Chapter 1

AFFINE GEOMETRIES OVER $GF[3]$: THEIR MINIMAL DEFINING SETS *

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Abstract Many distinct objects may exemplify a particular combinatorial structure, for example, a block design or latin square. When studying such objects with the same parameters, two questions arise naturally:
• Given two such objects, where and how do they differ?
• What portion of a particular object identifies it uniquely?

Here we consider triple systems and latin squares. The first question leads to the ideas of a *trade* in a triple system and of a *latin trade* in a latin square. The second question leads to the ideas of a *defining set*, in particular a *minimal* defining set, in a triple system and of a *uniquely completable set*, in particular a *critical* set, in a latin square. We study the relationship between latin squares and triple systems, especially that between the trades and defining sets of the triple system and the latin trades and critical sets of the square.

We apply these ideas and construct new families of minimal defining sets for triple systems associated with $AG(d, 3)$.

1. Background

In [**7**] Gower identified sets of $2d$ hyperplanes in $AG(d, 3)$, $d \geq 2$, the lines of which uniquely determine the incidence structure of the affine geometry. In this paper we go beyond the specific examples given by Gower, and determine general conditions which, when satisfied, identify

*Research supported by Australian Research Council Grants A49937047, A49802044

sets of hyperplanes for which the associated lines define the incidence structure. The Main Theorem of this paper is:

Theorem 1 *Let* $\mathbf{H}$ *be a set of hyperplanes of* $AG(d,3)$*,* $d \geq 2$*, whose equations are:*

$$\begin{cases} a_{11}x_{d-1} + a_{12}x_{d-2} + \dots + a_{1d-1}x_1 + a_{1d}x_0 = b_1 \\ a_{21}x_{d-1} + a_{22}x_{d-2} + \dots + a_{2d-1}x_1 + a_{2d}x_0 = b_2 \\ a_{31}x_{d-1} + a_{32}x_{d-2} + \dots + a_{3d-1}x_1 + a_{3d}x_0 = b_3 \\ \dots \\ \dots \\ \dots \\ a_{m1}x_{d-1} + a_{m2}x_{d-2} + \dots + a_{md-1}x_1 + a_{md}x_0 = b_m. \end{cases}$$

Let $\mathcal{H}$ *be the collection of blocks which lie within the hyperplanes of* $\mathbf{H}$*. Let* P *be the partial Steiner latin square corresponding to* $\mathcal{H}$*. Suppose that:*

(1) *there is one point of* $AG(d,3)$ *which is not incident with any hyperplane of* $\mathbf{H}$*;*

(2) P *is a uniquely completable set of order* 3^d*;*

(3) *for each block* $\{\bar{i},\bar{j},\bar{k}\}$ *of* $\mathcal{H}$ *there is a trade* T *of type one in* $AG(d,3)$ *such that* $T \cap \mathcal{H} = \{\{\bar{i},\bar{j},\bar{k}\}\}$*.*

Let $\mathbf{H}^*$ *be the extension of* $\mathbf{H}$ *to an analogous set of hyperplanes in* $AG(d+1,3)$*, and let* $\mathcal{H}^*$ *be the collection of blocks within the hyperplanes of* $\mathbf{H}^*$*. Finally, let* P^* *be the partial Steiner latin square corresponding to* $\mathcal{H}^*$*. Then:*

(1′) *there is one point of* $AG(d+1,3)$ *which is not incident with any hyperplane of* $\mathbf{H}^*$*;*

(2′) P^* *is a uniquely completable set of order* 3^{d+1}*;*

(3′) *for each block* $\{\bar{i},\bar{j},\bar{k}\}$ *of* $\mathcal{H}^*$ *there is a trade* T^* *of type one in* $AG(d+1,3)$ *such that* $T^* \cap \mathcal{H}^* = \{\{\bar{i},\bar{j},\bar{k}\}\}$*.*

We then develop examples which show that this theory can be applied to determine new families of defining sets. The key to the determination of the results is the representation of the lines of $AG(d,3)$ as blocks of a Steiner triple system (STS) or entries of an associated latin square. Through this approach the theory of direct products of latin squares is used to recursively develop sets of hyperplanes of $AG(d+1,3)$ from well chosen sets of hyperplanes of $AG(d,3)$. In this way it is relatively straightforward to prove that the corresponding lines (blocks) form a minimal defining set in $AG(d+1,3)$ (the Steiner triple system).

The examples of minimal defining sets in the Steiner triple system are interesting in their own right. Let $\mathcal{M}$ be the collection of all minimal defining sets of a design D and define the spectrum to be $spec(D) = \{m | M \text{ is a minimal defining set of } D \text{ and } |M| = m\}$. We say that the spectrum contains a *hole* if there exist $\ell < m < n$ such that $\ell, n \in spec(D)$, but $m \notin spec(D)$. Little is known about the spectra of designs or even whether there exist spectra with holes. As far as the authors are aware, the five Steiner triple systems set out in Table 1.1 are the only ones for which the spectra of minimal defining sets are fully known. However Ramsay **[15]** contains many partial results for the other 79 Steiner triple systems on 15 points. The results presented in this paper shed new light on the spectrum of the Steiner triple system corresponding to $AG(d,3)$ and document new techniques which may be adapted to construct defining sets for general designs.

v	Type of STS	$spec(D)$	Reference
7		$\{3\}$	**[9]**
9		$\{4,5\}$	**[8]**
13	cyclic	$\{9,10,11,12,13\}$	**[12]**,**[11]**
	noncyclic	$\{8,9,10,11,12,13\}$	**[12]**,**[11]**
15	$PG(3,2)$	$\{16,17,18,19,20,21,22\}$	**[15]**

Table 1.1 Spectra for minimal defining sets in some STSs of small order

2. Introduction

We start with basic material which allows us to develop our Main Theorem (Theorem 1). In this section definitions and results are illustrated with carefully chosen examples. These examples will be used extensively in Sections 3, 4 and 5.

Definition 2 Let $\mathcal{F}$ be the Galois field of order q, denoted by $GF(q)$. An *affine d-dimensional space over* $\mathcal{F}$ is denoted by $AG(d,q)$ and defined as follows:

- points are vectors of $\mathcal{F}^d$;
- lines are cosets of 1-dimensional subspaces of $\mathcal{F}^d$;
- planes are cosets of 2-dimensional subspaces of $\mathcal{F}^d$;
- m-flats are cosets of m-dimensional subspaces of $\mathcal{F}^d$;
- hyperplanes are cosets of $(d-1)$-dimensional subspaces of $\mathcal{F}^d$.

Two m-flats are said to be *parallel* if they are both cosets of the same subspace, and the set of all the cosets of a particular subspace is known collectively as a *parallel class.*

Definition 3 Let $\bar{x} = (x_{d-1}, x_{d-2}, x_{d-3}, \ldots, x_1, x_0)$ be a point in $AG(d, 3)$. The integer

$$x = 3^{d-1}x_{d-1} + 3^{d-2}x_{d-2} + 3^{d-3}x_{d-3} + \ldots + 3x_1 + x_0$$

is called the *integer representation* of $\bar{x}$ and the vector $\bar{x}$ is called the *vector representation* of x.

Moreover, for $b \in \{0, 1, 2\}$, we define $b\bar{x}$ to be the point

$$(b, x_{d-1}, x_{d-2}, x_{d-3}, \ldots, x_1, x_0)$$

in $AG(d+1, 3)$. In this paper we sometimes assume that the points of $AG(d, 3)$ are integers x for $0 \leq x \leq 3^d - 1$ and the lines (blocks) of $AG(d, 3)$ are of the form $\{x, y, z\}$, where $0 \leq x, y, z \leq 3^d - 1$.

Note that there is a one-to-one correspondence between the integers x and the points of $AG(d, 3)$, where $0 \leq x \leq 3^d - 1$.

Example 4 When $d = 2$ the correspondence between vector and integer representations is that given in the following table.

Vector representations	Integer representations
$\bar{0} = (0, 0)$	$0.3^1 + 0.3^0 = 0$
$\bar{1} = (0, 1)$	$0.3^1 + 1.3^0 = 1$
$\bar{2} = (0, 2)$	$0.3^1 + 2.3^0 = 2$
$\bar{3} = (1, 0)$	$1.3^1 + 0.3^0 = 3$
$\bar{4} = (1, 1)$	$1.3^1 + 1.3^0 = 4$
$\bar{5} = (1, 2)$	$1.3^1 + 2.3^0 = 5$
$\bar{6} = (2, 0)$	$2.3^1 + 0.3^0 = 6$
$\bar{7} = (2, 1)$	$2.3^1 + 1.3^0 = 7$
$\bar{8} = (2, 2)$	$2.3^1 + 2.3^0 = 8$

Since the vector space has nine vectors, the affine space has nine points. In the vector space, the equation $1x_1 + 0x_0 = 0$ defines a 1-dimensional subspace which contains the points with position vectors $(0, 0), (0, 1)$ and $(0, 2)$. The corresponding line in the affine space is incident with the points 0, 1 and 2. The other subspaces in the vector space are determined by the equations $0x_1 + 1x_0 = 0$, $1x_1 + 2x_0 = 0$ and $1x_1 + 1x_0 = 0$ respectively, and the corresponding lines are incident with the points {0,3,6}, {0,4,8}, {0,5,7}.

These four subspaces have two cosets each. For example, the subspace with equation $1x_1 + 0x_0 = 0$ has cosets $1x_1 + 0x_0 = 1$ and $1x_1 + 0x_0 = 2$ which correspond to the lines incident with the points {3,4,5} and {6,7,8} respectively. Altogether there are 12 lines.

Definition 5 Let m be a positive integer and let $0 \leq a_{ij} \leq 2$ and $0 \leq b_i \leq 2$ for all $1 \leq i,j \leq m$. Let **H** be a set of m hyperplanes of $AG(d,3)$ whose equations are:

$$\begin{cases} a_{11}x_{d-1} & + & a_{12}x_{d-2} & + & \dots & + & a_{1d-1}x_1 & + & a_{1d}x_0 & = & b_1 \\ a_{21}x_{d-1} & + & a_{22}x_{d-2} & + & \dots & + & a_{2d-1}x_1 & + & a_{2d}x_0 & = & b_2 \\ a_{31}x_{d-1} & + & a_{32}x_{d-2} & + & \dots & + & a_{3d-1}x_1 & + & a_{3d}x_0 & = & b_3 \\ & & & & \dots & & & & & & \\ & & & & \dots & & & & & & \\ & & & & \dots & & & & & & \\ a_{m1}x_{d-1} & + & a_{m2}x_{d-2} & + & \dots & + & a_{md-1}x_1 & + & a_{md}x_0 & = & b_m. \end{cases}$$

By the *extension of* **H**, denoted $\mathbf{H}^*$, we mean the following set of $m+2$ hyperplanes of $AG(d+1,3)$.

$$\begin{cases} 1x_d & + & 0x_{d-1} & + & 0x_{d-2} & + & \dots & + & 0x_1 & + & 0x_0 & = & 0 \\ 1x_d & + & 0x_{d-1} & + & 0x_{d-2} & + & \dots & + & 0x_1 & + & 0x_0 & = & 1 \\ 0x_d & + & a_{11}x_{d-1} & + & a_{12}x_{d-2} & + & \dots & + & a_{1d-1}x_1 & + & a_{1d}x_0 & = & b_1 \\ 0x_d & + & a_{21}x_{d-1} & + & a_{22}x_{d-2} & + & \dots & + & a_{2d-1}x_1 & + & a_{2d}x_0 & = & b_2 \\ 0x_d & + & a_{31}x_{d-1} & + & a_{32}x_{d-2} & + & \dots & + & a_{3d-1}x_1 & + & a_{3d}x_0 & = & b_3 \\ & & & & & & \dots & & & & & & \\ & & & & & & \dots & & & & & & \\ & & & & & & \dots & & & & & & \\ 0x_d & + & a_{m1}x_{d-1} & + & a_{m2}x_{d-2} & + & \dots & + & a_{md-1}x_1 & + & a_{md}x_0 & = & b_m. \end{cases}$$

Definition 6 A *triple system* is a pair $(V,\mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a collection of b 3-subsets of V (*blocks* or *triples*) such that each element of V is contained in precisely r blocks and each 2-subset of V is contained in precisely λ blocks. The numbers v, b, r and λ are the *parameters* of the triple system and simple counting arguments show that $r = \lambda(v-1)/2$ and $b = vr/3$. If $\lambda = 1$ the triple system is called a *Steiner* triple system $(STS(v))$ which has $r = (v-1)/2$ and $b = v(v-1)/6$.

A *partial* Steiner triple system is a pair $(V,\mathcal{B}')$ where $\mathcal{B}'$ is a collection of 3-subsets (blocks) of V such that each 2-subset of V is contained in at most one block.

Example 7 Let V and $\mathcal{B}$ respectively be the set of points and the set of lines of $AG(d,3)$. Then the pair $(V,\mathcal{B})$ forms a Steiner triple system on 3^d points, denoted by $STS_A(3^d)$. In particular, if $d=2$, then the set of points of the $STS(3^2)$ is $V = \{0,1,2,3,4,5,6,7,8\}$ and the set of blocks $\mathcal{B}$ are those of the following table, where each column shows the blocks of one parallel class.

$$\begin{array}{cccc} \{0,1,2\} & \{0,3,6\} & \{0,4,8\} & \{0,5,7\} \\ \{3,4,5\} & \{1,4,7\} & \{1,5,6\} & \{1,3,8\} \\ \{6,7,8\} & \{2,5,8\} & \{2,3,7\} & \{2,4,6\} \end{array}$$

Note that although every $STS(9)$ is isomorphic to $STS_A(3^2)$, there are in general many $STS(3^d)$ not isomorphic to $STS_A(3^d)$.

We now consider the idea of a defining set, introduced by Gray [9] in the more general context of balanced incomplete block designs. We need it only for Steiner triple systems.

Definition 8 A set of blocks which is a subset of a unique $STS(v)$ is said to be a *defining set* of that Steiner triple system. A *minimal* defining set is a defining set, no proper subset of which is a defining set. A *smallest* defining set is a defining set such that no other defining set has smaller cardinality.

Since $\mathcal{B}$ itself forms a defining set of the $STS(v)$ $(V, \mathcal{B})$, every Steiner triple system has a defining set. Since every defining set can be reduced, by deletion of blocks, to at least one minimal defining set, every Steiner triple system has at least one minimal defining set.

Example 9 In the $STS_A(3^2)$ of Example 7, the following set $\mathcal{S}$ of six blocks is a defining set.

$$\begin{array}{cccc} \{0,1,2\} & \{0,3,6\} & \{0,4,8\} & \\ \{3,4,5\} & \{1,4,7\} & & \{1,3,8\} \end{array}$$

Its subsets $\mathcal{S}_5 = \mathcal{S} \setminus \{\{1,4,7\}\}$ and $\mathcal{S}_4 = \mathcal{S} \setminus \{\{0,4,8\},\{1,3,8\}\}$ are minimal defining sets; $\mathcal{S}_4$ is a smallest defining set.

On the other hand, the following set of six blocks, $\mathcal{N}$,

$$\begin{array}{cccc} \{0,1,2\} & & & \\ \{3,4,5\} & & & \\ \{6,7,8\} & \{2,5,8\} & \{2,3,7\} & \{2,4,6\} \end{array}$$

is not a defining set since it can be completed to an $STS(9)$ by adjoining either the six remaining blocks of $\mathcal{B}$, or the six blocks of $\mathcal{F}$, given below.

$$\begin{array}{ccc} \{0,4,7\} & \{0,5,6\} & \{0,3,8\} \\ \{1,3,6\} & \{1,4,8\} & \{1,5,7\} \end{array}$$

Similarly, the following set $\mathcal{M}$ of six blocks, five of which lie in the set $\mathcal{B}$, is not a defining set of an $STS(9)$ since it cannot be completed to an $STS(9)$ at all.

$$\begin{array}{cccc} \{0,1,2\} & \{0,3,6\} & \{0,4,8\} & \{0,5,7\} \\ \{3,4,5\} & \{1,7,8\} & & \end{array}$$

Lemma 10 *(Gray [9]) Any defining set of an $STS(v)$ has at least $v-1$ elements of V occurring in its blocks.*

Example 11 In the defining sets of $STS_A(3^2)$ given in Example 9, the blocks of $\mathcal{S}$ contain all elements of V, those of $\mathcal{S}_4$ contain all except 8, and those of $\mathcal{S}_5$ all except 7.

Definition 12 Let T_1 and T_2 be two collections of m 3-sets of elements of the v-set V. If each 2-set of elements of V occurs in the triples of T_1 with precisely the same multiplicity that it occurs in the triples of T_2, then the collections T_1 and T_2 are said to be *mutually balanced.* If for a collection of 3-sets T_1, there exists a collection of 3-sets T_2 such that T_1 and T_2 are mutually balanced and have no common triple, they are disjoint and we say T_1 is a *Steiner trade* of *volume m.* If the triples of T_1 are triples of a Steiner triple system $STS(v)$, then $STS(v)$ is said to *contain* the trade T_1.

Example 13 In the $STS_A(3^2)$ of Example 7 with the defining sets of Example 9, let the set of six blocks $T_1 = \mathcal{B} \setminus \mathcal{N}$. Let $T_2 = \mathcal{F}$. Then T_1 is a trade of volume six contained in $STS_A(3^2)$.

Lemma 14 *(Gray [9]) In an $STS(v)$, every defining set has at least one block in common with every trade.*

Example 15 In the $STS_A(3^2)$ of Examples 7 and 9, $\mathcal{S}_4 \cap T_1 = \{\{0,3,6\}, \{1,4,7\}\}$ and $\mathcal{S}_5 \cap T_1 = \{\{0,3,6\}; \{0,4,8\}; \{1,3,8\}\}$.

Definition 16 There are only two non-isomorphic Steiner trades of volume 6 and block size 3; see [14] for example. These are called a *trade of type one,* such as

$$\tau_1 = \{\{0,2,5\},\{0,3,6\},\{0,4,7\},\{1,2,6\},\{1,3,7\},\{1,4,5\}\},$$

and a *trade of type two,* such as

$$\tau_2 = \{\{0,3,4\},\{0,5,6\},\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\}.$$

In this paper only trades of type one occur. Note that the six blocks of a trade of type one consist of three disjoint pairs, whereas any two blocks of a trade of type two intersect each other.

Definition 17 A *latin square L* of order n is an $n \times n$ array with entries chosen from a set N, of size n, such that each element of N occurs precisely once in each row and column. Similarly, a *partial latin square P* of order n is an $n \times n$ array with entries chosen from a set N, of size

n, such that each element of N occurs at most once in each row and column. Thus P may contain a number of empty cells. For convenience, a (partial) latin square will sometimes be represented as a set of ordered triples $(i,j;k)$, which is read to mean that element k occurs in cell (i,j) of the (partial) latin square L. For a (partial) latin square of order n we label entries, rows and columns by $0,1,2,\dots,n-1$. $|P|$ denotes the number of non-empty cells of the partial latin square P and is called the *size* of P, and the set of positions $\mathcal{S}_P = \{(i,j) \mid (i,j;k) \text{ for some } k \in N\}$ is said to determine the *shape* of P.

Example 18 Define the partial latin squares R, S and W, and the latin square L, each of order 3, as follows.

$$R = \begin{array}{|c|c|c|}\hline 0 & 2 & \\ \hline 2 & & \\ \hline & & \\ \hline\end{array} \quad S = \begin{array}{|c|c|c|}\hline 0 & & \\ \hline & 1 & \\ \hline & & \\ \hline\end{array} \quad W = \begin{array}{|c|c|c|}\hline 0 & 1 & \\ \hline & & 2 \\ \hline & & \\ \hline\end{array} \quad L = \begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline 2 & 1 & 0 \\ \hline 1 & 0 & 2 \\ \hline\end{array}$$

Then R and S can each be completed to L but to no other latin square of order 3, and W cannot be completed to any latin square of order 3. Note that throughout this paper S stands for this particular partial latin square of order 3.

Definition 19 Let M and N be two latin squares of orders m and n, with entries chosen from the sets $\{0,1,2,\dots,m-1\}$ and $\{0,1,2,\dots,n-1\}$, respectively. Suppose that P is a (partial) latin square in M and Q is a (partial) latin square in N. Let P^r be the array obtained from P by adding rm to the entry in each non-empty cell of P, for $r = 0,1,2,\dots,n-1$. Similarly, let M^r be the array obtained from M by adding rm to the entry in each cell of M, for $r = 0,1,2,\dots,n-1$. Then we define the *completable product* of Q and P, with respect to M and N, to be the (partial) latin square T of order mn obtained by replacing each cell containing the entry r of Q with the array M^r and each cell of $N \setminus Q$ with the array P^r. The completable product of Q with P will be denoted $Q \otimes P$. If Q is a latin square then the completable product corresponds to the definition of the direct product of the latin squares Q and N, usually denoted $Q \times N$.

Example 20 Let R and S be the partial latin squares and L the latin square defined in Example 18. We consider three examples with $L = M = N$, and a fourth subsquare of the first which cannot be written as a completable product.

(i) $L = P = Q$.

$$L_9 = Q \otimes P = L \times L =$$

L^0	L^2	L^1
L^2	L^1	L^0
L^1	L^0	L^2

$=$

0	2	1	6	8	7	3	5	4
2	1	0	8	7	6	5	4	3
1	0	2	7	6	8	4	3	5
6	8	7	3	5	4	0	2	1
8	7	6	5	4	3	2	1	0
7	6	8	4	3	5	1	0	2
3	5	4	0	2	1	6	8	7
5	4	3	2	1	0	8	7	6
4	3	5	1	0	2	7	6	8

(ii) $S = P = Q$.

$$L_{9,4} = Q \otimes P = S \otimes S =$$

L^0	S^2	S^1
S^2	L^1	S^0
S^1	S^0	S^2

$=$

0	2	1	6			3		
2	1	0		7			4	
1	0	2						
6			3	5	4	0		
	7		5	4	3		1	
			4	3	5			
3			0			6		
	4			1			7	

(iii) $R = P = Q$.

$$Q \otimes P = R \otimes R =$$

L^0	L^2	R^1
L^2	R^1	R^0
R^1	R^0	R^2

(iv) The following partial latin square, $L_{9,5}$, cannot be written as a completable product.

$$L_{9,5} =$$

0	2	1	6	8		3		4
2	1	0	8					3
1	0	2						
6	8		3	5	4	0		1
8			5	4	3			0
			4	3	5			
3			0			6		
4	3		1	0				8

Example 21 We also need some partial latin squares with special properties. We choose examples of order 5, where A belongs to the second main class of latin squares of order 5 (see [3], page 129) and each of the partial latin squares B, C and D can be completed to A.

$A =$

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

$B =$

	1			3
		5		
4				1
5			2	

$C =$

	1			3
	4		1	
		2		
5			2	

$D =$

				5
	1	4		
3				
4		2		
			2	

Definition 22 Let $(V, \mathcal{B})$ be a (partial) Steiner triple system of order v on the element set $V = \{0, 1, 2, \ldots, v-1\}$. We define the corresponding *(partial) Steiner latin square of order v* to be the array with entry k in the cell (i, j), $i \neq j$, if and only if $\{i, j, k\} \in \mathcal{B}$. Moreover, the cell (i, i) contains the entry i if and only if the element i occurs in a block. Note that some elements may not occur in any block of a partial Steiner triple system of order v.

Example 23 The Steiner triple system $STS_A(3^2)$ of Example 7 corresponds to the Steiner latin square L_9 given in Example 20, and its minimal defining sets $\mathcal{S}_5$ and $\mathcal{S}_4$ given in Example 9 to the partial squares $L_{9,5}$ and $L_{9,4}$ given in Example 20. Note that the element 7 missing from the blocks of $\mathcal{S}_5$ does not appear on the diagonal of $L_{9,5}$, nor does the element 8 missing from the blocks of $\mathcal{S}_4$ appear on the diagonal of $L_{9,4}$.

Definition 24 A *critical set* in a latin square L of order n is a set $\mathcal{C} = \{(i, j; k) \mid i, j, k \in N\}$, such that both of the following conditions hold:

(1) L is the only latin square of order n with element k in cell (i, j) for each $(i, j; k) \in \mathcal{C}$;

(2) no proper subset of $\mathcal{C}$ satisfies (1).

A *uniquely completable set* in a latin square L of order n is a partial latin square in L satisfying condition (1). Condition (2) guarantees that each entry of $\mathcal{C}$ is *necessary* for the completion to be unique.

Example 25 In Example 18 the partial latin square S is a critical set of the latin square L. In Example 20 the partial latin squares $L_{9,5}$ and $L_{9,4}$ are both critical sets of the latin square L_9. In Example 21 the partial latin squares B, C and D are all critical sets of the latin square A.

Definition 26 Let U be a uniquely completable set in the latin square L of order n. The adjunction of a triple $t = (r, c; s)$ is said to be *forced* (see [13]) during completion of a set P of triples ($|P| < n^2$, $U \subseteq P \subset L$) to the complete set of triples which represents L, if at least one of the following conditions holds:

(i) $\forall r' \neq r$, $\exists z \neq c$ such that $(r', z; s) \in P$ or $\exists z \neq s$ such that $(r', c; z) \in P$ (that is, in the partial completion F of L, each cell of column c except that in row r is either in a row of F which already contains the symbol s or is already filled with an element z distinct from s);

(ii) $\forall c' \neq c$, $\exists z \neq r$ such that $(z, c'; s) \in P$ or $\exists z \neq s$ such that $(r, c'; z) \in P$ (that is, in the partial completion F of L, each cell of row r except that in column c is either in a column of F which already contains the symbol s or is already filled with an element z distinct from s);

(iii) $\forall s' \neq s$, $\exists z \neq r$ such that $(z, c; s') \in P$ or $\exists z \neq c$ such that $(r, z; s') \in P$ (that is, in the partial completion F of L, every symbol except s already occurs either in column c or in row r of F).

The uniquely completable set U is called *strong* if we can define a sequence of sets of triples $U = F_1 \subset F_2 \subset F_3 \subset \cdots \subset F_r = L$ such that each triple $t \in F_{i+1} \setminus F_i$ is forced in F_i for $1 \leq i \leq r - 1$. If U is not strong, it is called *weak*. A completable set is *super-strong* if each triple in this sequence is forced by virtue of property (iii) alone. In particular, a critical set can be super-strong, strong but not super-strong, or weak.

Example 27 The critical sets $L_{9,5}$ and $L_{9,4}$ of Example 20 are both super-strong. In Example 21, the critical set B is super-strong, the critical set C is strong but not super-strong, and the critical set D is weak; see [1], [2].

Definition 28 Let P and P' be two partial latin squares of the same order, with the same size and shape. Then P and P' are said to be *mutually balanced* if the entries in each row (and column) of P are the same as those in the corresponding row (and column) of P'. They are said to be *disjoint* if no position in P' contains the same entry as the corresponding position in P. A *latin trade* (sometimes called a latin interchange) is a partial latin square I for which there exists another partial latin square I' of the same order, size and shape, with the property that I and I' are disjoint and mutually balanced. Then I' is called the *disjoint mate* of I. The smallest possible size for such partial latin squares is four, and such a latin trade is called an *intercalate.*

Example 29 The partial latin square I_1 is an example of an intercalate; that is a latin trade of size 4, with disjoint mate I_1'. The partial latin square I_2, is an example of a latin trade of order 3 and size 8, with disjoint mate I_2'.

$$I_1 = \begin{array}{|c|c|c|}\hline 0 & & 1 \\ \hline 1 & & 0 \\ \hline & & \\ \hline\end{array} \quad I_1' = \begin{array}{|c|c|c|}\hline 1 & & 0 \\ \hline 0 & & 1 \\ \hline & & \\ \hline\end{array} \quad I_2 = \begin{array}{|c|c|c|}\hline 0 & & 2 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 0 \\ \hline\end{array} \quad I_2' = \begin{array}{|c|c|c|}\hline 2 & & 0 \\ \hline 3 & 1 & 2 \\ \hline 0 & 3 & 1 \\ \hline\end{array}$$

In a given latin square L, every latin trade contained in L must intersect every critical set contained in L. More precisely, we have the following result.

Theorem 30 *(Donovan and Howse [4]) A partial latin square C of the latin square L is a critical set of L if and only if the following conditions hold:*

(1) *C intersects every latin trade that occurs in L;*

(2) *for each $(i, j; k) \in C$, there exists a latin trade I in L such that $I \cap C = \{(i, j; k)\}$.*

Lemma 31 *Consider τ_1 as given in Definition 16. Then τ_1 defines six disjoint latin trades each of size six.*

Proof. Define

$$\begin{aligned}
I_1 &= \{(0,2;5),(0,3;6),(0,4;7),(1,2;6),(1,3;7),(1,4;5)\} \\
I_2 &= \{(2,0;5),(3,0;6),(4,0;7),(2,1;6),(3,1;7),(4,1;5)\} \\
I_3 &= \{(0,5;2),(0,6;3),(0,7;4),(1,6;2),(1,7;3),(1,5;4)\} \\
I_4 &= \{(5,0;2),(6,0;3),(7,0;4),(6,1;2),(7,1;3),(5,1;4)\} \\
I_5 &= \{(5,2;0),(6,3;0),(7,4;0),(6,2;1),(7,3;1),(5,4;1)\} \\
I_6 &= \{(2,5;0),(3,6;0),(4,7;0),(2,6;1),(3,7;1),(4,5;1)\}.
\end{aligned}$$

It is easy to see that I_r for $1 \leq r \leq 6$ is a latin trade. Moreover, $I_r \cap I_s = \emptyset$ for $1 \leq r < s \leq 6$. These latin trades are shown in the following table.

		5	6	7	2	3	4
		6	7	5	4	2	3
5	6				0	1	
6	7					0	1
7	5				1		0
2	4	0		1			
3	2	1	0				
4	3		1	0			

□

In Section 3, we give some lemmas and constructions needed for the proof of the Main Theorem. The proof itself is given in Section 4.

3. Further Preliminaries

Lemma 32 *(Gower [6]) Let* P *be a strongly completable set in the latin square* M *of order* m *and let* Q *be a (strongly) completable set in the latin square* N *of order* n*. Then* $P \otimes Q$ *is a (strongly) completable set of order* mn *in the latin square* $M \times N$*.*

Example 33 Let S and L be as defined in Example 20. Then S is a super-strong critical set in L. Now let $P = Q = S$ and $M = N = L$. Then $L_{9,4} = Q \otimes P = S \otimes S$ as in Example 20, and the fact that it is a strongly completable set of order 9 is confirmed by Lemma 32.

The following result states the relationship between a completable set and a defining set.

Lemma 34 *(Gower [5]) Let* $(V, \mathcal{B})$ *be a Steiner triple system of order* v *and let* $\mathcal{B}' \subseteq \mathcal{B}$*. Suppose that* P *is the partial Steiner latin square corresponding to* $(V, \mathcal{B}')$*. If* P *is a completable set of order* v *then* $\mathcal{B}'$ *is a defining set in* $(V, \mathcal{B})$*.*

Unfortunately the partial Steiner latin square corresponding to a defining set of an $STS(v)$ need not be uniquely completable. For an example of this, see the Appendix.

Example 35 Consider the latin square L_9 and the partial latin squares S and $L_{9,5}$ discussed in Example 18. Note that $L_{9,5}$ is a partial Steiner latin square. The partial Steiner triple system corresponding to $L_{9,5}$ consists of the blocks $\{0,1,2\}$, $\{0,3,6\}$, $\{0,4,8\}$, $\{1,3,8\}$ and $\{3,4,5\}$, which constitute the set S_5 given in Example 9 as a minimal defining set of the $STS(9)$ given in Example 7. Lemma 34 confirms that these blocks yield a defining set of order 9, and it is also easy to check that all the blocks are necessary for unique completion.

Now by Lemma 32, $R = S \otimes L_{9,5}$ is a strongly completable set of order 27. (Indeed, we see later that R is a critical set by Theorem 46, Section 6.) One can observe that R is a partial Steiner latin square. The partial Steiner triple system corresponding to R consists of the following 67 blocks.

{0, 1, 2}	{0, 3, 6}	{0, 4, 8}	{0, 5, 7}	{0, 9, 18}
{0, 10, 20}	{0, 11, 19}	{0, 12, 24}	{0, 13, 26}	{0, 15, 21}
{0, 17, 22}	{1, 3, 8}	{1, 4, 7}	{1, 5, 6}	{1, 9, 20}
{1, 10, 19}	{1, 11, 18}	{1, 12, 26}	{1, 17, 21}	{2, 3, 7}
{2, 4, 6}	{2, 5, 8}	{2, 9, 19}	{2, 10, 18}	{2, 11, 20}
{3, 4, 5}	{3, 9, 24}	{3, 10, 26}	{3, 12, 21}	{3, 13, 23}
{3, 14, 22}	{3, 15, 18}	{3, 17, 19}	{4, 9, 26}	{4, 12, 23}
{4, 13, 22}	{4, 14, 21}	{4, 17, 18}	{5, 12, 22}	{5, 13, 21}
{5, 14, 23}	{6, 7, 8}	{6, 9, 21}	{6, 12, 18}	{6, 15, 24}
{8, 9, 22}	{8, 10, 21}	{8, 12, 19}	{8, 13, 18}	{8, 17, 26}
{9, 10, 11}	{9, 12, 15}	{9, 13, 17}	{9, 14, 16}	{10, 12, 17}
{10, 13, 16}	{10, 14, 15}	{11, 12, 16}	{11, 13, 15}	{11, 14, 17}
{12, 13, 14}	{15, 16, 17}	{18, 19, 20}	{18, 21, 24}	{18, 22, 26}
{19, 21, 26}	{21, 22, 23}			

By Lemma 34, these blocks yield a defining set of order 27. (Indeed, these blocks yield a minimal defining set in $AG(3,3)$ by Theorem 44, Section 5.)

Theorem 36 *(Gower [7]) Let* $\mathbf{H}$ *be the following* $2d$ *hyperplanes in* $AG(d,3)$, $d \geq 2$.

$$
\begin{cases}
1x_{d-1} + 0x_{d-2} + \ldots + 0x_1 + 0x_0 = 0 \\
1x_{d-1} + 0x_{d-2} + \ldots + 0x_1 + 0x_0 = 1 \\
0x_{d-1} + 1x_{d-2} + \ldots + 0x_1 + 0x_0 = 0 \\
0x_{d-1} + 1x_{d-2} + \ldots + 0x_1 + 0x_0 = 1 \\
\ldots \\
\ldots \\
\ldots \\
0x_{d-1} + 0x_{d-2} + \ldots + 1x_1 + 0x_0 = 0 \\
0x_{d-1} + 0x_{d-2} + \ldots + 1x_1 + 0x_0 = 1 \\
0x_{d-1} + 0x_{d-2} + \ldots + 0x_1 + 1x_0 = 0 \\
0x_{d-1} + 0x_{d-2} + \ldots + 0x_1 + 1x_0 = 1.
\end{cases}
$$

Then the lines contained within these hyperplanes correspond to blocks of the $PSTS(3^d)$ which are a minimal defining set for this Steiner triple system.

Theorem 37 *(Gower [7]) The hyperplanes of Theorem 36 can be replaced by another set of $2d$ hyperplanes of $AG(d,3)$ provided these new hyperplanes are such that:*

(1) they belong to d parallel classes with precisely two hyperplanes chosen from each of these parallel classes;

(2) no line of $AG(d,3)$ is contained within any d of them;

(3) there is a point of $AG(d,3)$ which is incident with none of these $2d$ hyperplanes.

By Lemma 10, it is necessary for a set of hyperplanes in $AG(d,3)$ which provides a defining set in $AG(d,3)$ to cover all points of $AG(d,3)$ except possibly one. We state this fact in the following lemma and give a simple proof for it.

Lemma 38 *Let $\mathbf{H}$ be a set of hyperplanes in $AG(d,3)$, where $d \geq 2$. Suppose that the points $\bar{i}$ and $\bar{j}$ of $AG(d,3)$ are not incident with any hyperplane of $\mathbf{H}$. Then the lines within the hyperplanes of $\mathbf{H}$ do not form a defining set.*

Proof. Let $\bar{k} \neq -\bar{i} - \bar{j}$ be a point of $AG(d,3)$. Let T consist of the following blocks:

$$\begin{array}{lll} \{\bar{i},\ \bar{k},\ -\bar{i}-\bar{k}\}, & \{\bar{i},\ -\bar{i}+\bar{j}+\bar{k},\ \ -\bar{j}-\bar{k}\}, & \{\bar{i},\ \bar{i}-\bar{j}+\bar{k},\ \bar{i}+\bar{j}-\bar{k}\}, \\ \{\bar{j},\ \bar{k},\ -\bar{j}-\bar{k}\}, & \{\bar{j},\ -\bar{i}+\bar{j}+\bar{k},\ \bar{i}+\bar{j}-\bar{k}\}, & \{\bar{j},\ \bar{i}-\bar{j}+\bar{k},\ \ -\bar{i}-\bar{k}\}. \end{array}$$

First we note that T is a type one trade. Secondly, for each $\{\bar{x},\bar{y},\bar{z}\}$ of T we have $\bar{x}+\bar{y}+\bar{z}=\bar{0}$. So T is a trade in $AG(d,3)$. Thirdly, no block of T lies in any hyperplane of $\mathbf{H}$. Now the result follows. □

Lemma 39 *Consider the set of hyperplanes $\mathbf{H}$ in $AG(d,3)$ defined in Definition 5. Let L be the Steiner latin square corresponding to the blocks of $AG(d,3)$. Let $\mathcal{H}$ be the collection of blocks contained within the hyperplanes of $\mathbf{H}$. Suppose that P is the partial Steiner latin square of order 3^d corresponding to $\mathcal{H}$. Let $\mathcal{H}^*$ be the set of blocks within the hyperplanes of $\mathbf{H}^*$, the extension of $\mathbf{H}$. Then the partial Steiner latin square P^* corresponding to $\mathcal{H}^*$ is*

$$P^* = S \otimes P = \begin{array}{|c|c|c|} \hline L^0 & P^2 & P^1 \\ \hline P^2 & L^1 & P^0 \\ \hline P^1 & P^0 & P^2 \\ \hline \end{array}$$

where S is as in Example 18.

Proof. First for each block of $\mathcal{H}^*$ we find the corresponding elements of $S \otimes P$. Note that each block of $\mathcal{H}^*$ fills precisely six off-diagonal cells of P^*. Conversely, for each off-diagonal element of $S \otimes P$ we give its corresponding block of $\mathcal{H}^*$. Finally, for a diagonal element $(i, i; i)$ of $S \otimes P$ we prove that $\bar{i}$ occurs in a block of $\mathcal{H}^*$.
Let $\{\bar{i}, \bar{j}, \bar{k}\}$ be a block of $\mathcal{H}^*$ and let $\bar{i} = (i_d, i_{d-1}, i_{d-2}, \ldots, i_1, i_0)$, $\bar{j} = (j_d, j_{d-1}, j_{d-2}, \ldots, j_1, j_0)$, and $\bar{k} = (k_d, k_{d-1}, k_{d-2}, \ldots, k_1, k_0)$. Suppose that i, j and k are integer representations of $\bar{i}$, $\bar{j}$ and $\bar{k}$, respectively. Without loss of generality, we can assume $i < j < k$. This leads to

$$(i_d, j_d, k_d) \in \{(0,0,0), (1,1,1), (2,2,2), (0,1,2)\}.$$

Define $\bar{i'} = (i_{d-1}, i_{d-2}, \ldots, i_0)$, $\bar{j'} = (j_{d-1}, j_{d-2}, \ldots, j_0)$, and $\bar{k'} = (k_{d-1}, k_{d-2}, \ldots, k_0)$. Then $\bar{i'} + \bar{j'} + \bar{k'} = \bar{0}$ since $\bar{i} + \bar{j} + \bar{k} = \bar{0}$. So $\{\bar{i'}, \bar{j'}, \bar{k'}\}$ is a block of $AG(d, 3)$.

Case $(i_d, j_d, k_d) = (0,0,0)$**:** It is obvious that the points $\bar{i}$, $\bar{j}$ and $\bar{k}$ are in the first hyperplane of $\mathbf{H}^*$. So the block $\{\bar{i}, \bar{j}, \bar{k}\}$ lies in this hyperplane. On the other hand, the integer representations for $\bar{i'}$, $\bar{j'}$ and $\bar{k'}$ are i, j and k, respectively. Moreover, $(i_d, j_d, k_d) = (0,0,0)$ implies that $0 \le i, j, k < 3^d$. So $(i, j; k) \in L^0$.

Case $(i_d, j_d, k_d) = (1,1,1)$**:** It is obvious that the points $\bar{i}$, $\bar{j}$ and $\bar{k}$ are in the second hyperplane of $\mathbf{H}^*$. So the block $\{\bar{i}, \bar{j}, \bar{k}\}$ lies in this hyperplane. On the other hand, the integer representations for $\bar{i'}$, $\bar{j'}$ and $\bar{k'}$ are $i - 3^d$, $j - 3^d$ and $k - 3^d$, respectively. Moreover, $(i_d, j_d, k_d) = (1,1,1)$ implies that $3^d \le i, j, k < 2.3^d$. So $(i - 3^d, j - 3^d; k) \in L^1$.

Case $(i_d, j_d, k_d) = (2,2,2)$**:** It is obvious that the points $\bar{i}$, $\bar{j}$ and $\bar{k}$ cannot be in the first or second hyperplane of $\mathbf{H}^*$. So the block $\{\bar{i}, \bar{j}, \bar{k}\}$ must be in one of the last $m-2$ hyperplanes of $\mathbf{H}^*$. Let $\{\bar{i}, \bar{j}, \bar{k}\}$ be in the nth hyperplane of $\mathbf{H}^*$, where $3 \le n \le m+2$. Then the points $\bar{i'}$, $\bar{j'}$ and $\bar{k'}$ are in the $(n-2)$nd hyperplane of $\mathbf{H}$. So the block $\{\bar{i'}, \bar{j'}, \bar{k'}\}$ is in $\mathcal{H}$. Moreover, $(i_d, j_d, k_d) = (2,2,2)$ implies that $2.3^d \le i, j, k < 3^{d+1}$. So $(i - 2.3^d, j - 2.3^d; k) \in P^2$.

Case $(i_d, j_d, k_d) = (0,1,2)$**:** It is obvious that the point $\bar{k}$ cannot be in the first or second hyperplane of $\mathbf{H}^*$. So the block $\{\bar{i}, \bar{j}, \bar{k}\}$ must be in one of the last $m-2$ hyperplanes of $\mathbf{H}^*$. Let $\{\bar{i}, \bar{j}, \bar{k}\}$ be in the nth hyperplane of $\mathbf{H}^*$, where $3 \le n \le m+2$. Then the points $\bar{i'}$, $\bar{j'}$ and $\bar{k'}$ are in the $(n-2)$nd hyperplane of $\mathbf{H}$. Now if $i \not\equiv j \pmod{3^d}$ then

$\bar{i'} \neq \bar{j'}$. So $\{\bar{i'}, \bar{j'}, \bar{k'}\}$ is a block in $\mathcal{H}$. Moreover, $(i_d, j_d, k_d) = (0, 1, 2)$ implies that $0 \le i < 3^d$, $3^d \le j < 2.3^d$, and $2.3^d \le k < 3^{d+1}$. So $(i, j - 3^d, k) \in P^2$. Finally, if $i \equiv j \pmod{3^d}$ then $\bar{i'} = \bar{j'} = \bar{k'}$. So the cell $(i, j - 3^d)$ is a diagonal cell in P. Now by the definition of P^2 we have $(i, j - 3^d; k - 2.3^d) \in P$ if and only if $(i, j - 3^d; k) \in P^2$. Note that in this case the six off-diagonal elements corresponding to the block $\{\bar{i}, \bar{j}, \bar{k}\}$ are in the two P^0s, the two P^1s and the two (off-diagonal) P^2s in the latin square above.

Suppose that $(i, j; k)$ is an element of $S \otimes P$. We consider the following cases:

First, assume $i = j$. Then $(i, j; k)$ is a diagonal element of $S \otimes P$. So $\bar{i'}$ is incident with a hyperplane of $\mathbf{H}$ and $\bar{i}$ is incident with one of the last m hyperplanes of $\mathbf{H}^*$. Therefore $\bar{i}$ occurs in one of the blocks of $\mathcal{H}^*$.

Now we assume that $i \neq j$.

Case $0 \le i, j, k < 3^d$: One can see that $\bar{i}$, $\bar{j}$ and $\bar{k}$ are incident with the first hyperplane of $\mathbf{H}^*$. Since $i \neq j$, $\{\bar{i}, \bar{j}, \bar{k}\}$ is a block of $\mathcal{H}^*$.

Case $3^d \le i, j, k < 2.3^d$: One can see that $\bar{i}$, $\bar{j}$ and $\bar{k}$ are incident with the second hyperplane of $\mathbf{H}^*$. Since $i \neq j$, $\{\bar{i}, \bar{j}, \bar{k}\}$ is a block of $\mathcal{H}^*$.

Case $2.3^d \le i, j, k < 3^{d+1}$: Suppose that $\bar{i} = (i_d, i_{d-1}, \dots, i_1, i_0)$, $\bar{j} = (j_d, j_{d-1}, \dots, j_1, j_0)$, and $\bar{k} = (k_d, k_{d-1}, \dots, k_1, k_0)$. Then since $2.3^d \le i, j, k < 3.3^d$ we have $i_d = j_d = k_d = 2$. Since $i \neq j$ and $(i - 2.3^d, j - 2.3^d; k) \in P^2$ we have $\{\bar{i'}, \bar{j'}, \bar{k'}\} \in \mathcal{H}$, where $\bar{i'} = (i_{d-1}, \dots, i_1, i_0)$, $\bar{j'} = (j_{d-1}, \dots, j_1, j_0)$, and $\bar{k'} = (k_{d-1}, \dots, k_1, k_0)$. So the block $\{\bar{i'}, \bar{j'}, \bar{k'}\}$ lies in one of the hyperplanes of $\mathbf{H}$. Therefore, the block $\{\bar{i}, \bar{j}, \bar{k}\}$ lies in one of the last m hyperplanes of $\mathbf{H}^*$. Hence, $\{\bar{i}, \bar{j}, \bar{k}\} \in \mathcal{H}^*$.

The remaining six cases are as follows.

1	$0 \le i < 3^d$	$3^d \le j < 2.3^d$	$2.3^d \le k < 3^{d+1}$
2	$0 \le i < 3^d$	$2.3^d \le j < 3^{d+1}$	$3^d \le k < 2.3^d$
3	$3^d \le i < 2.3^d$	$0 \le j < 3^d$	$2.3^d \le k < 3^{d+1}$
4	$3^d \le i < 2.3^d$	$2.3^d \le j < 3^{d+1}$	$0 \le k < 3^d$
5	$2.3^d \le i < 3^{d+1}$	$0 \le j < 3^d$	$3^d \le k < 2.3^d$
6	$2.3^d \le i < 3^{d+1}$	$3^d \le j < 2.3^d$	$0 \le k < 3^d$

These cases are similar to the case $2.3^d \le i, j, k < 3^{d+1}$ and we leave them to the reader. □

4. Proof of the Main Theorem

First we prove the following lemmas, in which we let $\mathbf{H}$, $\mathbf{H}^*$, $\mathcal{H}$ and $\mathcal{H}^*$ be as defined in Definition 5 and applied in Theorem 1.

Lemma 40 *Let $\{\bar{\imath}, \bar{\jmath}, \bar{k}\}$ be a block of $\mathcal{H}$ and let*

$$T = \{\{\bar{\imath}, \bar{\jmath}, \bar{k}\}, \{\bar{\imath}, \bar{e}, \bar{f}\}, \{\bar{\imath}, \bar{g}, \bar{h}\}, \{\bar{\ell}, \bar{\jmath}, \bar{e}\}, \{\bar{\ell}, \bar{k}, \bar{g}\}, \{\bar{\ell}, \bar{f}, \bar{h}\}\}$$

be a trade in $AG(d, 3)$ such that $T \cap \mathcal{H} = \{\{\bar{\imath}, \bar{\jmath}, \bar{k}\}\}$. Then there exist trades T_0^* and T_1^*, of type one, in $AG(d+1, 3)$ such that $T_0^* \cap \mathcal{H}^* = \{\{0\bar{\imath}, 0\bar{\jmath}, 0\bar{k}\}\}$ and $T_1^* \cap \mathcal{H}^* = \{\{1\bar{\imath}, 1\bar{\jmath}, 1\bar{k}\}\}$.

Proof. First define

$$\begin{aligned} T_0^* &= \{\{0\bar{\imath}, 0\bar{\jmath}, 0\bar{k}\}, \{0\bar{\imath}, 1\bar{e}, 2\bar{f}\}, \{0\bar{\imath}, 1\bar{g}, 2\bar{h}\}, \\ &\quad \{2\bar{\ell}, 0\bar{\jmath}, 1\bar{e}\}, \{2\bar{\ell}, 0\bar{k}, 1\bar{g}\}, \{2\bar{\ell}, 2\bar{f}, 2\bar{h}\}\}. \end{aligned}$$

Since for each block $\{\bar{r}, \bar{s}, \bar{t}\} \in T$ we have $\bar{r} + \bar{s} + \bar{t} = \bar{0}$ it follows that for each block $\{\bar{x}, \bar{y}, \bar{z}\} \in T_0^*$ we have $\bar{x} + \bar{y} + \bar{z} = \bar{0}$. Therefore, the blocks of T_0^* are in $AG(d+1, 3)$. Using the fact that $T \cap \mathcal{H} = \{\{\bar{\imath}, \bar{\jmath}, \bar{k}\}\}$, it is straightforward to see that $T_0^* \cap \mathcal{H}^* = \{\{0\bar{\imath}, 0\bar{\jmath}, 0\bar{k}\}\}$.
Next define

$$\begin{aligned} T_1^* &= \{\{1\bar{\imath}, 1\bar{\jmath}, 1\bar{k}\}, \{1\bar{\imath}, 0\bar{e}, 2\bar{f}\}, \{1\bar{\imath}, 0\bar{g}, 2\bar{h}\}, \\ &\quad \{2\bar{\ell}, 1\bar{\jmath}, 0\bar{e}\}, \{2\bar{\ell}, 1\bar{k}, 0\bar{g}\}, \{2\bar{\ell}, 2\bar{f}, 2\bar{h}\}\}. \end{aligned}$$

Similarly, the blocks of T_1^* are in $AG(d+1, 3)$ and $T_1^* \cap \mathcal{H}^* = \{\{1\bar{\imath}, 1\bar{\jmath}, 1\bar{k}\}\}$. □

Lemma 41 *Let $\{\bar{\imath}, \bar{\jmath}, \bar{k}\}$ be a block in $AG(d, 3) \setminus \mathcal{H}$. Then there exist trades T_0^* and T_1^* of type one in $AG(d+1, 3)$ such that $T_0^* \cap \mathcal{H}^* = \{\{0\bar{\imath}, 0\bar{\jmath}, 0\bar{k}\}\}$ and $T_1^* \cap \mathcal{H}^* = \{\{1\bar{\imath}, 1\bar{\jmath}, 1\bar{k}\}\}$.*

Proof. First define

$$\begin{aligned} T_0^* &= \{\{0\bar{\imath}, 0\bar{\jmath}, 0\bar{k}\}, \{0\bar{\imath}, 1\bar{\jmath}, 2\bar{k}\}, \{0\bar{\imath}, 2\bar{\jmath}, 1\bar{k}\}, \\ &\quad \{2\bar{\imath}, 0\bar{\jmath}, 1\bar{k}\}, \{2\bar{\imath}, 1\bar{\jmath}, 0\bar{k}\}, \{2\bar{\imath}, 2\bar{\jmath}, 2\bar{k}\}\}. \end{aligned}$$

Obviously, any block of T_0^* is a block of $AG(d+1, 3)$. Moreover, the fact that $\{\bar{\imath}, \bar{\jmath}, \bar{k}\}$ is not a block of $\mathcal{H}$ leads to $T_0^* \cap \mathcal{H}^* = \{\{0\bar{\imath}, 0\bar{\jmath}, 0\bar{k}\}\}$.
Next define

$$\begin{aligned} T_1^* &= \{\{1\bar{\imath}, 1\bar{\jmath}, 1\bar{k}\}, \{1\bar{\imath}, 0\bar{\jmath}, 2\bar{k}\}, \{1\bar{\imath}, 2\bar{\jmath}, 0\bar{k}\}, \\ &\quad \{2\bar{\imath}, 0\bar{\jmath}, 1\bar{k}\}, \{2\bar{\imath}, 1\bar{\jmath}, 0\bar{k}\}, \{2\bar{\imath}, 2\bar{\jmath}, 2\bar{k}\}\}. \end{aligned}$$

Similarly, any block of T_1^* is a block of $AG(d+1,3)$ and $T_1^* \cap \mathcal{H}^* = \{\{1\bar{i}, 1\bar{j}, 1\bar{k}\}\}$. $\square$

Lemma 42 *Let $\bar{\gamma} = (\gamma_{d-1}, \gamma_{d-2}, \dots, \gamma_1, \gamma_0)$ be a point of $AG(d,3)$ which occurs in one of the blocks of $\mathcal{H}$. Then there exists a trade T^* of type one in $AG(d+1,3)$ such that $T^* \cap \mathcal{H}^* = \{\{0\bar{\gamma}, 1\bar{\gamma}, 2\bar{\gamma}\}\}$.*

Proof. Suppose that the point $\bar{\theta}$ of $AG(d,3)$ is not incident with any hyperplane of $\mathbf{H}$. So it does not occur in any block of $\mathcal{H}$. Now define

$$\begin{aligned} T^* \; = \; & \{\{0\bar{\gamma}, 1\bar{\gamma}, 2\bar{\gamma}\}, \{0\bar{\gamma}, 1\bar{\theta}, 2\bar{\delta}\}, \{0\bar{\gamma}, 1\bar{\delta}, 2\bar{\theta}\}, \\ & \{0\bar{\theta}, 1\bar{\gamma}, 2\bar{\delta}\}, \{0\bar{\theta}, 1\bar{\delta}, 2\bar{\gamma}\}, \{0\bar{\theta}, 1\bar{\theta}, 2\bar{\theta}\}\} \end{aligned}$$

where $\bar{\delta} = -\bar{\gamma} - \bar{\theta}$. First note that for each block $\{\bar{x}, \bar{y}, \bar{z}\}$ in T^* we have $\bar{x} + \bar{y} + \bar{z} = \bar{0}$. So T^* is a trade of type one in $AG(d+1,3)$. Secondly, it is easy to see that $T^* \cap \mathcal{H}^* = \{\{0\bar{\gamma}, 1\bar{\gamma}, 2\bar{\gamma}\}\}$. $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1 (1′) Suppose $\bar{\theta}$ is the point which is not incident with any hyperplane of $\mathbf{H}$. Then $2\bar{\theta}$ is not incident with any hyperplane of $\mathbf{H}^*$.

(2′) Let L be the Steiner latin square corresponding to the blocks of $AG(d,3)$. Then, by Lemma 39, P^* is the product of $S = \{(0,0;0), (1,1;1)\}$ and P; that is:

$$P^* = S \otimes P = \begin{array}{|c|c|c|} \hline L^0 & P^2 & P^1 \\ \hline P^2 & L^1 & P^0 \\ \hline P^1 & P^0 & P^2 \\ \hline \end{array}$$

Since the partial latin square S is a super-strong critical set of order three, by Lemma 32, P^* is a completable set of order 3^{d+1}. So the blocks of $\mathcal{H}^*$ form a defining set in $AG(d+1,3)$.

(3′) Let $\{\bar{i}, \bar{j}, \bar{k}\}$ be a block of $\mathcal{H}^*$ and let $\bar{i} = (i_d, i_{d-1}, i_{d-2}, \dots, i_1, i_0)$, $\bar{j} = (j_d, j_{d-1}, j_{d-2}, \dots, j_1, j_0)$, and $\bar{k} = (k_d, k_{d-1}, k_{d-2}, \dots, k_1, k_0)$. Suppose that i, j and k are integer representations of $\bar{i}$, $\bar{j}$ and $\bar{k}$, respectively. Without loss of generality, we can assume $i < j < k$. This leads to

$$(i_d, j_d, k_d) \in \{(0,0,0), (1,1,1), (2,2,2), (0,1,2)\}.$$

Define

$$\bar{i'} = (i_{d-1}, i_{d-2}, \dots, i_0),$$
$$\bar{j'} = (j_{d-1}, j_{d-2}, \dots, j_0),$$

and

$$\bar{k'} = (k_{d-1}, k_{d-2}, \dots, k_0).$$

Then $\bar{i'} + \bar{j'} + \bar{k'} = \bar{0}$ since $\bar{i} + \bar{j} + \bar{k} = \bar{0}$. So $\{\bar{i'}, \bar{j'}, \bar{k'}\}$ is a block of $AG(d, 3)$.

Case $(i_d, j_d, k_d) = (0, 0, 0)$**:** If $\{\bar{i'}, \bar{j'}, \bar{k'}\} \in \mathcal{H}$ we take T_0^* as defined in Lemma 40. If the block $\{\bar{i'}, \bar{j'}, \bar{k'}\}$ is not in $\mathcal{H}$ we take T_0^* as defined in Lemma 41. In both cases $T_0^* \cap \mathcal{H}^* = \{\{\bar{i}, \bar{j}, \bar{k}\}\}$.

Case $(i_d, j_d, k_d) = (1, 1, 1)$**:** This case is quite similar to the case $(i_d, j_d, k_d) = (0, 0, 0)$ and the proof is left to the reader.

Case $(i_d, j_d, k_d) = (2, 2, 2)$**:** By Lemma 39 the block $\{\bar{i'}, \bar{j'}, \bar{k'}\}$ is in $\mathcal{H}$. So there is a trade T of type one in $AG(d, 3)$ such that $T \cap \mathcal{H} = \{\{\bar{i'}, \bar{j'}, \bar{k'}\}\}$. Now define

$$T^* = \{\{2\bar{x}, 2\bar{y}, 2\bar{z}\} |\ \{x, y, z\} \in T\}.$$

Then T^* is a trade of type one in $AG(d+1, 3)$ and $T^* \cap \mathcal{H}^* = \{\{\bar{i}, \bar{j}, \bar{k}\}\}$.

Case $(i_d, j_d, k_d) = (0, 1, 2)$**:** First let $i \not\equiv j \pmod{3^d}$. Then by Lemma 39 the block $\{\bar{i'}, \bar{j'}, \bar{k'}\}$ is in $\mathcal{H}$. So there is a trade T of type one in $AG(3, d)$ such that $T \cap \mathcal{H} = \{\{\bar{i'}, \bar{j'}, \bar{k'}\}\}$. Now define

$$T^* = \{\{0\bar{x}, 1\bar{y}, 2\bar{z}\} |\ \{x, y, z\} \in T\}.$$

Then T^* is a trade of type one in $AG(d+1, 3)$ and $T^* \cap \mathcal{H}^* = \{\{\bar{i}, \bar{j}, \bar{k}\}\}$. Secondly, if $i \equiv j \pmod{3^d}$ then $\bar{i'} = \bar{j'} = \bar{k'}$ and $\bar{i'}$ occurs in a block of $\mathcal{H}$. So if T^* is the trade defined in Lemma 42 then $T^* \cap \mathcal{H}^* = \{\{\bar{i}, \bar{j}, \bar{k}\}\}$. This completes the proof. □

Remark 43 Consider $\mathcal{H}$ as in Theorem 1. Then by Lemma 34 and Condition (2) in Theorem 1 we see that $\mathcal{H}$ is a defining set in $AG(d, 3)$. Now Condition (3) in Theorem 1 guarantees that each block of $\mathcal{H}$ is necessary for unique completion. Therefore $\mathcal{H}$ is a minimal defining set in $AG(d, 3)$.

5. Sets of good hyperplanes in $AG(d, 3)$

In this section we show first that Theorem 36 is a simple corollary of Theorem 1. Then we introduce two sets of hyperplanes which provide minimal defining sets of different sizes in $AG(d, 3)$. These two sets are also different from the set introduced in Theorem 36. Note that for simplicity we use integer representation for the points of $AG(d, 3)$ in this section.

Proof of Theorem 36 If $d = 2$ then $\mathbf{H}$ consists of the following four hyperplanes of $AG(2,3)$:

$$\begin{cases} 1x_1 & + & 0x_0 & = & 0 \\ 1x_1 & + & 0x_0 & = & 1 \\ 0x_1 & + & 1x_0 & = & 0 \\ 0x_1 & + & 1x_0 & = & 1. \end{cases}$$

One can see that the set of blocks which lie within these hyperplanes is

$$\mathcal{S}_4 = \{\{0,1,2\},\{3,4,5\};\{0,3,6\},\{1,4,7\}\}.$$

Moreover, the partial Steiner latin square corresponding to $\mathcal{S}_4$ is the partial Steiner latin square $L_{9,4}$ given in Example 20. Now it is straightforward to check that:

(1) the point 8 ($\bar{8} = (2,2)$) is not incident with any hyperplane of $\mathbf{H}$;

(2) $L_{9,4}$ is a completable set of order 9;

(3) for each block $\{i,j,k\}$ of $\mathcal{H}$ there is a trade T of type one in $AG(2,3)$ such that $T \cap \mathcal{H} = \{\{i,j,k\}\}$.

So by Remark 43 we see that $\mathcal{H}$ is a minimal defining set in $AG(2,3)$. Now for $d \geq 3$ Theorem 36 follows by Theorem 1 and an induction on d. □

Theorem 44 *Let* $\mathbf{H}$ *consist of the following* $2d+1$ *hyperplanes of* $AG(d,3)$, *where* $d \geq 2$.

$$\begin{cases} 1x_{d-1} & + & 0x_{d-2} & + & 0x_{d-3} & + & \dots & + & 0x_1 & + & 0x_0 & = & 0 \\ 1x_{d-1} & + & 0x_{d-2} & + & 0x_{d-3} & + & \dots & + & 0x_1 & + & 0x_0 & = & 1 \\ 0x_{d-1} & + & 1x_{d-2} & + & 0x_{d-3} & + & \dots & + & 0x_1 & + & 0x_0 & = & 0 \\ 0x_{d-1} & + & 1x_{d-2} & + & 0x_{d-3} & + & \dots & + & 0x_1 & + & 0x_0 & = & 1 \\ 0x_{d-1} & + & 0x_{d-2} & + & 1x_{d-3} & + & \dots & + & 0x_1 & + & 0x_0 & = & 0 \\ 0x_{d-1} & + & 0x_{d-2} & + & 1x_{d-3} & + & \dots & + & 0x_1 & + & 0x_0 & = & 1 \\ & & & & & & \dots & & & & & & \\ & & & & & & \dots & & & & & & \\ & & & & & & \dots & & & & & & \\ 0x_{d-1} & + & 0x_{d-2} & + & 0x_{d-3} & + & \dots & + & 1x_1 & + & 0x_0 & = & 0 \\ 0x_{d-1} & + & 0x_{d-2} & + & 0x_{d-3} & + & \dots & + & 1x_1 & + & 0x_0 & = & 1 \\ 0x_{d-1} & + & 0x_{d-2} & + & 0x_{d-3} & + & \dots & + & 0x_1 & + & 1x_0 & = & 0 \\ 0x_{d-1} & + & 0x_{d-2} & + & 0x_{d-3} & + & \dots & + & 1x_1 & + & 2x_0 & = & 0 \\ 0x_{d-1} & + & 0x_{d-2} & + & 0x_{d-3} & + & \dots & + & 1x_1 & + & 1x_0 & = & 1 \end{cases}$$

Then $\mathcal{H}$, *the collection of blocks contained within the hyperplanes of* $\mathbf{H}$, *is a minimal defining set in* $AG(d,3)$.

Proof. If $d = 2$ then $\mathbf{H}$ consists of the following five hyperplanes of $AG(2,3)$:

$$\begin{cases} 1x_1 & + & 0x_0 & = & 0 \\ 1x_1 & + & 0x_0 & = & 1 \\ 0x_1 & + & 1x_0 & = & 0 \\ 1x_1 & + & 2x_0 & = & 0 \\ 1x_1 & + & 1x_0 & = & 1. \end{cases}$$

It is easy to see that the blocks which lie in these hyperplanes are

$$\mathcal{S}_5 = \{\{0,1,2\}, \{0,3,6\}, \{0,4,8\}, \{1,3,8\}, \{3,4,5\}\}.$$

Moreover, the partial Steiner latin square corresponding to $\mathcal{H}$ is the partial Steiner latin square $L_{9,5}$ given in Example 20. Now it is straightforward to check that:

(1) the point 7 ($\bar{7} = (2,1)$) is not incident with any hyperplane of $\mathbf{H}$;

(2) $L_{9,5}$ is a completable set of order 9;

(3) for each block $\{i,j,k\}$ of $\mathcal{H}$ there is a trade T of type one in $AG(2,3)$ such that $T \cap \mathcal{H} = \{\{i,j,k\}\}$.

So by Remark 43 we see that $\mathcal{H}$ is a minimal defining set in $AG(2,3)$. Now for $d \geq 3$ the result follows by Theorem 1 and an induction on d. $\square$

Theorem 45 *Let* $\mathbf{H}$ *consist of the following* $2d+1$ *hyperplanes of* $AG(d, 3)$*, where* $d \geq 3$.

$$\begin{cases} 1x_{d-1} & + & 0x_{d-2} & + & \dots & + & 0x_2 & + & 0x_1 & + & 0x_0 & = & 0 \\ 1x_{d-1} & + & 0x_{d-2} & + & \dots & + & 0x_2 & + & 0x_1 & + & 0x_0 & = & 1 \\ 0x_{d-1} & + & 1x_{d-2} & + & \dots & + & 0x_2 & + & 0x_1 & + & 0x_0 & = & 0 \\ 0x_{d-1} & + & 1x_{d-2} & + & \dots & + & 0x_2 & + & 0x_1 & + & 0x_0 & = & 1 \\ 0x_{d-1} & + & 0x_{d-2} & + & \dots & + & 0x_2 & + & 0x_1 & + & 0x_0 & = & 0 \\ 0x_{d-1} & + & 0x_{d-2} & + & \dots & + & 0x_2 & + & 0x_1 & + & 0x_0 & = & 1 \\ & & & & \dots & & & & & & & & \\ & & & & \dots & & & & & & & & \\ & & & & \dots & & & & & & & & \\ 0x_{d-1} & + & 0x_{d-2} & + & \dots & + & 0x_2 & + & 1x_1 & + & 0x_0 & = & 0 \\ 0x_{d-1} & + & 0x_{d-2} & + & \dots & + & 0x_2 & + & 1x_1 & + & 0x_0 & = & 1 \\ 0x_{d-1} & + & 0x_{d-2} & + & \dots & + & 0x_2 & + & 0x_1 & + & 1x_0 & = & 0 \\ 0x_{d-1} & + & 0x_{d-2} & + & \dots & + & 1x_2 & + & 1x_1 & + & 2x_0 & = & 0 \\ 0x_{d-1} & + & 0x_{d-2} & + & \dots & + & 1x_2 & + & 1x_1 & + & 1x_0 & = & 1 \end{cases}$$

Then $\mathcal{H}$, the collection of blocks contained within the hyperplanes of **H**, *is a minimal defining set in $AG(d,3)$.*

Proof. If $d = 3$ then **H** consists of the following seven hyperplanes of $AG(3,3)$:

$$\begin{cases} 1x_2 + 0x_1 + 0x_0 = 0 \\ 1x_2 + 0x_1 + 0x_0 = 1 \\ 0x_2 + 1x_1 + 0x_0 = 0 \\ 0x_2 + 1x_1 + 0x_0 = 1 \\ 0x_2 + 0x_1 + 1x_0 = 0 \\ 1x_2 + 1x_1 + 2x_0 = 0 \\ 1x_2 + 1x_1 + 1x_0 = 1 \end{cases}$$

It is straightforward to see that the blocks which lie in these hyperplanes are as follows.

{0, 1, 2}	{0, 3, 6}	{0, 4, 8}	{0, 5, 7}	{0, 9, 18}
{0, 10, 20}	{0, 11, 19}	{0, 12, 24}	{0, 14, 25}	{0, 15, 21}
{1, 3, 8}	{1, 4, 7}	{1, 5, 6}	{1, 9, 20}	{1, 10, 19}
{1, 11, 18}	{1, 14, 24}	{1, 16, 22}	{2, 3, 7}	{2, 4, 6}
{2, 5, 8}	{2, 9, 19}	{2, 10, 18}	{2, 11, 20}	{3, 4, 5}
{3, 9, 24}	{3, 12, 21}	{3, 13, 23}	{3, 14, 22}	{3, 15, 18}
{3, 16, 20}	{4, 10, 25}	{4, 12, 23}	{4, 13, 22}	{4, 14, 21}
{4, 15, 20}	{5, 12, 22}	{5, 13, 21}	{5, 14, 23}	{6, 7, 8}
{6, 9, 21}	{6, 12, 18}	{6, 15, 24}	{8, 9, 22}	{8, 10, 21}
{8, 14, 20}	{8, 15, 25}	{8, 16, 24}	{9, 10, 11}	{9, 12, 15}
{9, 13, 17}	{9, 14, 16}	{10, 12, 17}	{10, 13, 16}	{10, 14, 15}
{11, 12, 16}	{11, 13, 15}	{11, 14, 17}	{12, 13, 14}	{15, 16, 17}
{18, 19, 20}	{18, 21, 24}	{20, 21, 25}	{20, 22, 24}	{21, 22, 23}

Let P be the partial Steiner latin square corresponding to $\mathcal{H}$. Then:

(1) the point 26 ($\bar{26} = (2,2,2)$) is not incident with any hyperplane of **H**;

(2) a backtrack search shows that P is a completable set of order 27;

(3) a modification of the backtrack search given in [15] shows that for each block $\{i,j,k\}$ of $\mathcal{H}$ there is a trade T of type one in $AG(3,3)$ such that $T \cap \mathcal{H} = \{\{i,j,k\}\}$.

So by Remark 43 we see that $\mathcal{H}$ is a minimal defining set in $AG(3,3)$. Now for $d \geq 4$ the result follows by Theorem 1 and an induction on d. □

6. Related critical sets

In [7], Gower shows that the number of blocks in the minimal defining set in Theorem 36 is $\frac{1}{6}(3^d(3^d-1)-7^d+1)$. In this section we calculate the number of blocks in minimal defining sets obtained in Theorems 44 and 45. Moreover, we prove that the partial Steiner latin squares corresponding to the minimal defining sets obtained in Theorems 36, 44 and 45 are critical sets.

Theorem 46 *Let* $\mathbf{H}$*,* $\mathcal{H}$ *and* P *be as in Theorem 1. Then* P *is a critical set of order* 3^d*.*

Proof. By Condition (2) we only need to prove that each entry of P is necessary for completion. First consider the off-diagonal entry $(i,j;k)$ of P. By Condition (3) there is a trade T of type one in $AG(d,3)$ such that $T \cap \mathcal{H} = \{\{i,j,k\}\}$. Now by an extension of Lemma 31 we see that the entry $(i,j;k)$ is necessary. Secondly, consider the entry $(i,i;i)$ of P. By Condition (1) there is a point, j say, of $AG(d,3)$ which is not incident with any hyperplane of $\mathbf{H}$. So the cell (j,j) of P is empty. Let $\bar{k} = -\bar{i} - \bar{j}$, where $\bar{i}$ and $\bar{j}$ are vector representations of i and j respectively. Define the latin trade

$$I = \{(i,i;i),(i,k;j),(j,k;i),(j,j;j),(k,j;i),(k,i;j)\}.$$

Since $\{i,j,k\}$ is not a block of $\mathcal{H}$ and the cell (j,j) of P is empty it follows that $I \cap P = \{(i,i;i)\}$. So the entry $(i,i;i)$ of P is necessary. □

Theorem 47 *For all* $d \geq 2$ *the partial Steiner latin square* P *corresponding to the minimal defining set in Theorem 36 is a critical set of size* $9^d - 7^d$*.*

Proof. First note that by Theorem 46 P is a critical set of order 3^d. Since each block of the minimal defining set fills exactly six off-diagonal cells of the partial Steiner latin square and there is exactly one point of $AG(d,3)$ which is not incident with any hyperplanes in Theorem 36 it follows that the number of entries is

$$\begin{aligned} & 6(\frac{1}{6}(3^d(3^d-1)-7^d+1))+(3^d-1) \\ = \; & (3^d-1)(3^d+1)-7^d+1 = 9^d-7^d. \end{aligned}$$

□

Theorem 48 *For all* $d \geq 2$ *the partial Steiner latin square* P *in Theorem 44 is a critical set of size* $9^d - 43.7^{d-2}$*.*

Proof. First note that by Theorem 46 P is a critical set of order 3^d. If $d = 2$ the number of entries in the partial latin square P (see the partial latin square $L_{9,5}$ given in Example 20) is $38 = 3^4 - 43.7^0$. Now by an induction on d the proof for $d \geq 3$ follows. □

Since any block $\{i, j, k\}$ of $\mathcal{H}$ corresponds to exactly six off-diagonal entries of P and there is exactly one point of $AG(d, 3)$ which is not in any block of $\mathcal{H}$ we have the following result on the number of blocks of $\mathcal{H}$.

Theorem 49 *Let $\mathcal{H}$ be as in Theorem 44. Then the number of blocks in $\mathcal{H}$ is*

$$\frac{1}{6}(3^d(3^d - 1) - 7^{d-2} + 1) - 7^{d-1}.$$

Theorem 50 *For all $d \geq 3$ the partial Steiner latin square P in Theorem 45 is a critical set of size $9^d - 313.7^{d-3}$.*

Proof. First note that by Theorem 46 P is a critical set of order 3^d. If $d = 3$ then P is the partial Steiner latin square corresponding to the blocks given in Theorem 45. The number of entries in P is $416 = 3^6 - 313$. Now by an induction on d the proof for $d \geq 4$ follows. □

As for Theorem 49 we can prove the following result.

Theorem 51 *Let $\mathcal{H}$ be as in Theorem 45. Then the number of blocks in $\mathcal{H}$ is*

$$\frac{1}{6}(3^d(3^d - 1) - 7^{d-3} + 1) - 52.7^{d-3}.$$

7. Appendix

The following 12 blocks form a definig set for the noncyclic $STS(13)$ on $\{0, 1, 2, \ldots, 12\}$.

$\{1, 7, 5\}$	$\{1, 6, 12\}$	$\{2, 4, 10\}$	$\{2, 6, 8\}$	$\{3, 4, 9\}$
$\{3, 5, 11\}$	$\{4, 7, 8\}$	$\{5, 8, 9\}$	$\{6, 7, 9\}$	$\{7, 10, 11\}$
$\{8, 11, 12\}$	$\{9, 10, 12\}$			

Let P be the partial Steiner latin square corresponding to this defining set. Then P has precisely 9 completions. Here we present a completion which is not a Steiner latin square.

0	9	7	8	1	2	3	12	10	11	5	6	4
4	1	9	10	11	7	12	5	3	0	8	2	6
5	11	2	7	10	12	8	0	6	1	4	9	3
6	8	12	3	9	11	10	2	0	4	1	5	7
12	0	10	9	4	6	11	8	7	3	2	1	5
10	7	0	11	12	5	4	1	9	8	6	3	2
11	12	8	0	5	10	6	9	2	7	3	4	1
2	5	3	12	8	1	9	7	4	6	11	10	0
3	10	6	1	7	9	2	4	8	5	0	12	11
1	2	11	4	3	8	7	6	5	9	12	0	10
8	3	4	6	2	0	5	11	1	12	10	7	9
9	4	1	5	6	3	0	10	12	2	7	11	8
7	6	5	2	0	4	1	3	11	10	9	8	12

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