajectory starting with a positive y value reaches the threshold if it stays in the region $|x-x^*| < \tilde{\tilde{x}}$ for at least $\frac{\alpha}{(1-\epsilon')\tilde{\tilde{\delta}}} < \frac{\alpha(1+2\epsilon')}{\tilde{\tilde{\delta}}} =: \tilde{\tilde{n}}$ consecutive iterates. Thus, in this setting, we can take $S = B_{x^*}(\tilde{\tilde{x}}, \tilde{\tilde{n}} + n_0)$ as a *surely escaping* set, where $n_0 = \mathcal{O}(1)$ is as in Lemma 4.1. By Lemmas 3.1 and 3.3, it follows that there is a constant $U = U(T)$ such that for sufficiently large α, an upper bound on the logarithm of the average bursting time is

$$\log\tau \leq \Lambda\frac{(1+3\epsilon')\alpha}{\tilde{\tilde{\delta}}} + \log U - \log\tilde{\tilde{x}}.$$

Since $\epsilon' > 0$ may be arbitrarily small and $\tilde{\tilde{\delta}}$ can be made arbitrarily close to 1 for p, q and s sufficiently small, the statement follows. ☐

Proposition 4.3. For any $\epsilon > 0$, if p, q and s are sufficiently small and $p\alpha$ sufficiently large, an upper bound on the logarithm of the average bursting time is given by:

$$\frac{\log\tau}{\frac{1}{p}\log(1+p\alpha)} < (1+\epsilon)\Lambda.$$

Proof. Using the Lipschitz assumptions on f, g and h, we can find $\tilde{x} > 0$ to be specified later, and $\tilde{\delta} > 0$ so that for $|x-x^*| < \tilde{x}$ we have $f(x) + (h(x)-\sigma)p > 1 + \tilde{\gamma}p$ for some $0 < \tilde{\gamma} < 1-\sigma$ and $g(x) - \zeta > \tilde{\delta} > 0$. By the choices of $\tilde{\kappa}$ and $\tilde{x}$, every trajectory with initial condition $y_0 \geq -\tilde{\kappa}q$ that is in the region $|x-x^*| < \tilde{x}$ for n_0 iterates we have that $y_{n_0} > 0$. Hence, if the trajectory remains in the region $|x - x^*| < \tilde{x}$ for another iterate, we will have that

$$\begin{aligned} y_{n_0+1} \quad &\geq (f(x_{n_0}) + h(x_{n_0})p)y_n + g(x_{n_0})q - \sigma|py_{n_0}| - \zeta q \\ &\geq (1+\tilde{\gamma}p)y_{n_0} + \tilde{\delta}q, \end{aligned}$$

and if $|x - x^*| < \tilde{x}$ for another n consecutive iterates,

$$y_{n_0+n} \quad \geq (1+\tilde{\gamma}p)^n\tfrac{\tilde{\delta}q}{\tilde{\gamma}p} - \tfrac{\tilde{\delta}q}{\tilde{\gamma}p}.$$

Hence, all orbits that remain in the region $|x - x^*| < \tilde{x}$ for time

$$\tilde{n} := n_0 + \frac{\log\frac{\tilde{\gamma}p\alpha+\tilde{\delta}}{\tilde{\delta}}}{\log(1+\tilde{\gamma}p)}$$

will reach the threshold αq within $\tilde{n}$ steps.

Thus, in this setting, we can take $S = B_{x^*}(\tilde{x}, \tilde{n})$ as a *surely escaping* set. By Lemma 3.3, we know that there are constants $C = C(T)$ and $\varphi = \varphi(T)$ such that:

$$\mu(S) \geq C\tilde{x}^{\varphi}\chi^{-\tilde{n}}.$$

Thus, by Lemma 3.1 we have that there is a constant $\tilde{U} = \tilde{U}(T)$ such that an upper bound on the average bursting time $\tau = \tau(Y)$ is:

$$\tau \leq \tilde{U}\tilde{x}^{-\varphi}\chi^{\tilde{n}} + \tilde{n}. \tag{10}$$

Therefore, there is a constant $U = U(T)$ such that for any $\epsilon > 0$, if p, q and s are sufficiently small and α sufficiently large, we have:

$$\log\tau \leq \left(1+\frac{\epsilon}{2}\right)\Lambda\left(n_0 + \frac{\log\frac{\tilde{\gamma}p\alpha+\tilde{\delta}}{\tilde{\delta}}}{\log(1+\tilde{\gamma}p)}\right) + \log U(1-\log\tilde{x}).$$

Furthermore, we can make $\tilde{\gamma}$ and $\tilde{\delta}$ arbitrarily close to 1 by making $\frac{\tilde{x}}{p}, p, q$ and s sufficiently small and $p\alpha$ sufficiently large. Choosing $\tilde{x} = \frac{p}{\log(1+p\alpha)}$, the proposition follows. □

4.1.2 Lower bound for the bursting time.

To get a lower bound for the bursting time, we need to consider different *escape routes*. For a given y_0, in order for a trajectory starting at height less than $\tilde{y}_0 q$ to escape, it needs to get total expansion by a factor of $\alpha_0 := \frac{\alpha}{\tilde{y}_0}$. This expansion can be achieved in one long sequence of expansive iterates, which corresponds to the case presented in the previous subsection, or in several expansive sequences.

An important characteristic of our model is that the linearized contraction rate between any two expansive sequences is bounded above by some factor $0 < c < 1$ independent of p and α, and that it takes a long time to recover from it. These consideration will allow us to show that the measure of initial conditions that initiate an escape is comparable to the measure of initial conditions that escape in just one sequence of expansive iterates.

The goal of this section is to find a set $S' \subset X$ that every escaping orbit must visit in order to escape. More precisely, the last time a trajectory lies below an intermediate threshold (specified below) before escaping, its x coordinate must lie in S'. In order to define S', we will consider the x dynamics in symbolic terms. For this, we fix a Markov partition $\mathcal{R}$ for T, as in §3.1. Growth in the y term happens when a trajectory spends a long time in the expansive neighborhood of x^*. When a transition from expansive to non-expansive sequence (or vice versa) occurs, there is a contraction as described above. We will represent a point x in X by two sequences of numbers: $\xi_0(x), \xi_1(x), \ldots$, indicating the number of consecutive iterates the x trajectory spends in a Markov rectangle containing the fixed point x^* and $\tilde{\xi}_1(x), \tilde{\xi}_2(x), \ldots$, indicating the number of consecutive iterates the x trajectory spends outside of it. We also let $N_k := \xi_0 + \xi_1 + \cdots + \xi_k$ and $\tilde{N}_k := \tilde{\xi}_1 + \cdots + \tilde{\xi}_k$. All of these numbers can be thought of as random variables on the Borel probability space (X, μ). Our set S' will be defined in terms of consecutive sequences of ξ.

Remark 4.4. In the case of the maps $T(x) = mx \pmod 1$, there exists a Markov partition for which the sequence of ξ_j corresponds to a sequence of iid geometric random variables on the Borel probability space (X, μ). In this case, calculations can be done directly, using only properties of elementary discrete probability distributions.

In general, the random variables ξ_i are not independent. However, the exponential cluster property (also known as ψ mixing property with exponential decay) used in §3.1.3 allows one to show, in our parameter range, that the total probability of escape can be still compared with the probability of escaping through only one long sequence of consecutive expanding iterates. This was estimated in §4.1.1.

First, we establish a lower bound in the average bursting time in terms of $p\alpha$, that is of special interest in the case when the multiplicative effect is negligible (small $p\alpha$). Later, in Proposition 4.6, we establish a sharper lower bound for the multiplicative case (large $p\alpha$).

We set $\Delta = \|g\|_\infty + \zeta = \mathcal{O}(1)$, so that $q\Delta$ is a global upper bound on the kick. For the Markov partition $\mathcal{R}$, we let R_0 be the rectangle containing x^*, and define $\Delta_0 = \sup_{x\in R_0}\{g(x)\} + \zeta$, so that $q\Delta_0$ bounds the kick on R_0. We recall that the partition $\mathcal{R}$ can be chosen with arbitrarily small radius. Hence, Δ_0 can be made as close to $1+\zeta$ as desired. Also, let $\Gamma = \sup_{x\in R_0}\{h(x)+\sigma\}$; then Γ can be made arbitrarily close to $1+\sigma$.

Proposition 4.5. Let $l(z) = \frac{z}{e^z-1}$. Then, for any $\epsilon > 0$ such that if α is sufficiently large and p, q and s are sufficiently small, we have:

$$\log\tau \geq (1-\epsilon)l(p\alpha)\Lambda\alpha.$$

Proof. To establish this, we let $\tilde{\Delta}_0 = \frac{\Delta_0}{l((1+\sigma)p\alpha)}$. We also fix $B > 0$ as in the statement of Lemma 3.5 and choose time 0 to be the last time that the y trajectory is below Bq, so that for $n \geq 1$, y_n never goes back below Bq before exceeding the threshold Y. Notice that for $x_n \in R_0$,

$$y_{n+1} \leq \big(f(x_n) + (h(x_n)+\sigma)p\big)y_n + q\Delta_0 \leq (1+\Gamma p)y_n + q\Delta_0.$$

First, we consider only escaping trajectories for which $\xi_i \leq \alpha$ for all i before escaping. The measure of the remaining escaping trajectories will be included directly in the final estimate. Then, by induction on ξ, for $\xi \leq \alpha$,

$$y_\xi - y_0(1+\Gamma p)^\xi \leq q\Delta_0\frac{(1+\Gamma p)^\xi - 1}{\Gamma p} \leq q\Delta_0\frac{e^{\Gamma p\xi}-1}{\Gamma p} \leq q\tilde{\Delta}_0\xi.$$

Next, for $x_n \notin R_0$ we have $y_{n+1} \leq cy_n + q\Delta$, so by induction on $\tilde{\xi}$,

$$y_{\xi+\tilde{\xi}} \leq c^{\tilde{\xi}}(y_0(1+\Gamma p)^\xi + q\tilde{\Delta}_0\xi) + \frac{q\Delta}{1-c}.$$

Let $\tilde{c} = c(1+\Gamma p)^\alpha$. Then $\tilde{c} < 1$ if p, q and s are sufficiently small and $p\alpha \leq \frac{1}{2}\log\frac{1}{c}$. By induction, we obtain:

$$y_{N_k+\tilde{N}_k} \leq \tilde{c}^k y_0 + \frac{q\Delta}{(1-c)(1-\tilde{c})} + q\tilde{\Delta}_0\Big(\sum_{j=0}^{k}\tilde{c}^{k-j}\xi_j\Big).$$

If the threshold $Y = \alpha q$ is reached within $k > 1$ expansive sequences, then recalling that $y_0 \leq Bq$ we must have:

$$\sum_{j=0}^{k}\tilde{c}^{k-j}\xi_j \geq \frac{\alpha - \tilde{c}^k B}{\tilde{\Delta}_0} - \frac{\Delta}{\tilde{\Delta}_0(1-c)(1-\tilde{c})}.$$

In this context, we will say that a trajectory escapes by route k if $k+1$ is the smallest integer for which the above holds. The set S' mentioned above consists of the union of trajectories that may initiate an escape by route k over all $k \in \mathbb{N}$, and those for which there exists some i with $\xi_i > \alpha$ before escaping. We let ι be the smallest so that $\xi_\iota > \alpha$, and denote its measure by $\hat{\mu}_\iota$.

To bound $\hat{\mu}_\iota$, we use the proof of Lemma 3.5 with $t = \alpha$ and $c = \tilde{c}$ defined above. Adding over ι, we get that $\sum_{\iota \in \mathbb{N}} \hat{\mu}_\iota \leq \hat{A}\chi^{-\alpha}$, for some constant $\hat{A}$ independent of p, q and α.

Given any $\epsilon > 0$, if α sufficiently large (depending on ϵ but independent of p and q), we can apply Lemma 3.5 with $t = (1-\epsilon)\frac{\alpha}{\Delta_0} > B$ and $t_* = B$. We obtain that the measure μ_k of x trajectories that initiate an escape by route k decays exponentially with k. For $k = 1$, Lemma 3.4 implies that $\mu_1 \leq A\chi^{-(1-\epsilon)\frac{\alpha}{\Delta_0}}$.

Combining the previous two paragraphs, we get that there exists some constant $\tilde{L}$ such that the total measure $\mu(S')$ is bounded by

$$\mu(S') \leq \tilde{L}\chi^{-(1-\epsilon)\frac{\alpha}{\Delta_0}}.$$

Recalling that $\sigma \to 0$ as $(p, q, s) \to (0, 0, 0)$ and that Δ_0 can be chosen arbitrarily close to 1 by choosing $\mathcal{R}$ appropriately, and (p, q, s) sufficiently close to $(0, 0, 0)$, we combine the previous estimate with Lemma 3.2, and conclude that for α sufficiently large, $p\alpha \leq \frac{1}{2}\log\frac{1}{c}$ and p, q and s sufficiently small we have:

$$\log\tau \geq (1-\epsilon)l(p\alpha)\Lambda\alpha.$$

□

Recall that $K(p,\alpha) := \frac{1}{p\alpha\tilde{l}(p\alpha)} e^{\frac{p}{2}\chi^{\frac{\log\frac{1}{c}}{4p(1+\tilde{l}(p\alpha))} - \frac{\Delta\tilde{l}(p\alpha)}{4c(1-c)(1+\tilde{l}(p\alpha))}}}$.

Proposition 4.6. Let $\epsilon > 0$. For sufficiently small p, q, s and sufficiently large α such that $K(p,\alpha) > 1$, a lower bound on the scaling of $\log\tau$ is:

$$\log\tau \geq (1-\epsilon)\big(1 - \tilde{l}(p\alpha)\big)\Lambda\frac{\log(1+p\alpha)}{p},$$

where $\tilde{l}(p\alpha) \to 0$ as $p\alpha \to \infty$.

Remark 4.7. The restriction on the size of α in terms of p can be improved by taking into account the fact that typically, trajectories spend a long time outside of the expanding region before coming back to it. This would allow larger thresholds α. However, sufficiently large values of α would still need to be excluded. Therefore, we only present the argument as stated in Proposition 4.6.

Proof of Proposition 4.6. Now we fix $B > 0$, to be specified later, and for the moment choose time 0 to be the first time that the y trajectory exceeds Bq and y_n never goes back below Bq before escaping. After a sequence of expansions corresponding to a block of length ξ followed by a contraction, similarly to the proof of Proposition 4.5, we have:

$$\begin{aligned}
\frac{y_{\xi+\tilde{\xi}}}{y_0} &\leq \left((1+\Gamma p)^\xi + \tfrac{\Delta_0}{B}\tfrac{(1+\Gamma p)^\xi - 1}{\Gamma p}\right)c + \tfrac{\Delta}{B(1-c)} \\
&\leq (1+\Gamma p)^\xi\left(1 + \tfrac{\Delta}{B(1-c)c} + \tfrac{\Delta_0}{B}\tfrac{1-(1+\Gamma p)^{-\xi}}{\Gamma p}\right)c \\
&\leq (1+\Gamma p)^\xi\left(1 + \tfrac{\Delta}{B(1-c)c} + \tfrac{\Delta_0}{B}\xi\right)c.
\end{aligned}$$

Let $E(p) := \log(1 + \Gamma p)$. Then, by induction on k,

$$\log\left(\frac{y_{N_k+\tilde N_k}}{y_0}\right) \le N_k E(p) + k\log c + \sum_{j=1}^k \log\left(1 + \tfrac{\Delta}{B(1-c)c} + \tfrac{\Delta_0}{B}\xi_j\right)$$
$$\le N_k\left(E(p) + \tfrac{\Delta_0}{B}\right) + k\left(\log c + \tfrac{\Delta}{B(1-c)c}\right).$$

Therefore, for a trajectory to initiate an escape without returning to the region $y \le Bq$ before reaching the threshold, we need to have the following inequality holding for some k, l:

$$\log\frac{\alpha}{B} \le (N_{k+l} - N_{l-1})\left(E(p) + \frac{\Delta_0}{B}\right) + k\left(\log c + \frac{\Delta}{B(1-c)c}\right).$$

Equivalently,

$$N_{k+l} - N_{l-1} \ge \frac{\log\alpha_0}{E(p)+\frac{\Delta_0}{B}} - k\frac{\log c + \frac{\Delta}{B(1-c)c}}{E(p)+\frac{\Delta_0}{B}}$$
$$=: M_0(\alpha, p, B) + k\beta(p, B) =: M_k(\alpha, p, B).$$

We will say that such a trajectory escapes by route k. This condition depends only on the x dynamics and will be used to bound the total measure of trajectories that initiate an escape by route k from above. In this setting, we define the set $S' \subset X$ as the union of all trajectories that can initiate an escape by route k over all $k \in \mathbb{N}$.

By Lemma 3.6, we know that if β is sufficiently large and M_0 is not exponentially large in β, there is a constant $0 < \tilde\theta < 1$ such that

$$\mu(N_{k+l} - N_{l-1} \ge M_0 + k\beta) \le A\tilde\theta^k \chi^{-M_0}.$$

Now, we set $B = \frac{\alpha\log(1+p\alpha)}{1+p\alpha}$, and let $\tilde l(z) = \frac{1+z}{z\log(1+z)}$. For p, q and s sufficiently small, the restriction in the sizes relative of M_0 and β is satisfied as long as

$$K(p,\alpha) := \frac{1}{p\alpha\tilde l(p\alpha)} e^{\frac{p}{2}} \chi^{\frac{\log\frac{1}{c}}{4p(1+\tilde l(p\alpha))} - \frac{\Delta\tilde l(p\alpha)}{4c(1-c)(1+\tilde l(p\alpha))}} > 1.$$

Then, by Lemma 3.2, the measure of S' is bounded by $\mu(S') \le \frac{A}{1-\tilde\theta}\chi^{-M_0}$. In consequence, for sufficiently small p, q and s we have:

$$\log\tau \ge \Lambda\frac{\log(1+p\alpha) - \log\log(1+p\alpha)}{\log(1+\Gamma p) + \Delta_0\frac{1+p\alpha}{\alpha\log(1+p\alpha)}} + \log\frac{A}{1-\tilde\theta} - \log 4$$
$$\ge (1-\epsilon)\Lambda\frac{\log(1+p\alpha)}{p(1+\tilde l(p\alpha))} \ge (1-\epsilon)(1-\tilde l(p\alpha))\Lambda\frac{\log(1+p\alpha)}{p}.$$

□

If parameters p, α satisfy condition $(\star)$, upper and lower bounds from Propositions 4.3 and 4.6 combined yield Theorem 4.

Remark 4.8. In case the bifurcating orbit is periodic, $\{x_1^*, \ldots, x_d^*\}$, the corresponding f and h in the analogue of Equation (3) for T^d have the same value at all points of the bifurcating periodic orbit. Furthermore, there are smooth conjugacies between the fiber maps restricted to small neighborhoods of the d fixed points. In general, we would not be able to normalize g simultaneously at all d points; instead, we would normalize g so that its maximum value on the periodic orbit is 1. The estimates for the lower bound would need to be modified accordingly. The ones for the upper bound remain valid.

4.2 Proof of scaling laws

In this section, we extend the linear analysis presented in §4.1 to the nonlinear setting, and complete the proof of the results stated in §2.2. We also obtain results that are valid in a parameter range broader than that of Theorems 1 and 2, as claimed in the introduction.

With the normalizations described in §2.1 and after possibly rescaling y, the y dynamics on the fiber over the fixed point x^* is described as follows. In the case of transcritical bifurcations (general case), Equation (4) becomes

$$y_{n+1} = (1+p)y_n \pm y_n^2 + \mathcal{O}(qy_n + p^2 y_n + pq + q^2 + y_n^3), \tag{11}$$

and in the case of pitchfork bifurcations (symmetric case), Equation (5) becomes

$$y_{n+1} = (1+p)y_n \pm y_n^3 + \mathcal{O}(qy_n + p^2 y_n + pq + q^2 + y_n^4). \tag{12}$$

In [ZHO03], Zimin, Hunt and Ott have classified the effect of the nonlinearities depending on whether they accelerate or confine the burst. They call them hard and soft transitions, respectively. We will analyze these two scenarios. We also distinguish between multiplicative (drift-dominated) and additive (noise-dominated) bubbling phenomena, which occur depending on the relative sizes of the parameters p and q. Roughly speaking, when the effect of p is dominant, we call it multiplicative bubbling, and when it is negligible, we call it additive bubbling.

We note that the analysis from §4.1.2 is applicable in the nonlinear setting since it deals with a lower bound for the bursting time. On the other hand, we have to adjust the upper bound estimates from §4.1.1 to incorporate nonlinear terms.

4.2.1 Asymmetric case: generic transcritical bifurcation.

Here, we show two scaling laws valid for generic asymmetric bubbling bifurcations. They are valid for a threshold Y independent of p and q in the hard transition case ($a(x^*)g(x^*) > 0$), proportional to p in the multiplicative case of soft transition ($a(x^*)g(x^*) < 0$), and to $\sqrt{q}$ in the additive case of soft transition, as will be shown in the proofs. In this setting, the y dynamics of the fixed point x^* can be written as (11).

Multiplicative bubbling.

Proposition 4.9. If $p^2 > 4q > \frac{p}{k(p)}$, there exists a constant $\tilde{C} > 1$ independent of p, q such that if (p, q) is sufficiently close to $(0, 0)$,

$$\tilde{C}^{-1}\Lambda \le \frac{\log \tau(Y)}{\frac{1}{p}\log(1 + \frac{p^2}{q})} \le \tilde{C}\Lambda.$$

Furthermore, for any $\epsilon > 0$, if $(p, \frac{q}{p^2})$ is sufficiently close to $(0, 0)$,

$$(1-\epsilon)\Lambda < \frac{\log \tau(Y)}{\frac{1}{p}\log\frac{p^2}{q}} < (1+\epsilon)\Lambda.$$

Proof. Assume is $p^2 \geq 4q$. The attracting fixed point of the y dynamics is our threshold of interest in the soft transition case ($a = -1$). Since $y_* \approx p$, we set $\alpha = r\frac{p}{q}$, for some $0 < r \leq 1$. In this case, the parameter s introduced in §4.1 is simply $r = s$, so $s \to 0$ if $r \to 0$.

The hard transition case ($a = 1$), where no attractor is given by the local analysis, corresponds to the scenario where a linear regime takes place and then it is replaced by a nonlinear one. We set a threshold $\alpha = r\frac{p}{q}$ to separate the linear and nonlinear behaviors, for some $r > 0$ independent of p and q.

Let us take an initial condition $y_0 = \frac{q}{p}$. Assuming p is small and q is small but not extremely small compared to p, $\frac{1}{p}k(p) > \frac{1}{q}$, Theorem 4 implies the following scaling for sufficiently small r:

$$C^{-1}\Lambda < \frac{\log \tau(rp)}{\frac{1}{p}\log(1 + \frac{rp^2}{q})} < C\Lambda,$$

which, in turn, implies:

$$C^{-1}\frac{\log(1 + \frac{rp^2}{q})}{\log(1 + \frac{p^2}{q})}\Lambda < \frac{\log \tau(rp)}{\frac{1}{p}\log(1 + \frac{p^2}{q})} < C\Lambda.$$

In the hard transition case, the burst is not confined in a small region. It may be of order one. In this setting, we also investigate the average bursting time associated to a threshold Y, which is determined by the y value at which the higher order terms become significant, for example of size $\frac{1}{3}y^2$. To bound $\log \tau(Y)$ from below, we use $\tau(Y) \geq \tau(rp)$ for $rp < Y$. We choose a threshold $Y \leq 1$, that is reached for all sufficiently small values of p and q and such that the higher order terms are bounded by $\frac{1}{3}y^2$ for $rp < y < Y$.

To find an upper bound on the scaling of $\log \tau$, we extend the analysis in §4.1.1. There we found $\tilde{n}$ such that if x spends $\tilde{n}$ consecutive iterates in the region $|x - x^*| < \tilde{x}$, then at the end of those iterations, $y \geq \alpha q = rp$. We can guarantee that $y \geq Y$ if x spends t additional iterates in the region $|x - x^*| \leq \tilde{x}$, where we determine t as follows. When $|x_n - x^*| < \tilde{x}$ and $rp \leq y_n \leq Y$, $y_n + \frac{2}{3}y_n^2 \leq y_{n+1} \leq (1 + 2p)y_n + \frac{4}{3}y_n^2$. Hence, we have that $y_{n+1} \leq \frac{8}{3}y_n$.

Calling the time at which y exceeds rp time 0, we can bound from below the solution of our original difference equation with the solution $y(t)$ of a differential equation inductively if we can check $y(0) = rp$ and $y(n+1) \leq y(n) + \frac{2}{3}y(n)^2$. For values $n \leq \frac{32}{3rp} - \frac{32}{3}$, this is the case for the solution of

$$\dot{y} = \frac{3}{32}y^2, \quad y(0) = rp.$$

This solution is given by $y(t) = \frac{1}{\frac{1}{rp} - \frac{3}{32}t}$.

From this, we conclude that an extra $t = \frac{32}{3}\frac{(Y - rp)}{Yrp} \leq \frac{32}{3rp} - \frac{32}{3} < \frac{32}{3rp}$ iterates in the non-contracting region would oblige a burst of size Y. Thus, proceding as in (10), for p, q and r sufficiently small, we have the following bounds:

$$C^{-1}\frac{\log(1 + \frac{rp^2}{q})}{\log(1 + \frac{p^2}{q})}\Lambda < \frac{\log \tau(Y)}{\frac{1}{p}\log(1 + \frac{p^2}{q})} < C\left(1 + \frac{1}{r\log(1 + \frac{p^2}{q})}\right)\Lambda.$$

In particular, if we fix a sufficiently small value for r, the first statement follows.

We obtain the second statement, corresponding to the asymptotic scaling for $\log \tau(Y)$ in the parameter regime considered in [ZHO03], $p^2 \gg q$ as follows. For any $\epsilon > 0$, if (p, q) is sufficiently close to $(0, 0)$ and $\frac{p^2}{q}$ sufficiently large, we can let $r = \frac{1}{\log\log \frac{p^2}{q}}$ and obtain from the previous bounds:

$$(1-\epsilon)\Lambda < \frac{\log \tau(Y)}{\frac{1}{p}\log \frac{p^2}{q}} < (1+\epsilon)\Lambda. \qquad \square$$

Additive bubbling.

Proposition 4.10. If $p^2 < 4q$, there exists a constant $\tilde{C} > 1$ independent of p, q such that if (p, q) is sufficiently close to $(0, 0)$,

$$\tilde{C}^{-1}\Lambda \leq \frac{\log \tau(Y)}{\frac{1}{q^{\frac{1}{2}}}} \leq \tilde{C}\Lambda.$$

Proof. Assume $p^2 < 4q$. In the case of a soft transition ($q > 0$), the attracting fixed point for the y dynamics is $y_* \approx \sqrt{q}$. Our threshold of interest is of the order of $\sqrt{q}$. Hence, we choose $\alpha = r\frac{1}{\sqrt{q}}$. In this case, $r = \sqrt{s}$ and condition s is small when r is small.

In the hard transition case, the linear term is negligible with respect to the kick. Therefore nonlinear terms become significant when the kick becomes negligible, and no intermediate regime is governed by the expansive linear term. In this setting, we investigate the threshold $\alpha q = r\sqrt{q} \approx y_*$, which separates the constant and nonlinear behaviors.

In both cases we first require to reach $\alpha = \frac{r}{\sqrt{q}}$, for some $0 < r \leq 1$ sufficiently small, corresponding to the predominance of the linear regime. From Theorem 4, if p, q and r are sufficiently small, we obtain:

$$C^{-1}(1-r)\Lambda < \frac{\log \tau(r\sqrt{q})}{\frac{r}{\sqrt{q}}} \leq C\Lambda.$$

In the hard transition case, by reasoning similarly to the multiplicative case, we obtain that to pass from the linear setting to the threshold Y of order 1, $\frac{32}{3r\sqrt{q}}$ extra iterates in the non-contracting region suffice. Hence, we have:

$$C^{-1}r(1-r)\Lambda \leq \frac{\log \tau(Y)}{\frac{1}{q^{\frac{1}{2}}}} \leq \Lambda\left(Cr + \frac{32}{3r}\right).$$

Hence, if we fix a sufficiently small value for r, Proposition 4.10 follows. $\square$

4.2.2 Symmetric case: generic pitchfork bifurcation.

Here, we show two scaling laws valid for generic pitchfork bubbling bifurcations. They are valid for a threshold Y independent of p and q in the hard transition case ($a(x^*) > 0$), proportional to $\sqrt{p}$ in the multiplicative case of soft transition ($a(x^*) < 0$), and to $\sqrt[3]{q}$ in the additive case of soft transition, as will be shown in the proofs. In this setting, the y dynamics of the fixed point x^* can be written as (12).

Multiplicative bubbling.

Proposition 4.11. If $p^3 > \frac{27}{4}q^2$ and $q > \frac{\sqrt{p}}{k(p)}$, there exists a constant $\tilde{C} > 1$ independent of p, q such that if (p, q) is sufficiently close to $(0, 0)$,

$$\tilde{C}^{-1}\Lambda \le \frac{\log \tau(Y)}{\frac{1}{p}\log(1 + \frac{p^{\frac{3}{2}}}{q})} \le \tilde{C}\Lambda.$$

Furthermore, for any $\epsilon > 0$, if $(p, \frac{p^{\frac{3}{2}}}{q})$ is sufficiently close to $(0, 0)$,

$$(1 - \epsilon)\Lambda < \frac{\log \tau(Y)}{\frac{1}{p}\log(1 + \frac{p^{\frac{3}{2}}}{q})} < (1 + \epsilon)\Lambda.$$

Proof. Assume $p^3 \ge \frac{27}{4}q^2$. The soft transition case occurs when $a = -1$. In this situation, the cubic equation has three real roots and the continuation of the fixed point 0, $y_* \approx \sqrt{p}$, is stable. In this case, we set $\alpha = r\frac{\sqrt{p}}{q}$, where r corresponds to $\sqrt{s}$, and therefore s is small when r is.

The hard transition case occurs when $a = 1$. The threshold corresponding to $\alpha = r\frac{\sqrt{p}}{q}$ corresponds to the transition between linear and nonlinear behaviors.

The analysis is similar to the previous subsection. Let us take an initial condition $y_0 = \frac{q}{p}$. Assuming that p is small and $\frac{1}{\sqrt{p}}k(p) > \frac{1}{q}$, by Theorem 4 we get:

$$C^{-1}\frac{\log(1 + r\frac{p^{\frac{3}{2}}}{q})}{\log(1 + \frac{p^{\frac{3}{2}}}{q})}\Lambda \le \frac{\log \tau(r\sqrt{p})}{\frac{1}{p}\log(1 + \frac{p^{\frac{3}{2}}}{q})} \le C\Lambda.$$

As in the asymmetric case, when the transition is hard, we are also interested in bursts up to order one, whose size Y is determined by higher order terms, but independent of p and q. We choose it in such a way that the higher order terms are bounded by $\frac{1}{3}y^3$ for $r\sqrt{p} \le y < Y$. In this case, if $|x_n - x^*| \le \tilde{x}$ and $y_n \ge r\sqrt{p}$, we know that $y_n + \frac{2}{3}y_n^3 \le y_{n+1} \le (1 + 2p)y_n + \frac{4}{3}y_n^3$.

As in §4.2.1, $\frac{y_n}{y_{n+1}} \ge \frac{3}{8}$, and we consider the differential equation:

$$\dot{y} = \frac{9}{256}y^3, \quad y(0) = r\sqrt{p},$$

with solution given by $y(t) = \sqrt{\frac{r^2 p}{1 - 2\frac{9}{256}r^2 pt}}$.

This function bounds from below the solution of our system up to $t = \frac{128}{9r^2 p} - \frac{128}{9}$. This is the time it takes the solution of the differential equation to reach Y. Hence, we get that an extra $t = \frac{128}{9r^2 p}$ iterates in the non-contracting region would oblige a burst of size Y. Thus, if p, q and r are sufficiently small, we have the following bounds:

$$C^{-1}\frac{\log(1 + r\frac{p^{\frac{3}{2}}}{q})}{\log(1 + \frac{p^{\frac{3}{2}}}{q})}\Lambda < \frac{\log \tau(Y)}{\frac{1}{p}\log(1 + \frac{p^{\frac{3}{2}}}{q})} < C\left(1 + \frac{1}{r^2\log(1 + \frac{p^{\frac{3}{2}}}{q})}\right)\Lambda.$$

Hence, if we fix a sufficiently small value for r, the first statement follows. Furthermore, we obtain the asymptotic scaling for $\log \tau(Y)$ in the parameter regime considered in [ZHO03], $p^{\frac{3}{2}} \gg q$, as follows. For any $\epsilon > 0$, if (p,q) is sufficiently close to $(0,0)$ and $\frac{p^{\frac{3}{2}}}{q}$ sufficiently large, we can let $r = \frac{1}{\log\log \frac{p^{\frac{3}{2}}}{q}}$ and obtain from the previous bounds:

$$(1-\epsilon)\Lambda < \frac{\log \tau(Y)}{\frac{1}{p}\log(1+\frac{p^{\frac{3}{2}}}{q})} < (1+\epsilon)\Lambda. \qquad \square$$

Additive bubbling.

Proposition 4.12. If $p^3 < \frac{27}{4}q^2$, there exists a constant $\tilde{C} > 1$ independent of p, q such that if (p,q) is sufficiently close to $(0,0)$,

$$\tilde{C}^{-1}\Lambda \leq \frac{\log \tau(Y)}{\frac{1}{q^{\frac{2}{3}}}} \leq \tilde{C}\Lambda.$$

Proof. Assume $p^3 < \frac{27}{4}q^2$. Analogously to the asymmetric case, we first consider the linear regime, determined by the fact that $y_* \approx \sqrt[3]{q}$. We set the threshold $\alpha q = r\sqrt[3]{q}$. In this setting, $r = \sqrt[3]{s}$, and from Theorem 4, for $\epsilon > 0$, if p, q and r are sufficiently small, we obtain:

$$C^{-1}(1-r)\Lambda \leq \frac{\log \tau(r\sqrt[3]{q})}{\frac{r}{q^{\frac{2}{3}}}} \leq C\Lambda.$$

As above, in the hard transition case, to pass from the linear setting to a threshold Y of order 1, $\frac{128}{9r^2 q^{\frac{2}{3}}}$ extra iterates in the non-contracting region suffice. Hence, for sufficiently small p, q and r we have:

$$C^{-1}r(1-\frac{3}{2}r)\Lambda \leq \frac{\log \tau(Y)}{\frac{1}{q^{\frac{2}{3}}}} \leq \left(Cr + \frac{128}{9r^2}\right)\Lambda.$$

Hence, if we fix a sufficiently small value for r, Proposition 4.12 follows. $\square$

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