On bipartite 2-factorisations of $K_n - I$ and the Oberwolfach problem

Darryn Bryant [*]
The University of Queensland
Department of Mathematics
Qld 4072
Australia

Peter Danziger [†]
Department of Mathematics
Ryerson University
Toronto, Ontario,
Canada M5B 2K3

Abstract

It is shown that if $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order n and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are non-negative integers such that $\alpha_1+\alpha_2+\cdots+\alpha_t = \frac{n-2}{2}$, $\alpha_1 \geq 3$ is odd, and α_i is even for $i = 2, 3, \ldots, t$, then there exists a 2-factorisation of $K_n - I$ in which there are exactly α_i 2-factors isomorphic to F_i for $i = 1, 2, \ldots, t$. This result completes the solution of the Oberwolfach problem for bipartite 2-factors.

1 Introduction

The Oberwolfach problem was posed by Ringel in the 1960s and is first mentioned in [19]. It relates to specification of tournaments and specifically to balanced seating arrangments at round tables. In this article we will provide a complete solution to the Oberwolfach problem in the case where there are an even number of seats at each table.

Let $n \geq 3$ and let F be a 2-regular graph of order n. When n is odd, the Oberwolfach problem OP(F) asks for a 2-factorisation of the complete graph K_n on n vertices

*Supported by the Australian Research Council
†Supported by NSERC

in which each 2-factor is isomorphic to F. When n is even, the Oberwolfach problem OP(F) asks for a 2-factorisation of $K_n - I$, the complete graph on n vertices with the edges of a 1-factor removed, in which each 2-factor is isomorphic to F.

In 1985, Häggkvist [21] settled OP(F) for any bipartite 2-regular graph F of order $n \equiv 2 \pmod{4}$. The result is an immediate consequence of Lemma 6 below, and the existence of Hamilton cycle decompositions of K_m for all odd m. Here we complete the solution of the Oberwolfach problem for bipartite 2-factors by dealing with the case $n \equiv 0 \pmod{4}$. To do this we prove the following more general result on bipartite 2-factorisations of $K_n - I$.

Theorem 1 *If $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order n and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are non-negative integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{n-2}{2}$, $\alpha_1 \geq 3$ is odd, and α_i is even for $i = 2, 3, \ldots, t$, then there exists a 2-factorisation of $K_n - I$ in which there are exactly α_i 2-factors isomorphic to F_i for $i = 1, 2, \ldots, t$.*

For $n \equiv 0 \pmod{4}$, we obtain a solution to the Oberwolfach problem OP(F) for any bipartite 2-regular graph F of order n by applying Theorem 1 with $t = 1$, $F_1 = F$ and $\alpha_1 = \frac{n-2}{2}$. Combining this with Häggkvist's result we have the following theorem.

Theorem 2 *If F is a bipartite 2-regular graph of order n then there is a 2-factorisation of $K_n - I$ in which each 2-factor is isomorphic to F. That is, for any bipartite 2-regular graph F,* OP(F) *has a solution.*

Throughout the paper, we will use the notation $[m_1, m_2, \ldots, m_t]$ to denote the 2-regular graph consisting of t (vertex-disjoint) cycles of lengths $m_1, m_2, \ldots, m_t$. A large number of special cases of the Oberwolfach problem have been solved, but the general problem remains, for the most part, completely open. However, a recent result [10] shows that for a sparse infinite family of values of n, OP(F) has a solution for any 2-regular graph F of order n. It is known that there is no solution to OP(F) for

$$F \in \{[3,3], [4,5], [3,3,5], [3,3,3,3]\},$$

but there is no other known instance of the Oberwolfach problem with no solution. In particular, a solution is known for all other instances with $n \leq 18$, see [2, 7, 17, 18, 25]. The special case of the Oberwolfach problem in which all the cycles in F are of uniform length has been solved completely, see [4, 5, 23, 25]. The case where all the cycles are of length 3 and n is odd is the famous Kirkman's schoolgirl problem, which was solved in 1971 [29]. A large number of other special cases of the Oberwolfach problem have been solved, see [8, 13, 20, 22, 26, 27, 28, 31].

A generalisation of the Oberwolfach problem, known as the Hamilton-Waterloo problem, asks for a 2-factorisation of K_n (n odd) or $K_n - I$ (n even) in which α_1 of the 2-factors are isomorphic to F_1 and α_2 of the 2-factors are isomorphic to F_2 for all non-negative α_1 and α_2 satisfying $\alpha_1 + \alpha_2 = \frac{n-1}{2}$ (n odd) or $\alpha_1 + \alpha_2 = \frac{n-2}{2}$ (n even). Results on the Hamilton-Waterloo problem can be found in [1, 11, 12, 15, 16, 21, 24].

If we apply Theorem 1 with $t = 2$ then we obtain the following result which settles the Hamilton-Waterloo problem for bipartite 2-factors of order $n \equiv 0 \pmod 4$ except in the case where all but one of the 2-factors are isomorphic. Note that for $n \equiv 0 \pmod 4$ the number of 2-factors in a 2-factorisation of $K_n - I$ is odd. When $n \equiv 2 \pmod 4$ and the number of 2-factors of each type is even, a solution to the Hamilton-Waterloo problem in the case of bipartite 2-factors can be obtained by applying Lemma 6 below, and using the existence of Hamilton cycle decompositions of K_m for all odd m (with $m = \frac{n}{2}$), see [21].

Theorem 3 *If F_1 and F_2 are two bipartite 2-regular graphs of order $n \equiv 0 \pmod 4$ and α_1 and α_2 are non-negative integers satisfying $\alpha_1 + \alpha_2 = \frac{n-2}{2}$, then there is a 2-factorisation of $K_n - I$ in which α_1 of the 2-factors are isomorphic to F_1 and α_2 of the 2-factors are isomorphic to F_2, except possibly when $n \equiv 0 \pmod 4$ and $1 \in \{\alpha_1, \alpha_2\}$.*

Recent surveys of results on the Oberwolfach problem, the Hamilton Waterloo Problem, and on 2-factorisations generally, are [9] and [30].

2 Preliminary results and notation

Let Γ be a finite group. A *Cayley subset* of Γ is a subset which does not contain the identity and which is closed under taking of inverses. If S is a Cayley subset of Γ, then the *Cayley graph* on Γ with *connection set* S, denoted $\mathrm{Cay}(\Gamma, S)$, has the elements of Γ as its vertices and there is an edge between vertices g and h if and only if $g = h + s$ for some $s \in S$.

We need the following two results on Hamilton cycle decompositions of Cayley graphs. The first was proved by Bermond et al [6], and the second by Dean [14]. Both results address the open question of whether every connected Cayley graph of even degree on a finite abelian group has a Hamilton cycle decomposition [3].

Theorem 4 ([6]) *Every connected 4-regular Cayley graph on a finite abelian group has a Hamilton cycle decomposition.*

Theorem 5 ([14]) *Every 6-regular Cayley graph on a cyclic group which has a generator of the group in its connection set has a Hamilton cycle decomposition.*

A Cayley graph on a cyclic group is called a *circulant graph* and we will be using these, and certain subgraphs of them, frequently. Thus, we introduce the following notation. The length of an edge $\{x, y\}$ in a graph with vertex set $\mathbb{Z}_m$ is defined to be either $x - y$ or $y - x$, whichever is in $\{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$ (calculations in $\mathbb{Z}_m$). When m is even we call $\{\{x, x + s\} : x = 0, 2, \ldots, m - 2\}$ the even edges of length s and we call $\{\{x, x + s\} : x = 1, 3, \ldots, m - 1\}$ the odd edges of length s.

For any $m \geq 3$ and any $S \subseteq \{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$, we denote by $\langle S \rangle_m$ the graph with vertex set $\mathbb{Z}_m$ and edge set consisting of the edges of length s for each $s \in S$, that is, $\langle S \rangle_m = \mathrm{Cay}(\mathbb{Z}_m, S \cup -S)$. For m even, if we wish to include in our graph only the even edges of length s then we give s the superscript "e". Similarly, if we wish to include only the odd edges of length s then we give s the superscript "o". For example, the graph $\langle \{1, 2^{\mathrm{o}}, 5^{\mathrm{e}}\} \rangle_{12}$ is shown in Figure 1.

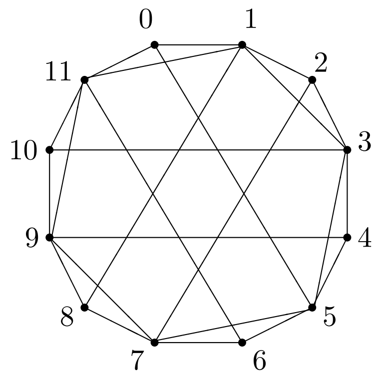

Figure 1: The graph $\langle\{1, 2^{\text{o}}, 5^{\text{e}}\}\rangle_{12}$

For any given graph K, the graph $K^{(2)}$ is defined by $V(K^{(2)}) = V(K) \times \mathbb{Z}_2$ and $E(K^{(2)}) = \{\{(x,a),(y,b)\} : \{x,y\} \in E(K),\ a,b \in \mathbb{Z}_2\}$. If $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ is a set of graphs then we define $\mathcal{F}^{(2)} = \{F_1^{(2)}, F_2^{(2)}, \ldots, F_t^{(2)}\}$. Observe that if $\mathcal{F}$ is a factorisation of K, then $\mathcal{F}^{(2)}$ is a factorisation of $K^{(2)}$.

Häggkvist proved the following very useful result in [21].

Lemma 6 ([21]) *For any $m > 1$ and for each bipartite 2-regular graph F of order $2m$, there exists a 2-factorisation of $C_m^{(2)}$ in which each 2-factor is isomorphic to F.*

Lemma 7 *For each even $m \geq 8$ there is a factorisation of K_m into $\frac{m-4}{2}$ Hamilton cycles and a copy of $\langle\{1, 3^{\text{e}}\}\rangle_m$.*

Proof The cases $m \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 4$ are dealt with separately. For $m \equiv 2 \pmod 4$ observe that the mapping

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & m-3 & m-2 & m-1 \\ 0 & \frac{m}{2} & \frac{m}{2}+1 & 1 & 2 & \frac{m}{2}+2 & \frac{m}{2}+3 & 3 & 4 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-1 & m-1 \end{pmatrix}$$

given by

$$\psi(x) = \begin{cases} \frac{x}{2} & x \equiv 0 \pmod 4 \\ \frac{m}{2} + \lfloor \frac{x}{2} \rfloor & x \equiv 1, 2 \pmod 4 \\ \frac{x-1}{2} & x \equiv 3 \pmod 4 \end{cases}$$

is an isomorphism from $\langle\{1,3^{\rm e}\}\rangle_m$ to $\langle\{1,\frac{m}{2}\}\rangle_m$. So in the case $m\equiv 2\,(\bmod\ 4)$ it is sufficient to show that $\langle\{2,3,\ldots,\frac{m}{2}-1\}\rangle_m$ has a Hamilton cycle decomposition. This is straightforward as $\{\langle\{2,3\}\rangle_m,\langle\{4,5\}\rangle_m,\ldots,\langle\{\frac{m}{2}-5,\frac{m}{2}-4\}\rangle_m,\langle\{\frac{m}{2}-3,\frac{m}{2}-2,\frac{m}{2}-1\}\rangle_m\}$ is a factorisation of $\langle\{2,3,\ldots,\frac{m}{2}-1\}\rangle_m$ in which each 4-factor has a Hamilton cycle decomposition by Theorem 4, and the 6-factor has a Hamilton cycle decomposition by Theorem 5 (since $\gcd(\frac{m}{2}-2,m)=1$ when $m\equiv 2\,(\bmod\ 4)$).

For the case $m\equiv 0\,(\bmod\ 4)$, observe that $\{\langle\{4,5\}\rangle_m,\langle\{6,7\}\rangle_m,\ldots,\langle\{\frac{m}{2}-2,\frac{m}{2}-1\}\rangle_m\}$ is a 4-factorisation of $\langle\{4,5,\ldots,\frac{m}{2}-1\}\rangle_m$ in which each 4-factor has a Hamilton cycle decomposition by Theorem 4. Thus it is sufficient to show that $\langle\{2,3^{\rm o},\frac{m}{2}\}\rangle_m$ has a Hamilton cycle decomposition. But it is easy to see that $\langle\{2,3^{\rm o},\frac{m}{2}\}\rangle_m\cong \mathrm{Cay}(\mathbb{Z}_{\frac{m}{2}}\times\mathbb{Z}_2,\{(1,0),(\frac{m}{4},0),(0,1)\})$ and so the result follows by Theorem 4. □

3 Factorisations of the graph G_{2m}

For each even $m\geq 8$ we denote by G_{2m} the 7-regular graph obtained from $\langle\{1,3^{\rm e}\}\rangle_m^{(2)}$ by adding the edge $\{(x,0),(x,1)\}$ for each $x\in\mathbb{Z}_m$. Observe that if $F_1,F_2,\ldots,F_{\frac{m-4}{2}}$ are the Hamilton cycles in the factorisation of the complete graph with vertex set $\mathbb{Z}_m$ given by Lemma 7, then the 7-factor that remains when $F_1^{(2)},F_2^{(2)},\ldots,F_{\frac{m-4}{2}}^{(2)}$ are removed from the complete graph with vertex set $\mathbb{Z}_m\times\mathbb{Z}_2$ is isomorphic to G_{2m}. In this section we will construct for each even $m\geq 8$ and each bipartite 2-regular graph F of order $2m$, a factorisation of G_{2m} into three 2-factors each isomorphic to F and a 1-factor.

We now define a family of subgraphs of G_{2m} which are used extensively in the results that follow. For each even $r\geq 2$ we define the graph J_{2r} (see Figure 2) to be the graph with vertex set

$$V(J_{2r})=\{u_1,u_2,\ldots,u_{r+2}\}\cup\{v_1,v_2,\ldots,v_{r+2}\}$$

and edge set

$$\begin{aligned} E(J_{2r}) = \ & \{\{u_i, v_i\} : i = 3, 4, \ldots, r+2\} \cup \\ & \{\{u_i, u_{i+1}\}, \{v_i, v_{i+1}\}, \{u_i, v_{i+1}\}, \{v_i, u_{i+1}\} : i = 2, 3, \ldots, r+1\} \cup \\ & \{\{u_i, u_{i+3}\}, \{v_i, v_{i+3}\}\{u_i, v_{i+3}\}, \{v_i, u_{i+3}\} : i = 1, 3, \ldots, r-1\}. \end{aligned}$$

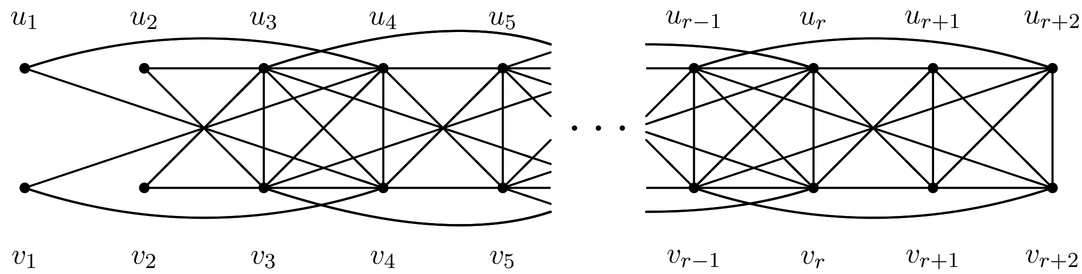

Figure 2: The graph J_{2r}

Notice that in the graph J_{2r} if we identify vertices u_1 with u_{r+1}, u_2 with u_{r+2}, v_1 with v_{r+1}, and v_2 with v_{r+2}, then the resulting graph is isomorphic to G_{2r}. For each integer $k \geq 0$ define the mapping ϕ_k on the vertices of J_{2r} by

$$\phi_k(u_i) = u_{i+k} \text{ and } \phi_k(v_i) = v_{i+k} \text{ for } i = 1, 2, \ldots, r+2,$$

and for any subgraph H of J_{2r} define $H^{(+k)}$ to be the graph with vertex set $\{\phi_k(x) : x \in V(H)\}$ and edge set $\{\{\phi_k(x), \phi_k(y)\} : \{x, y\} \in E(H)\}$. So in particular, $J_{2r} \cup J_{2s}^{(+r)} = J_{2(r+s)}$. For convenience we use the notation $J_{2r} \oplus J_{2s}$ to denote $J_{2r} \cup J_{2s}^{(+r)}$. More generally, if H is a subgraph of J_{2r} and H' is a subgraph of J_{2s} then we denote $H \cup H'^{(+r)}$ by $H \oplus_{(+r)} H'$, or just $H \oplus H'$ if the value of r is clear from the context.

A decomposition of a graph J is set of subgraphs whose edge sets partition the edge set of J. If $\mathcal{D} = \{H_i : i = 1, 2, \ldots, t\}$ is a decomposition of J_{2r} and $\mathcal{D}' = \{H_i' : i = 1, 2, \ldots, t\}$ is a decomposition of J_{2s}, then we define $\mathcal{D} \oplus \mathcal{D}'$ by $\mathcal{D} \oplus \mathcal{D}' = \{H_i \oplus H'_i : i = 1, 2, \ldots, t\}$. It is clear that $\mathcal{D} \oplus \mathcal{D}'$ is a decomposition of $J_{2(r+s)}$.

Let F be a 2-regular graph of order $2r$, and suppose there exists a decomposition $\{H_1, H_2, H_3, H_4\}$ of J_{2r} such that

(1) $V(H_1) = \{u_1, u_2, \ldots, u_r\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,

(2) $V(H_2) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_1, v_2, \ldots, v_r\}$,

(3) $V(H_3) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,

(4) $V(H_4) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,

(5) $H_1 \cong H_2 \cong H_3 \cong F$,

(6) H_4 is a 1-regular graph of order $2r$.

Then we shall write $J_{2r} \mapsto F$, or just $J \mapsto F$ if the value of r is clear from the context (usually from the order of F) and refer to such a decomposition as a decomposition $J \mapsto F$. The next Lemma gives a method for adjoining such decompositions in a natural way using the $\oplus$ operation.

Lemma 8 *If F and F' are 2-regular graphs such that $J \mapsto F$ and $J \mapsto F'$, then $J \mapsto F''$ where F'' is the union of vertex-disjoint copies of F and F'.*

Proof Let $\mathcal{D} = \{H_1, H_2, H_3, H_4\}$ be a decomposition $J \mapsto F$ and let $\mathcal{D}' = \{H_1', H_2', H_3', H_4'\}$ be a decomposition $J \mapsto F'$. Then $\mathcal{D} \oplus \mathcal{D}'$ is a decomposition $J \mapsto F''$. Properties (1)-(4) (in the definition of $J \mapsto F$) ensure that H_i and H_i' are vertex disjoint for $i \in \{1, 2, 3, 4\}$, and that the corresponding properties hold for $V(H_i \oplus H_i')$. □

Lemma 9 *Let $m \geq 8$ and let F be a 2-regular graph of order $2m$. If $J \mapsto F$, then there exists a factorisation of G_{2m} into three 2-factors each isomorphic to F and a 1-factor.*

Proof In J_{2m}, identify vertices u_1 with u_{m+1}, u_2 with u_{m+2}, v_1 with v_{m+1}, and v_2 with v_{m+2}. Clearly, the resulting graph is isomorphic to G_{2m}. If $\{H_1, H_2, H_3, H_4\}$ is the decomposition $J \mapsto F$, then H_1, H_2 and H_3 become the required 2-factors and H_4 becomes the required 1-factor. □

Lemma 10 *For each graph F in the following list we have $J \mapsto F$.*

- $[m]$ *for each* $m \in \{8, 12, 16, \ldots\}$
- $[4, m]$ *for each* $m \in \{4, 8, 12, \ldots\}$
- $[m, m']$ *for each* $m \in \{6, 10, 14, \ldots\}$ *and each* $m' \in \{6, 10, 14, \ldots\}$
- $[4, m, m']$ *for each* $m \in \{6, 10, 14, \ldots\}$ *and each* $m' \in \{6, 10, 14, \ldots\}$
- $[4, 4, 4]$

Proof The proof of this lemma is given at the end of Section 4. □

Lemma 11 *If F is a bipartite 2-regular graph of order $2m$ where $m \geq 8$ is even, then there is a factorisation of G_{2m} into three 2-factors each isomorphic to F, and a 1-factor.*

Proof Let F be a bipartite 2-regular graph of order $2m$ where $m \geq 8$ is even. We only need to show that there is a decomposition of F into 2-regular subgraphs $F_1, F_2, \ldots, F_t$ such that Lemma 10 covers $J \mapsto F_i$ for $i = 1, 2, \ldots, t$. The result then follows as we can use Lemma 8 to obtain $J \mapsto F$, and then use Lemma 9 to obtain the required factorisation of G_{2m}.

Observe that since m is even, the number of cycles of length congruent to $2 \pmod 4$ in F is even. If $F \cong [4, 4, \ldots, 4]$ then F can be decomposed into copies of $[4, 4]$ and $[4, 4, 4]$ ($m \geq 8$ implies $F \ncong [4]$). Hence we can assume that F has a subgraph which is isomorphic to either $[m]$ where $m \geq 8$ and $m \equiv 0 \pmod 4$, or to $[m, m']$ where $m \equiv m' \equiv 2 \pmod 4$.

If the number of 4-cycles in F is even, then clearly there is a decomposition of F in which each subgraph is either $[4, 4]$, $[m]$ where $m \geq 8$ and $m \equiv 0 \pmod 4$, or $[m, m']$ where $m \equiv m' \equiv 2 \pmod 4$. On the other hand, if the number of 4-cycles in F is odd, then there is a decomposition of F in which one subgraph is either $[4, m]$ where $m \geq 8$ and $m \equiv 0 \pmod 4$, or $[4, m, m']$ where $m \equiv m' \equiv 2 \pmod 4$, and each other subgraph is either $[4,4]$, $[m]$ where $m \geq 8$ and $m \equiv 0 \pmod 4$, or $[m, m']$ where $m \equiv m' \equiv 2 \pmod 4$. All of these are covered by Lemma 10. □

We can now prove our main Theorem, which we restate for convenience.

Theorem 1 *If $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order n and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are non-negative integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{n-2}{2}$, $\alpha_1 \geq 3$ is odd, and α_i is even for $i = 2, 3, \ldots, t$, then there exists a 2-factorisation of $K_n - I$ in which there are exactly α_i 2-factors isomorphic to F_i for $i = 1, 2, \ldots, t$.*

Proof The conditions guarantee that $n \equiv 0 \pmod 4$ and the result is known to hold for $n \in \{4, 8, 12\}$, see [25]. For $n \equiv 0 \pmod 4$ and $n \geq 16$, let $\mathcal{F}$ be the factorisation of $K_{\frac{n}{2}}$ given by Lemma 7 and let I be the 1-regular graph with $V(I) = V(K_{\frac{n}{2}}) \times \mathbb{Z}_2$ and $E(I) = \{\{(v,0),(v,1)\} : v \in V(K_{\frac{n}{2}})\}$. Then $\mathcal{F}^{(2)} \cup \{I\}$ is a factorisation of K_n into $\frac{n-8}{4}$ 4-factors, each isomorphic to $C_{\frac{n}{2}}^{(2)}$, a 6-factor isomorphic to $\langle\{1, 3^{\mathrm{e}}\}\rangle_{\frac{n}{2}}^{(2)}$, and a 1-factor. Moreover, the union of the 6-factor and the 1-factor is a 7-factor isomorphic to G_n. The result now follows by appropriate applications of Lemma 6 to each 4-factor, and Lemma 11 to the 7-factor. □

4 Decompositions of J_{2r}

The sole purpose of this section is to prove Lemma 10. In what follows, the cycle with vertices $x_1, x_2, \ldots, x_t$ and edges $x_1x_2, x_2x_3, \ldots, x_{t-1}x_t$ and x_tx_1 is denoted by $(x_1, x_2, \ldots, x_t)$, the path with vertices $x_1, x_2, \ldots, x_t$ and edges $x_1x_2, x_2x_3, \ldots, x_{t-1}x_t$ is denoted by $\langle x_1, x_2, \ldots, x_t\rangle$, and the 1-regular graph whose vertices are $x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t$ and edges are $x_1y_1, x_2y_2, \ldots, x_ty_t$ is denoted by $\{x_1y_1, x_2y_2, \ldots, x_ty_t\}$.

We obtain the necessary decompositions $J \mapsto F$ required for Lemma 10 by using the operation $\oplus$ to combine various ingredient decompositions, but in a slightly more general way than we did in Lemma 8. Now, our ingredient decompositions will be into subgraphs which are vertex-disjoint unions of paths and cycles, and each copy of F in our decomposition $J \mapsto F$ will be a union of some of these subgraphs. For example, a copy of $F = [4, 14, 18]$ in our decomposition $J \mapsto F$ might be $H_1 \oplus H_1' \oplus H_1'' \oplus H_1'''$ where H_1, H_1', H_1'' and H_1''' are as shown in Figure 3.

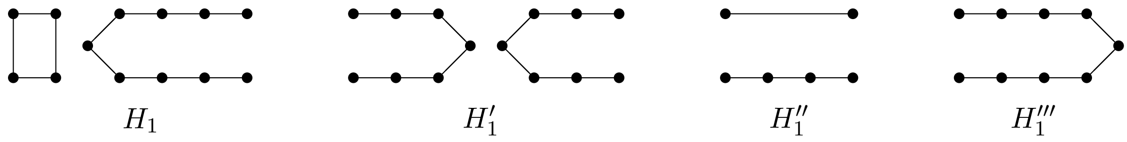

Figure 3: The vertex sets of the graphs H_1, H_1', H_1'' and H_1''' will be such that their union is isomorphic to the graph $[4, 14, 18]$.

The intuition behind the notation we will soon define is that in this example

- H_1 would be represented by "$[4, 8$",
- H_1' would be represented by "$6, 6$",
- H_1'' would be represented by "4" and
- H_1''' would be represented by "8]"

so that $H_1 \oplus H_1' \oplus H_1'' \oplus H_1'''$, being isomorphic to the graph $[4, 14, 18]$, is represented as $[4, 8 \ \oplus \ 6, 6 \ \oplus \ 4 \ \oplus \ 8]$.

We need to ensure that the vertex sets of the paths and cycles, and in particular the end-vertices of the paths, in our ingredient decompositions are such that the desired 2-regular graphs are indeed obtained when the ingredient decompositions are combined using the operation $\oplus$. It is routine, though somewhat tedious, to check that the following definitions ensure that this is the case.

We begin with decompositions into graphs which will act like H_1 in Figure 3. A decomposition $\{H_1, H_2, H_3, H_4\}$ of J_{2r} is called a *left-end decomposition* if each of H_1, H_2 and H_3 is the vertex-disjoint union of a number (possibly zero) of cycles and a path, H_4 is a 1-regular graph with vertex set $V(J_{2r}) \setminus \{u_1, u_2, v_1, v_2\}$, and

(1) $V(H_1) = V(J_{2r}) \setminus \{v_1, v_2, u_{r+2}\}$ and the end-vertices of the path in H_1 are v_{r+1} and v_{r+2}.

(2) $V(H_2) = V(J_{2r}) \setminus \{u_1, u_2\}$ and the end-vertices of the path in H_2 are u_{r+1} and v_{r+2}.

(3) $V(H_3) = V(J_{2r}) \setminus \{u_1, u_2, v_1, v_2\}$ and the end-vertices of the path in H_3 are u_{r+1} and v_{r+1}.

If $\mathcal{D}$ is a left-end decomposition $\{H_1, H_2, H_3, H_4\}$ of J_{2r} in which H_1 consists of cycles of lengths $a_1, a_2, \ldots, a_t$ and a path of length a, H_2 consists of cycles of lengths $b_1, b_2, \ldots, b_t$ and a path of length b, and H_3 consists of cycles of lengths $c_1, c_2, \ldots, c_t$ and a path of length c, then we will say that $\mathcal{D}$ is a *decomposition of type*

$$J \mapsto [a_1, a_2, \ldots, a_t, a + [b_1, b_2, \ldots, b_t, b + [c_1, c_2, \ldots, c_t, c.$$

Lemma 12 *There exist decompositions of the following types*

$J \mapsto [8 + [9 + [7$ *which we denote by* $\mathcal{L}_1$*.*

$J \mapsto [4, 8 + [4, 9 + [4, 7$ *which we denote by* $\mathcal{L}_2$*.*

$J \mapsto [6, 6 + [6, 7 + [6, 5$ *which we denote by* $\mathcal{L}_3$*.*

$J \mapsto [10, 6 + [10, 7 + [10, 5$ *which we denote by* $\mathcal{L}_4$*.*

$J \mapsto [4, 6, 6 + [4, 6, 7 + [4, 6, 5$ *which we denote by* $\mathcal{L}_5$*.*

$J \mapsto [4, 10, 6 + [4, 10, 7 + [4, 10, 5$ *which we denote by* $\mathcal{L}_6$*.*

Proof The decomposition $\mathcal{L}_1 = \{H_1, H_2, H_3, H_4\}$ is given by

$$H_1 = \langle v_5, u_5, u_4, u_1, v_4, u_3, u_2, v_3, v_6 \rangle \quad H_2 = \langle u_5, u_6, u_3, v_2, v_3, u_4, v_1, v_4, v_5, v_6 \rangle$$
$$H_3 = \langle u_5, v_6, u_3, u_4, v_4, v_3, u_6, v_5 \rangle \quad H_4 = \{u_3v_3, u_4v_5, v_4u_5, u_6v_6\}.$$

The decomposition $\mathcal{L}_2 = \{H_1, H_2, H_3, H_4\}$ is given by

$$H_1 = (u_1, u_4, u_5, v_4) \cup \langle v_7, u_7, v_6, v_3, u_2, u_3, u_6, v_5, v_8\rangle$$
$$H_2 = (v_1, u_4, v_5, v_4) \cup \langle u_7, u_8, u_5, v_6, u_3, v_2, v_3, u_6, v_7, v_8\rangle$$
$$H_3 = (u_3, u_4, v_3, v_4) \cup \langle u_7, v_8, u_5, u_6, v_6, v_5, u_8, v_7\rangle$$
$$H_4 = \{u_3v_3, u_4v_4, u_5v_5, u_6u_7, v_6v_7, u_8v_8\}.$$

The decomposition $\mathcal{L}_3 = \{H_1, H_2, H_3, H_4\}$ is given by

$$H_1 = (u_1, u_4, u_3, u_2, v_3, v_4) \cup \langle v_7, u_7, v_6, u_6, u_5, v_5, v_8\rangle$$
$$H_2 = (v_1, u_4, v_3, v_2, u_3, v_4) \cup \langle u_7, u_8, u_5, v_6, v_5, u_6, v_7, v_8\rangle$$
$$H_3 = (u_4, u_5, v_8, u_8, v_5, v_4)) \cup \langle u_7, u_6, v_3, u_3, v_6, v_7\rangle$$
$$H_4 = \{u_3u_6, v_3v_6, u_4v_5, v_4u_5, u_7v_8, v_7u_8\}.$$

The decomposition $\mathcal{L}_4 = \{H_1, H_2, H_3, H_4\}$ is given by

$$H_1 = (u_1, u_4, u_5, v_5, u_6, v_6, v_3, u_2, u_3, v_4) \cup \langle v_9, u_9, v_8, v_7, u_8, u_7, v_{10}\rangle$$
$$H_2 = (v_1, u_4, v_3, v_2, u_3, u_6, u_7, v_8, v_5, v_4) \cup \langle u_9, u_{10}, v_7, v_6, u_5, u_8, v_9, v_{10}\rangle$$
$$H_3 = (u_3, u_4, v_4, v_3, u_6, u_5, v_8, u_8, v_5, v_6) \cup \langle u_9, v_{10}, v_7, u_7, u_{10}, v_9\rangle$$
$$H_4 = \{u_3v_3, u_4v_5, v_4u_5, u_6v_7, v_6u_7, u_8u_9, v_8v_9, u_{10}v_{10}\}.$$

The decomposition $\mathcal{L}_5 = \{H_1, H_2, H_3, H_4\}$ is given by

$$H_1 = (u_1, u_4, u_5, v_4) \cup (u_2, u_3, u_6, v_5, v_6, v_3) \cup \langle v_9, u_9, v_8, v_7, u_8, u_7, v_{10}\rangle$$
$$H_2 = (v_1, u_4, v_5, v_4) \cup (v_2, u_3, v_6, u_5, u_6, v_3) \cup \langle u_9, u_{10}, v_7, u_7, v_8, u_8, v_9, v_{10}\rangle$$
$$H_3 = (u_3, u_4, v_3, v_4) \cup (u_6, u_7, u_{10}, v_{10}, v_7, v_6) \cup \langle u_9, u_8, v_5, u_5, v_8, v_9\rangle$$
$$H_4 = \{u_3v_3, u_4v_4, u_5u_8, v_5v_8, u_6v_7, v_6u_7, u_9v_{10}, v_9u_{10}\}.$$

The decomposition $\mathcal{L}_6 = \{H_1, H_2, H_3, H_4\}$ is given by

$$\begin{aligned}
H_1 = \quad & (u_1, u_4, u_5, v_4) \cup (u_2, u_3, v_6, v_5, v_8, u_8, v_7, u_7, u_6, v_3) \cup \\
& \langle v_{11}, u_{11}, v_{10}, v_9, u_{10}, u_9, v_{12}\rangle \\
H_2 = \quad & (v_1, u_4, v_5, v_4) \cup (v_2, u_3, u_6, u_5, u_8, u_9, v_{10}, v_7, v_6, v_3) \cup \\
& \langle u_{11}, u_{12}, v_9, v_8, u_7, u_{10}, v_{11}, v_{12}\rangle \\
H_3 = \quad & (u_3, u_4, v_3, v_4) \cup (u_5, v_6, u_6, v_5, u_8, u_7, v_{10}, u_{10}, v_7, v_8) \cup \\
& \langle u_{11}, v_{12}, v_9, u_9, u_{12}, v_{11}\rangle \\
H_4 = \quad & \{u_3v_3, u_4v_4, u_5v_5, u_6v_7, v_6u_7, u_8v_9, v_8u_9, u_{10}u_{11}, v_{10}v_{11}, u_{12}v_{12}\}.
\end{aligned}$$

□

Given the above usage of the term "left-end decomposition", the natural name for the decomposition given in the following lemma is a *right-end decomposition.* The subgraphs in the decomposition (other than the one that is 1-regular) are like H_1''' in Figure 3.

Lemma 13 *There exists a decomposition* $\{H_1, H_2, H_3, H_4\}$ *of* J_8 *where*

(1) H_1 *is a path of length* 8 *from* v_1 *to* v_2 *with vertex set* $V(J_8) \setminus \{u_1, u_5, u_6\}$

(2) H_2 *is a path of length* 7 *from* u_1 *to* v_2 *with vertex set* $V(J_8) \setminus \{u_2, v_1, v_5, v_6\}$

(3) H_3 *is a path of length* 9 *from* u_1 *to* v_1 *with vertex set* $V(J_8) \setminus \{u_2, v_2\}$

(4) H_4 *is a* 1*-regular graph with vertex set* $V(J_8) \setminus \{u_1, u_2, v_1, v_2\}$.

We denote the decomposition given in Lemma 13 by $\mathcal{R}$ and we say that $\mathcal{R}$ is a decomposition of type

$$J \mapsto 8] + 7] + 9].$$

Proof The decomposition $\mathcal{R} = \{H_1, H_2, H_3, H_4\}$ is given by

$$H_1 = \langle v_1, v_4, u_4, v_5, v_6, v_3, u_2, u_3, v_2 \rangle \qquad H_2 = \langle u_1, u_4, u_3, u_6, u_5, v_4, v_3, v_2 \rangle$$
$$H_3 = \langle u_1, v_4, u_3, v_6, u_5, v_5, u_6, v_3, u_4, v_1 \rangle \quad H_4 = \{u_3v_3, u_4u_5, v_4v_5, u_6v_6\}.$$

□

The following lemma gives a decomposition in which the subgraphs (other than the one that is 1-regular) are like H_1'' in Figure 3.

Lemma 14 *There exists a decomposition* $\{H_1, H_2, H_3, H_4\}$ *of* J_4 *where*

(1) H_1 *is a pair of vertex-disjoint paths from* v_1 *and* v_2 *to* v_3 *and* v_4 *such that* $V(H_1) = V(J_4) \setminus \{u_1, u_4\}$.

(2) H_2 *is a pair of vertex-disjoint paths from* u_1 *and* v_2 *to* u_3 *and* v_4 *such that* $V(H_2) = V(J_4) \setminus \{u_2, v_1\}$.

(3) H_3 *is a pair of vertex-disjoint paths from* u_1 *and* v_1 *to* u_3 *and* v_3 *such that* $V(H_3) = V(J_4) \setminus \{u_2, v_2\}$.

(4) H_4 *is a* 1*-regular graph with vertex set* $V(J_4) \setminus \{u_1, u_2, v_1, v_2\}$.

We denote the decomposition given in Lemma 14 by $\mathcal{C}$ and we say that $\mathcal{C}$ is a decomposition of type

$$J \mapsto 4 + 4 + 4.$$

Proof The decomposition $\mathcal{C} = \{H_1, H_2, H_3, H_4\}$ is given by

$$H_1 = \langle v_1, v_4 \rangle \cup \langle v_2, u_3, u_2, v_3 \rangle \quad H_2 = \langle u_1, u_4, u_3 \rangle \cup \langle v_2, v_3, v_4 \rangle$$
$$H_3 = \langle u_1, v_4, u_3 \rangle \cup \langle v_1, u_4, v_3 \rangle \quad H_4 = \{u_3v_3, u_4v_4\}.$$

□

The following lemma gives a decomposition in which the subgraphs (other than the one that is 1-regular) are like H_1' in Figure 3.

Lemma 15 *There exists a decomposition* $\{H_1, H_2, H_3, H_4\}$ *of* J_{12} *where*

(1) H_1 *is a pair of vertex-disjoint paths, one of length* 6 *from* v_1 *to* v_2*, and one of length* 6 *from* v_7 *to* v_8 *such that* $V(H_1) = V(J_{12}) \setminus \{u_1, u_8\}$.

(2) H_2 *is a pair of vertex-disjoint paths, one of length* 5 *from* u_1 *to* v_2*, and one of length* 7 *from* u_7 *to* v_8 *such that* $V(H_2) = V(J_{12}) \setminus \{u_2, v_1\}$.

(3) H_3 *is a pair of vertex-disjoint paths, one of length* 7 *from* u_1 *to* v_1*, and one of length* 5 *from* u_7 *to* v_7 *such that* $V(H_3) = V(J_{12}) \setminus \{u_2, v_2\}$.

(4) H_4 *is a* 1*-regular graph with vertex set* $V(J_{12}) \setminus \{u_1, u_2, v_1, v_2\}$.

We denote the decomposition given in Lemma 15 by $\mathcal{M}$ and we say that $\mathcal{M}$ is a decomposition of type

$$J \mapsto 6, 6 + 5, 7 + 7, 5.$$

Proof The decomposition $\mathcal{M} = \{H_1, H_2, H_3, H_4\}$ is given by

$$\begin{aligned} H_1 &= \langle v_1, v_4, u_4, v_3, u_2, u_3, v_2 \rangle \cup \langle v_7, u_7, v_6, v_5, u_6, u_5, v_8 \rangle \\ H_2 &= \langle u_1, u_4, u_5, v_6, v_3, v_2 \rangle \cup \langle u_7, u_8, v_5, v_4, u_3, u_6, v_7, v_8 \rangle \\ H_3 &= \langle u_1, v_4, v_3, u_6, v_6, u_3, u_4, v_1 \rangle \cup \langle u_7, v_8, v_5, u_5, u_8, v_7 \rangle \\ H_4 &= \{u_3v_3, u_4v_5, v_4u_5, u_6u_7, v_6v_7, u_8v_8\}. \end{aligned}$$

□

For convenience, we summarise the results of Lemmas 12, 13, 14 and 15 in the following lemma.

Lemma 16 *The following decompositions exist and are of the type indicated.*

$\mathcal{L}_1 : J \mapsto [8 + [9 + [7$	$\mathcal{L}_2 : J \mapsto [4, 8 + [4, 9 + [4, 7$
$\mathcal{L}_3 : J \mapsto [6, 6 + [6, 7 + [6, 5$	$\mathcal{L}_4 : J \mapsto [10, 6 + [10, 7 + [10, 5$
$\mathcal{L}_5 : J \mapsto [4, 6, 6 + [4, 6, 7 + [4, 6, 5$	$\mathcal{L}_6 : J \mapsto [4, 10, 6 + [4, 10, 7 + [4, 10, 5$
$\mathcal{C} : J \mapsto 4 + 4 + 4$	$\mathcal{M} : J \mapsto 6, 6 + 5, 7 + 7, 5$
$\mathcal{R} : J \mapsto 8] + 7] + 9]$	

Most of the decompositions $J \mapsto F$ that we need can be obtained from the decompositions given in Lemma 16 by using the $\oplus$ operation, but we also need a few other small decompositions and these are given in the following lemma.

Lemma 17 *The following decompositions exist.*

$$J \mapsto [8] \quad J \mapsto [12] \quad J \mapsto [4, 4] \quad J \mapsto [4, 8] \quad J \mapsto [4, 12] \quad J \mapsto [6, 6]$$
$$J \mapsto [6, 10] \quad J \mapsto [10, 10] \quad J \mapsto [4, 4, 4] \quad J \mapsto [4, 6, 6] \quad J \mapsto [4, 6, 10] \quad J \mapsto [4, 10, 10]$$

Proof The decomposition $J \mapsto [8]$ is given by

$$H_1 = (u_1, u_4, v_3, u_2, u_3, v_6, v_5, v_4) \quad H_2 = (v_1, u_4, u_3, v_2, v_3, u_6, u_5, v_4)$$
$$H_3 = (u_3, v_4, v_3, v_6, u_5, u_4, v_5, u_6) \quad H_4 = \{u_3v_3, u_4v_4, u_5v_5, u_6v_6\}.$$

The decomposition $J \mapsto [12]$ is given by

$$H_1 = (u_1, u_4, u_5, v_5, v_8, v_7, u_6, v_6, v_3, u_2, u_3, v_4)$$
$$H_2 = (v_1, u_4, v_3, v_2, u_3, u_6, u_5, u_8, u_7, v_6, v_5, v_4)$$
$$H_3 = (u_3, u_4, v_4, v_3, u_6, v_5, u_8, v_7, u_7, v_8, u_5, v_6)$$
$$H_4 = \{u_3v_3, u_4v_5, v_4u_5, u_6u_7, v_6v_7, u_8v_8\}.$$

The decomposition $J \mapsto [4, 4]$ is given by

$$H_1 = (u_1, u_4, v_5, v_4) \cup (u_2, u_3, v_6, v_3) \quad H_2 = (v_1, u_4, u_5, v_4) \cup (v_2, u_3, u_6, v_3)$$
$$H_3 = (u_3, u_4, v_3, v_4) \cup (u_5, u_6, v_5, v_6) \quad H_4 = \{u_3v_3, u_4v_4, u_5v_5, u_6v_6\}.$$

The decomposition $J \mapsto [4, 8]$ is given by

$$H_1 = (u_1, u_4, v_5, v_4) \cup (u_2, u_3, u_6, u_5, v_8, v_7, v_6, v_3)$$
$$H_2 = (v_1, u_4, u_5, v_4) \cup (v_2, u_3, v_6, u_7, u_8, v_5, u_6, v_3)$$
$$H_3 = (u_3, u_4, v_3, v_4) \cup (u_5, v_6, v_5, v_8, u_7, u_6, v_7, u_8)$$
$$H_4 = \{u_3v_3, u_4v_4, u_5v_5, u_6v_6, u_7v_7, u_8v_8\}.$$

The decomposition $J \mapsto [4, 12]$ is given by

$$H_1 = (u_1, u_4, u_5, v_4) \cup (u_2, u_3, u_6, v_5, v_8, u_8, v_9, v_{10}, v_7, u_7, v_6, v_3)$$
$$H_2 = (v_1, u_4, v_5, v_4) \cup (v_2, u_3, v_6, u_5, u_8, u_7, u_{10}, u_9, v_8, v_7, u_6, v_3)$$
$$H_3 = (u_3, u_4, v_3, v_4) \cup (u_5, u_6, v_6, v_5, u_8, v_7, u_{10}, v_9, u_9, v_{10}, u_7, v_8)$$
$$H_4 = \{u_3v_3, u_4v_4, u_5v_5, u_6u_7, v_6v_7, u_8u_9, v_8v_9, u_{10}v_{10}\}.$$

The decomposition $J \mapsto [6, 6]$ is given by

$$H_1 = (u_1, u_4, v_3, u_2, u_3, v_4) \cup (u_5, v_5, v_8, v_7, u_6, v_6)$$
$$H_2 = (v_1, u_4, u_3, v_2, v_3, v_4) \cup (u_5, u_6, v_5, v_6, u_7, u_8)$$
$$H_3 = (u_3, v_3, u_6, u_7, v_7, v_6) \cup (u_4, v_4, u_5, v_8, u_8, v_5)$$
$$H_4 = \{u_3u_6, v_3v_6, u_4u_5, v_4v_5, u_7v_8, v_7u_8\}.$$

The decomposition $J \mapsto [6, 10]$ is given by

$$H_1 = (u_7, u_8, v_7, v_8, v_9, v_{10}) \cup (u_1, u_4, u_5, v_5, u_6, v_6, v_3, u_2, u_3, v_4)$$
$$H_2 = (u_5, u_8, u_9, u_{10}, v_7, v_6) \cup (v_1, u_4, v_3, v_2, u_3, u_6, u_7, v_8, v_5, v_4)$$
$$H_3 = (u_7, u_{10}, v_9, u_9, v_{10}, v_7) \cup (u_3, u_4, v_4, v_3, u_6, u_5, v_8, u_8, v_5, v_6)$$
$$H_4 = \{u_3u_4, u_4v_5, v_4u_5, u_6v_7, v_6u_7, u_8v_9, v_8u_9, u_{10}v_{10}\}.$$

The decomposition $J \mapsto [10, 10]$ is given by

$$H_1 = (u_1, u_4, u_5, v_5, u_6, v_6, v_3, u_2, u_3, v_4) \cup (u_7, u_8, v_7, v_8, u_9, v_9, v_{12}, v_{11}, u_{10}, v_{10})$$
$$H_2 = (v_1, u_4, v_3, v_2, u_3, u_6, u_7, v_8, v_5, v_4) \cup (u_5, v_6, v_7, u_{10}, u_9, u_{12}, u_{11}, v_{10}, v_9, u_8)$$
$$H_3 = (u_3, u_4, v_4, v_3, u_6, u_5, v_8, u_8, v_5, v_6) \cup (u_7, v_7, v_{10}, u_9, v_{12}, u_{11}, v_{11}, u_{12}, v_9, u_{10})$$
$$H_4 = \{u_3u_4, u_4v_5, v_4u_5, u_6v_7, v_6u_7, u_8u_9, v_8v_9, u_{10}u_{11}, v_{10}v_{11}, u_{12}v_{12}\}.$$

The decomposition $J \mapsto [4, 4, 4]$ is given by

$$H_1 = (u_1, u_4, u_5, v_4) \cup (u_2, u_3, u_6, v_3) \cup (v_5, v_6, v_7, v_8)$$
$$H_2 = (v_1, u_4, v_5, v_4) \cup (v_2, u_3, v_6, v_3) \cup (u_5, u_6, u_7, u_8)$$
$$H_3 = (u_3, u_4, v_3, v_4) \cup (u_5, v_6, u_7, v_8) \cup (v_5, u_6, v_7, u_8)$$
$$H_4 = \{u_3v_3, u_4v_4, u_5v_5, u_6v_6, u_7v_7, u_8v_8\}.$$

The decomposition $J \mapsto [4, 6, 6]$ is given by

$$H_1 = (u_1, u_4, u_5, v_4) \cup (u_2, u_3, u_6, v_5, v_6, v_3) \cup (u_7, v_7, v_{10}, v_9, u_8, v_8)$$
$$H_2 = (v_1, u_4, v_5, v_4) \cup (v_2, u_3, v_6, u_5, u_6, v_3) \cup (u_7, u_8, v_7, v_8, u_9, u_{10})$$
$$H_3 = (u_3, u_4, v_3, v_4) \cup (u_5, v_5, u_8, u_9, v_9, v_8) \cup (u_6, v_6, u_7, v_{10}, u_{10}, v_7)$$
$$H_4 = \{u_3v_3, u_4v_4, u_5u_8, v_5v_8, u_6u_7, v_6v_7, u_9v_{10}, v_9u_{10}\}.$$

The decomposition $J \mapsto [4, 6, 10]$ is given by

$$H_1 = (u_1, u_4, u_5, v_4) \cup (u_9, u_{10}, v_9, v_{10}, v_{11}, v_{12}) \cup (u_2, u_3, v_6, v_5, v_8, u_8, v_7, u_7, u_6, v_3)$$
$$H_2 = (v_1, u_4, v_5, v_4) \cup (u_7, v_8, v_9, u_{12}, u_{11}, u_{10}) \cup (v_2, u_3, u_6, u_5, u_8, u_9, v_{10}, v_7, v_6, v_3)$$
$$H_3 = (u_3, u_4, v_3, v_4) \cup (u_9, v_9, v_{12}, u_{11}, v_{11}, u_{12}) \cup (u_5, v_6, u_6, v_5, u_8, u_7, v_{10}, u_{10}, v_7, v_8)$$
$$H_4 = \{u_3v_3, u_4v_4, u_5v_5, u_6v_7, v_6u_7, u_8v_9, v_8u_9, u_{10}v_{11}, v_{10}u_{11}, u_{12}v_{12}\}.$$

The decomposition $J \mapsto [4, 10, 10]$ is given by

$$\begin{aligned}
H_1 = \;& (u_1, u_4, u_5, v_4) \cup (u_2, u_3, v_6, v_5, v_8, u_8, v_7, u_7, u_6, v_3) \cup \\
& (u_9, u_{10}, v_9, v_{10}, u_{11}, v_{11}, v_{14}, v_{13}, u_{12}, v_{12}) \\
H_2 = \;& (v_1, u_4, v_5, v_4) \cup (v_2, u_3, u_6, u_5, u_8, u_9, v_{10}, v_7, v_6, v_3) \cup \\
& (u_7, v_8, v_9, u_{12}, u_{11}, u_{14}, u_{13}, v_{12}, v_{11}, u_{10}) \\
H_3 = \;& (u_3, u_4, v_3, v_4) \cup (u_5, v_6, u_6, v_5, u_8, u_7, v_{10}, u_{10}, v_7, v_8) \cup \\
& (u_9, v_9, v_{12}, u_{11}, v_{14}, u_{13}, v_{13}, u_{14}, v_{11}, u_{12}) \\
H_4 = \;& \{u_3v_3, u_4v_4, u_5v_5, u_6v_7, v_6u_7, u_8v_9, v_8u_9, u_{10}u_{11}, v_{10}v_{11}, u_{12}u_{13}, v_{12}v_{13}, u_{14}v_{14}\}.
\end{aligned}$$

□

We are now able to prove Lemma 10, which we restate for convenience.

Lemma 10 *For each graph F in the following list we have $J \mapsto F$.*

- $[m]$ *for each* $m \in \{8, 12, 16, \ldots\}$
- $[4, m]$ *for each* $m \in \{4, 8, 12, \ldots\}$
- $[m, m']$ *for each* $m \in \{6, 10, 14, \ldots\}$ *and each* $m' \in \{6, 10, 14, \ldots\}$
- $[4, m, m']$ *for each* $m \in \{6, 10, 14, \ldots\}$ *and each* $m' \in \{6, 10, 14, \ldots\}$
- $[4, 4, 4]$

Proof First we deal with the case $J \mapsto [m]$ for all $m \geq 8$ with $m \equiv 0 \pmod 4$ and the case $J \mapsto [4, m]$ for all $m \geq 4$ with $m \equiv 0 \pmod 4$. The decompositions $J \mapsto [8]$, $J \mapsto [12]$, $J \mapsto [4, 4]$, $J \mapsto [4, 8]$ and $J \mapsto [4, 12]$ are given in Lemma 17, and for each $m \geq 16$ with $m \equiv 0 \pmod 4$, the decompositions $\mathcal{L}_1 \oplus \mathcal{C} \oplus \mathcal{C} \oplus \cdots \oplus \mathcal{C} \oplus \mathcal{R}$ and $\mathcal{L}_2 \oplus \mathcal{C} \oplus \mathcal{C} \oplus \cdots \oplus \mathcal{C} \oplus \mathcal{R}$ with $\frac{m-16}{4}$ occurrences of $\mathcal{C}$ gives us $J \mapsto [m]$ and $J \mapsto [4, m]$ respectively.

We now deal with the cases $J \mapsto [m, m']$ and $J \mapsto [4, m, m']$ for all $m, m' \geq 6$ with $m \equiv m' \equiv 2 \pmod 4$. We can assume without loss of generality that $m \leq m'$. The decompositions $J \mapsto [6, 6]$, $J \mapsto [6, 10]$, $J \mapsto [10, 10]$, $J \mapsto [4, 6, 6]$, $J \mapsto [4, 6, 10]$ and

$J \mapsto [4, 10, 10]$ are given in Lemma 17. For each $m' \geq 14$ with $m' \equiv 2 \pmod 4$, the decompositions

- $\mathcal{L}_3 \oplus \mathcal{C} \oplus \mathcal{C} \oplus \cdots \oplus \mathcal{C} \oplus \mathcal{R}$,
- $\mathcal{L}_4 \oplus \mathcal{C} \oplus \mathcal{C} \oplus \cdots \oplus \mathcal{C} \oplus \mathcal{R}$,
- $\mathcal{L}_5 \oplus \mathcal{C} \oplus \mathcal{C} \oplus \cdots \oplus \mathcal{C} \oplus \mathcal{R}$ and
- $\mathcal{L}_6 \oplus \mathcal{C} \oplus \mathcal{C} \oplus \cdots \oplus \mathcal{C} \oplus \mathcal{R}$

with $\frac{m-14}{4}$ occurrences of $\mathcal{C}$ give us $J \mapsto [6, m']$, $J \mapsto [10, m]$, $J \mapsto [4, 6, m]$ and $J \mapsto [4, 10, m']$ respectively. The decompositions

$$\mathcal{L}_1 \oplus \mathcal{C} \oplus \mathcal{C} \oplus \cdots \oplus \mathcal{C} \oplus \mathcal{M} \oplus \mathcal{C} \oplus \mathcal{C} \oplus \cdots \oplus \mathcal{C} \oplus \mathcal{R}$$

and

$$\mathcal{L}_2 \oplus \mathcal{C} \oplus \mathcal{C} \oplus \cdots \oplus \mathcal{C} \oplus \mathcal{M} \oplus \mathcal{C} \oplus \mathcal{C} \oplus \cdots \oplus \mathcal{C} \oplus \mathcal{R},$$

each with $\frac{m-14}{4}$ occurrences of $\mathcal{C}$ followed by $\mathcal{M}$ and then a further $\frac{m'-14}{4}$ occurrences of $\mathcal{C}$, give us $J \mapsto [m, m']$ and $J \mapsto [4, m, m']$ respectively for all $m, m' \geq 14$ with $m \equiv m' \equiv 2 \pmod 4$.

Finally the decomposition $J \mapsto [4, 4, 4]$ is given in Lemma 17. □

References

[1] P. Adams, E. J. Billington, D. Bryant and S. I. El-Zanati, On the Hamilton-Waterloo problem, *Graphs Combin.*, **18** (2002), 31–51.

[2] P. Adams and D. Bryant, Two-factorisations of complete graphs of orders fifteen and seventeen, *Australas. J. Combin.*, **35** (2006), 113–118.

[3] B. Alspach, Research problems, Problem 59, *Discrete Math.*, **50** (1984), 115.

[4] B. Alspach and R. Häggkvist, Some observations on the Oberwolfach problem, *J. Graph Theory*, **9** (1985), 177–187.

[5] B. Alspach, P. J. Schellenberg, D. R. Stinson and D. Wagner, The Oberwolfach problem and factors of uniform odd length cycles, *J. Combin. Theory Ser. A*, **52** (1989), 20–43.

[6] J-C. Bermond, O. Favaron, and M. Mahéo, Hamiltonian decomposition of Cayley graphs of degree 4, *J. Combin. Theory Ser. B*, **46** (1989), 142–153.

[7] P. A. Bolstad, The Oberwolfach problem: a history and some new results,, *M.Sc. Thesis*, Simon Fraser University, (1990).

[8] D. Bryant, On the Oberwolfach problem with two similar length cycles, *Graphs Combin.*, **17** (2001), 199–206.

[9] D. Bryant and C. A. Rodger, Cycle decompositions, in *The CRC Handbook of Combinatorial Designs, 2nd edition* (Eds. C. J. Colbourn, J. H. Dinitz), CRC Press, Boca Raton (2007), 373–382.

[10] D. Bryant and V. Scharaschkin, Complete solutions to the Oberwolfach problem for an infinite set of orders, *J. Combin. Theory Ser. B*, (to appear).

[11] M. Buratti and G. Rinaldi, On sharply vertex transitive 2-factorizations of the complete graph, *J. Combin. Theory Ser. A*, **111** (2005), 245–256.

[12] P. Danziger, G. Quattrocchi and B. Stevens, The Hamilton-Waterloo problem for cycle sizes 3 and 4, *J. Combin. Des.*, (to appear).

[13] I. J. Dejter, F. Franek, E. Mendelsohn, and A. Rosa, Triangles in 2-factorizations, *J. Graph Theory*, **26** (1997), 83–94.

[14] M, Dean, On Hamilton cycle decomposition of 6-regular circulant graphs, *Graphs Combin.*, **22** (2006), 331–340.

[15] I. J. Dejter, D. A. Pike and C. A. Rodger, The directed almost resolvable Hamilton-Waterloo problem, *Australas. J. Combin.*, **18** (1998), 201–208.

[16] J. H. Dinitz and A. C. H. Ling, The Hamilton - Waterloo problem: The case of triangle-factors and one Hamilton cycle, *J. Combin. Des.*, **17** (2009), 160–176.

[17] F. Franek, J. Holub and A. Rosa, Two-factorizations of small complete graphs II: the case of 13 vertices, *J. Combin. Math. Combin. Comput.*, **51** (2004), 89–94.

[18] F. Franek and A. Rosa, Two-factorizations of small complete graphs, *J. Statist. Plann. Inference*, **86** (2000), 435–442.

[19] R. K. Guy, Unsolved combinatorial problems, Proceedings of the Conference on Combinatorial Mathematics and Its Applications, Oxford, 1967 (ed. D. J. A. Welsh, Academic Press, New York, 1971) p. 121.

[20] P. Gvozdjak, On the Oberwolfach problem for cycles with multiple lengths, PhD thesis, Simon Fraser University, 2004.

[21] R. Häggkvist, A lemma on cycle decompositions, *Ann. Discrete Math.* **27** (1985), 227–232.

[22] A. J. W. Hilton and M. Johnson, Some results on the Oberwolfach problem, *J. London Math. Soc.*, **64** (2001), 513–522.

[23] D. Hoffman and P. Schellenberg, The existence of C_k-factorizations of $K_{2n} - F$, *Dicrete Math.*, **97** (1991), 243–250.

[24] P. Horak, R. Nedela and A. Rosa, The Hamilton-Waterloo problem: the case of Hamilton cycles and triangle-factors, *Discrete Math.*, **284** (2004), 181–188.

[25] C. Huang, A. Kotzig and A. Rosa, On a variation of the Oberwolfach problem, *Discrete Math.*, **27** (1979), 261–277.

[26] E. Köhler, Das Oberwolfacher problem, *Mitt. Math. Ges. Hamburg*, **10** (1973), 52–57.

[27] E. Köhler, Über das Oberwolfacher problem, in “Beiträge zur Geometrischen Algebra” (H. Arnold, W. Benz, H. Wefelscheid: eds.), Birkhäuser Verlag, Basel, (1977), 189-201.

[28] M. A. Ollis, Some cyclic solutions to the three table Oberwolfach problem, *Elec. J. Combin.*, **12**, R58, (2005) 7pp.

[29] D. K. Ray-Chaudhuri and R. M. Wilson, Solution of Kirkman's schoolgirl problem, in *Proc. Symp. Pure Math.*, **19**, Amer. Math. Soc., Providence RI, (1971), 187–204.

[30] A. Rosa, Two-factorizations of the complete graph, *Rend. Sem. Mat. Messina II*, **9** (2003), 201–210.

[31] Q. Sui and B. Du, The Oberwolfach Problem for a unique 5-cycle and all other of length 3, *Utilitas Math.*, **65** (2004), 243–254.