Critical Sets for Families of Latin Squares

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Abstract. In this paper I give details of new constructions for critical sets in latin squares. These latin squares, of order n, are such that they can be partitioned into four subsquares each of which is based on the addition table of the integers modulo $n/2$, an isotopism of this or a conjugate.

1 Introduction

In this paper I focus on latin squares, of order n, which can be partitioned into four latin subsquares of order $\frac{n}{2}$. The subsquares will be labled L_1, L_2, L_3 and L_4 as shown on the left in Table 1 and will be refered to as quadrants 1, 2, 3 and 4 respectively.

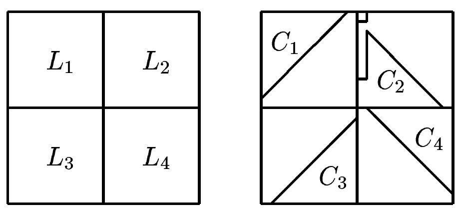

Table 1

Let C_1, C_2, C_3 and C_4 represent critical sets in L_1, L_2, L_3 and L_4 respectively. It will be shown that if one chooses the critical sets C_1, C_2, C_3 and C_4 with care, then one can essentially combine these critical sets to form a critical set in L. The basic shape of the critical sets C_1, C_2, C_3 and C_4 is illustrated on the right in Table 1. One of the interesting feature of the results presented in this paper, is that there is some degree of freedom when choosing L_1, L_2, L_3 and L_4. It will be shown that this basic pattern can be used to construct critical sets for an infinte number of classes of latin squares.

To achieve this I will need examples of critical sets which are complete in all entries above or below the main or back diagonal. So in Section 3, Theorem 3.5, I construct a new class of critical sets, with the desired property. This class is based on examples taken from the paper [**12**] by Stinson and van Rees. I will show that these examples can be generalised. Donovan and Cooper, in [**5**], have also constructed critical sets which comprise of all entries above or below the diagonal. Details of their critical sets are given in Theorem 2.4. Their result will be used throughout this paper.

If one takes a partial latin square with the basic format described above, then the position of elements in C_2 and C_3 play a major role in determining the completion of the partial latin squre. The significance of these entries will be discussed in Section 4.

Recently Donovan and Hoffman [**7**] constructed critical sets for the multiplication table of the group of order $4h$, $h \geq 2$, with generating relations $a^{2h} = 1$, $b^2 = a^h$ and $ba = a^{-1}b$. These critical sets conform to the basic shape described above. However, Donovan and Hoffman's result only applies to a small class of latin squares. In Section 5, I will extend Donovan and Hoffman's ideas and construct critical sets for an infinte number of classes of latin squares.

Finally, in Section 6, I will illustrate the flexibility of the given techniques, by constructing critical sets for a different class of latin squares.

The main results of this paper are presented in Theorems 3.5, 5.8, 6.9 and 6.10. It should be noted that in order to meet the requirements of the referee all examples have been deleted and only brief proofs are given. For full details contact the author for a copy of the corresponding technical report.

I begin the paper with the necessary background information.

2 Background

A *latin square* L of order n is an $n \times n$ array with entries chosen from a set, N, of size n, in such a way that each element of N occurs precisely once in each row and column of the array. (See [**4**].) Throughout this paper, a latin square L will be represented as a set of ordered triples $\{(i, j; k) \mid \text{cell } (i, j) \text{ contains } k\}$. By way of example, the array, $\mathcal{H}_1$, given in Table 2 is a latin square of order 7. Here $N = \{0, \ldots, 6\}$.

0	1	2	3	4		0	1	2	3	*
1	2	3	4	0		1	2	3	4	*
2	3	4	0	1		2	3	4	*	*
3	4	0	1	2		3	4	*	*	*
4	0	1	2	3		*	*	*	*	*
		$\mathcal{H}_1$						$C_{\mathcal{H}1}$		

Table 2

This latin square is the addition table for the integers modulo 7 and can be written as the set $\{(i, j; i + j(\text{mod } 7)) \mid i, j \in \{0, \ldots, 6\}\}$. In general, the addition table for the integers modulo g can be represented by the set $\{(i, j; i + j(\text{mod } g)) \mid i, j \in \{0, \ldots, g - 1\}\}$.

A latin square L' is said to be *isotopic* to L if L' can be obtained from L by permuting the rows and/or the columns and/or the entries of L. That is, L' is said to be isotopic to L if there exists permutations α, β, γ such that $L' = \{(i\alpha, j\beta; k\gamma) \mid (i, j; k) \in L\}$. Then $(\alpha,\ \beta,\ \gamma)$ is said to be an *isotopism* from L to L'.

There are six conjugate latin squares associated with each latin square L. The reader will find the definition of these in [**4**]. In this paper I require one of these conjugates and so define it as follows:

$$L^{-1} = \{(i, k; j) \mid (i, j; k) \in L\},$$

is a conjugate of the latin square L. The latin square $\mathcal{H}_2 = \{(i, j; j - i(\text{mod } h)) \mid i, j \in \{0, \ldots, h - 1\}\}$ is a conjuage of the addition table for the integers modulo h. The latin square $\mathcal{H}_4 = \{(i, j; j - i + f(\text{mod } h)) \mid i, j \in \{0, \ldots, h - 1\}\}$ where $f \in \{1, \ldots, h - 1\}$ is an isotopism of this conjugate.

If a latin square L contains an $s \times s$ subarray S and if S is a latin square of order s, then S is said to be a *latin subsquare* of L.

A *partial latin square* P, of order n, is an $n \times n$ array with some cells containing entries chosen from the set N, in such a way that each element of N occurs at most once in each row and at most once in each column of the array. A partial latin square $\mathcal{C} = \{(i, j; k) \mid \text{ cell } (i, j) \text{ contains } k\}$, of order n, is said to have a *completion* to a latin square L, if L is a latin square of order n which has element k in cell (i, j), for each $(i, j; k) \in \mathcal{C}$. The partial latin square $\mathcal{C}$ is said to have a *unique completion* to L if L is unique. The partial latin square $C_{\mathcal{H}1}$ given in Table 2 has a unique completion to $\mathcal{H}_1$, the addition table of the integers modulo 7. (Note $*$ indicates an empty cell.)

A *critical set* in a latin square L, of order n, is a set $\mathcal{C} \subseteq L$ such that,

1. $\mathcal{C}$ has a unique completion to L, and
2. no proper subset of $\mathcal{C}$ satisfies 1.

Donovan and Cooper [**6**] showed that if, in the example $\mathcal{C}_{\mathcal{H}1}$ given in Table 2, one removes any element from the partial latin square then what is left is contained in at least two latin squares of order 7. Thus the above partial latin square is an example of a critical set.

Relevant results relating to critical sets in latin squares are as follows.

LEMMA 2.1 *If L is a latin square of order n, S a subsquare in L and $\mathcal{C}$ a critical set in L, then $\mathcal{C} \cap S$ must have a unique completion in S.*

In 1995 Donovan, Cooper, Seberry and Nott [**5**] proved the following two results.

LEMMA 2.2 *Let L be a latin square with critical set $\mathcal{C}$. Let (α, β, γ) be an isotopism from the critical set $\mathcal{C}$ onto $\mathcal{C}'$. Then $\mathcal{C}'$ is a critical set in the latin square L' isotopic to L.*

LEMMA 2.3 *Let L be a latin square with critical set $\mathcal{C}$ and let $\mathcal{C}'$ be a conjugate of $\mathcal{C}$. Then $\mathcal{C}'$ is a critical set in the corresponding conjugate L' of L.*

Recently Donovan and Cooper [**6**] extended work of Curran and van Rees [**3**] and Cooper, Donovan and Seberry [**1**] and constructed infinite families of critical sets for the addition tables of the integers modulo n. Their result is as follows.

THEOREM 2.4 *Let L be the addition table for the integers modulo n, then the set*

$$\begin{aligned}\mathcal{C} \;=\; & \{(i,j;i+j) \mid i = 0,\ldots,r \text{ and } j = 0,\ldots,r-i\} \cup \\ & \{(i,j;i+j) \mid i = r+2,\ldots,n-1 \text{ and } j = r+1-i,\ldots,n-1\}\end{aligned}$$

where $\frac{n-3}{2} \leq r \leq n-2$, is a critical set in L.

COROLLARY 2.4.1 *Let L be the addition table for the integers modulo n, and*

$$\begin{aligned}\mathcal{P} \;=\; & \{(i,j;i+j) \mid i = 0,\ldots,r \text{ and } j = 1,\ldots,r-i\} \cup \\ & \{(i,j;i+j) \mid i = r+2,\ldots,n-1 \text{ and } j = r+1-i,\ldots,n-1\}\end{aligned}$$

where $\frac{n-3}{2} \leq r \leq n-2$. If one removes any element from $\mathcal{P}$ then the remaining partial latin square completes to L and to a latin square which agrees with L in column 1.

See also the work by Smetaniuk [**11**], Sittampalam (with Keedwell) [**10**], Burgess, see [**9**], Donovan and Hoffman [**7**], Howse [**8**] and Cooper, Donovan and Gower, [**1**].

3 A new class of critical sets

In this section assume all arithmetic is done modulo m unless otherwise stated.

Here I will generalise an example of a critical set given by Stinson and van Rees in Lemma 3.7 of the paper [**12**] and thus construct a new class of critical sets.

Let $\mathcal{L}$ be a latin square given by the set

$$\begin{aligned}\mathcal{L} = \{(u(i), u(j); u(i-j))\} \quad &\cup \quad \{(u(i), v(j); v(j-i))\} \quad \cup \\ \{(v(i), u(j); v(i-j))\} \quad &\cup \quad \{(v(i), v(j); u(j-i))\}\end{aligned}$$

where $i, j = 0, \ldots m-1$, $m \geq 2$. Note that the rows and columns of this latin square are indexed by the set $\{u(0), \ldots, u(m-1), v(0), \ldots, v(m-1)\}$, in the given order and all arithmetic is done modulo m.

THEOREM 3.5 *The partial latin square*

$$\begin{aligned}\mathcal{C}_{\mathcal{L}} \;=\; & \{(u(r), v(s); v(s-r)) \mid r = 0, \ldots, m-1, s = 0, \ldots, m-1\} \quad \cup \\ & \{(u(i), u(j); u(i-j)), (v(i), u(j); v(i-j)), (v(i), v(j); u(j-i))\},\end{aligned}$$

where $i = 0, \ldots, m-2, j = i+1, \ldots, m-1$ *and* $m \geq 2$*, is a critical set in the latin square* $\mathcal{L}$ *defined above.*

Proof.

Note that $\mathcal{C}_{\mathcal{L}}$ is complete in rows $u(0)$ to $u(m-1)$ of columns $v(0)$ to $v(m-1)$ and so the unique completion of $\mathcal{C}_{\mathcal{L}}$ to $\mathcal{L}$ follows immediately from Lemmas 2.2 and 2.3 and Theorem 2.4.

For $i = 0, \ldots, m-1$ and $j = i, \ldots, m-1$ the existence of the 2×2 subsquares on the elements $(u(i), u(i); u(0)), (u(i), v(j); v(j-i))$ implies that each of the elements $(u(i), v(j); v(j-i))$ is necessary for the unique completion of $\mathcal{C}_{\mathcal{L}}$. Similarly for each $i = 1, \ldots, m-1, j = 0, \ldots, i-1$ the 2×2 subsquare on the elements $(u(i), u(0); u(t)), (u(i), v(j); v(j-i))$ implies the entry $(u(i), v(j); v(j-i))$ is also necessary. The necessity of the remaining elements can be obtained from Lemmas 2.2 and 2.3 and Theorem 2.4. □

Results establishing the existence of critical sets of size is greater than $(2m)^2/2$ are relatively rare. Therefore the above result is of interest as this critical set is of size

$$m^2 + 3\frac{m^2 - m}{2} = \frac{5m^2 - 3m}{2}.$$

Infact for latin squares of orders 6, 8, and 10 this construction produces the largest known critical set.

This critical sets will be of use in Section 6.

4 A partial completion

In this section, two of the four latin subsquares L_1, L_2, L_3 and L_4 will be specifically defined, as will their critical sets. The subsquares L_2 and L_3 will be taken to be the addition table of the integers modulo h or a conjugate of this, while L_1 and L_4 will be thought of as general latin squares of order h. It will be shown that the subsquares L_2 and L_3 and their critical sets play a major role in the completion of the partial latin square.

In this section it should be assumed that all arithmetic is done modulo h unless otherwise stated.

Let $\mathcal{H}_1 = \{(s(i), s(j); s(k)) \mid 0 \le i, j \le h-1\}$ and $\mathcal{H}_4 = \{(s(i), s(j); s(k)) \mid 0 \le i, j \le h-1\}$ be two latin squares of order h, where $h \ge 3$.

Let $\mathcal{H}$ be the latin square, of order $2h$, represented by the set

$$\begin{aligned}\mathcal{H} &= \{(s(i), s(j); s(k)) \mid (s(i), s(j); s(k)) \in \mathcal{H}_1\} \cup \{(s(i), t(j); t(j-i))\} \\ &\cup \\ &\quad \{(t(i), s(j); t(i+j))\} \cup \{(t(i), t(j); s(k)) \mid (s(i), s(j); s(k)) \in \mathcal{H}_4\},\end{aligned}$$

where $i, j = 0, \ldots, h-1$.

Finally let $\mathcal{C}_{\mathcal{H}}$ be the partial latin square represented by the set

$$\begin{aligned}&\{(s(i), s(0); s(k)) \mid (s(i), s(0); s(k)) \in \mathcal{H}_1 \ \wedge \ i = 0, \ldots, h-2\} \quad \cup \\ &\{(s(0), t(0); t(0))\} \quad \cup \quad \{(s(h-1), t(0); t(1))\} \quad \cup \\ &\{(s(i), t(j); t(j-i)) \mid i = 2, \ldots, h-1, j = 1, \ldots, i-1\} \quad \cup \\ &\{(t(i), s(j); t(i+j)) \mid i = 1, \ldots, h-1, j = h-i, \ldots, h-1\} \quad \cup \\ &\{(t(i), t(j); s(k)) \mid (s(i), s(j); s(k)) \in \mathcal{H}_4, \qquad \wedge \\ &i = 0, \ldots, h-2, j = i+1, \ldots, h-1\ \}.\end{aligned}$$

LEMMA 4.6 *Any completion of the partial latin square $\mathcal{C}_{\mathcal{H}}$ agrees with $\mathcal{H}$ in rows $s(0)$ to $s(h-1)$ of columns $t(1)$ to $t(h-1)$.*

Proof. This result is established by first noting that the element $t(0)$ must occur in cell $(t(0), s(0))$. Then after noting that column $s(0)$ has a unique completion, it is easy to see that columns $t(1)$ to $t(h-1)$ of quadrant 2 have a unique completion and so the result follows. □

This partial completion will be used, in the next section.

5 Infinitely many critical sets

In this section the latin squares $\mathcal{H}_1$ and $\mathcal{H}_4$, of Section 4, are replaced by the addition table for the integers modulo h and an isotopism of one of its conjugates. Critical sets will then be constructed for these latin squares. Once again all arithmetic will be done modulo h unless otherwise stated.

It will be shown that the following example of a critical set can be generalised to construct critical sets for a class of latin squares.

0	1	2	3	*	5	*	*	*	*
1	2	3	*	*	9	*	*	*	*
2	3	*	*	*	8	9	*	*	*
3	*	*	*	*	7	8	9	*	*
*	*	*	*	*	6	7	8	9	*
*	*	*	*	*	*	1	2	3	4
*	*	*	*	5	*	*	1	2	3
*	*	*	5	6	*	*	*	1	2
*	*	5	6	7	*	*	*	*	1
*	5	6	7	8	*	*	*	*	*

THEOREM 5.7 *Let $\mathcal{H}$ be a latin square of order $2h$, $h \geq 3$, defined as follows:*

$$\begin{aligned}\mathcal{H} = \{(s(i), s(j); s(i+j))\} \;&\cup\; \{(s(i), t(j); t(j-i))\} \;\cup \\ \{(t(i), s(j); t(i+j))\} \;&\cup\; \{(t(i), t(j); s(j-i))\},\end{aligned}$$

where $i, j = 0, \ldots, h-1$. Let $\mathcal{C}_{\mathcal{H}}$ be the partial latin square represented by the set

$$\begin{aligned}\mathcal{C}_{\mathcal{H}} \;=\; & \{(s(i), s(j); s(i+j)) \mid i = 0, \ldots, h-2, j = 0, \ldots, h-2-i\} \quad \cup \\ & \{(s(0), t(0); t(0))\} \quad \cup \\ & \{(s(i), t(j); t(j-i)) \mid i = 1, \ldots, h-1, j = 0, \ldots, i-1\} \quad \cup \\ & \{(t(i), s(j); t(i+j)) \mid i = 1, \ldots, h-1, j = h-i, \ldots, h-1\} \quad \cup \\ & \{(t(i), t(j); s(j-i)) \mid i = 0, \ldots, h-2, j = i+1, \ldots, h-1\}.\end{aligned}$$

Then $\mathcal{C}_{\mathcal{H}}$ is a critical set in $\mathcal{H}$.

Proof. The proof of Lemma 4.6 can be used to show that the completion of $\mathcal{C}_{\mathcal{H}}$ will agree with $\mathcal{H}$ in all entries in rows $s(0)$ to $s(h-1)$ of columns $t(0)$ to $t(h-1)$. At this point one can apply Lemma 2.3 and Theorem 2.4 to prove that $\mathcal{C}_{\mathcal{H}}$ has a unique completion to $\mathcal{H}$.

Lemmas 2.1 and 2.3 and Theorem 2.4 can be used to prove that each of the elements in the first, third and fourth quadrants is necessary for the unique completion of $\mathcal{C}_{\mathcal{H}}$ to $\mathcal{H}$.

Lemmas 2.1 and 2.3 and Corollary 2.4.1 can be used to prove that each of the elements of the set $\{(s(i), t(j); t(j-i)) \mid i = 2, \ldots, h-1, j = 1, \ldots, i-1\}$ is necessary for the unique completion of $\mathcal{C}_{\mathcal{H}}$ to $\mathcal{H}$.

Next consider the elements of the set $\{(s(i), t(0); t(h-i)) \mid i = 1, \ldots, h-1\}$. Fix i and assume that entry $(s(i), t(0); t(h-i))$ is removed from $\mathcal{C}_{\mathcal{H}}$. The result is a partial latin square that will complete to $\mathcal{H}$ and to a latin square which agrees with $\mathcal{H}$ in all entries except a 2×2 subsquare on $(s(i), s(h-i); s(0)), (s(i), t(0); t(h-i))$. Hence each of these entries is necessary for the unique completion of $\mathcal{C}_{\mathcal{H}}$ to $\mathcal{H}$.

Finally consider element $(s(0), t(0); t(0))$ of $\mathcal{C}_{\mathcal{H}}$. If $(s(0), t(0); t(0))$ is removed from $\mathcal{C}_{\mathcal{H}}$, the result is a partial latin square that will complete to $\mathcal{H}$ and to a latin square which agrees with $\mathcal{H}$ in all entries except

$$\begin{array}{cc}
(s(0), s(h-1); s(h-1)), & (s(0), t(0); t(0)), \\
(s(0), t(h-1); t(h-1)), & \\
(s(i), s(h-1-i); s(h-1)), & (s(i), s(h-i); s(0)), \\
(s(h-1), s(0); s(h-1)), & (s(h-1), s(1); s(0)), \\
(s(h-1), t(h-1); t(0)), & \\
(t(0), s(0); t(0)), & (t(0), s(h-1); t(h-1)), \\
(t(0), t(0); s(0)), & \\
(t(i), t(i-1); s(h-1)), & (t(i), t(i); s(0)), \\
(t(h-1), t(h-2); s(h-1)), & (t(h-1), s(0); t(h-1)), \\
(t(h-1), t(h-1); s(0)), &
\end{array}$$

where $i = 1, \ldots, h-2$.

The relevant entries in quadrant 1 are on or below the back diagonal and in quadrant 4 the relevant entries are all on or below the main diagonal. Therefore one can easily check that these elements intersect $\mathcal{C}_{\mathcal{H}}$ in element $(s(0), t(0); t(0))$ alone. Thus $(s(0), t(0); t(0))$ is necessary for the unique completion of $\mathcal{C}_{\mathcal{H}}$ to $\mathcal{H}$.

Consequently $\mathcal{C}_{\mathcal{H}}$ is a critical set in $\mathcal{H}$. □

This last result can be generalized and critical sets can be constructed for an infinte number of classes of latin squares.

THEOREM 5.8 *Let $h \geq 3$ and let f range over the values $1, \ldots, h-2$. Take $\mathcal{H}$ to be a latin square of order h defined as follows:*

$$\begin{aligned}
\mathcal{H} = \{(s(i), s(j); s(i+j))\} \;&\cup\; \{(s(i), t(j); t(j-i))\} \;\cup \\
\{(t(i), s(j); t(i+j))\} \;&\cup\; \{(t(i), t(j); s(f+j-i))\},
\end{aligned}$$

where $i, j = 0, \ldots, h-1$. Let $\mathcal{C}_{\mathcal{H}}$ be the partial latin square represented by the set

$$\begin{aligned}
\mathcal{C}_{\mathcal{H}} \;=\; & \{(s(i), s(j); s(i+j)) \mid i = 0, \ldots, h-2, j = 0, \ldots, h-2-i\} \;\cup \\
& \{(s(0), t(0); t(0))\} \;\cup \\
& \{(s(i), t(0); t(h-i)) \mid i = f+1, \ldots, h-1\} \;\cup \\
& \{(s(i), t(j); t(j-i)) \mid i = 2, \ldots, h-1, j = 1, \ldots, i-1\} \;\cup \\
& \{(t(i), s(j); t(i+j)) \mid i = 1, \ldots, h-1, j = h-i, \ldots, h-1\} \;\cup \\
& \{(t(i), t(j); s(f+j-1)) \mid i = 0, \ldots, h-2, j = i+1, \ldots, h-1\}.
\end{aligned}$$

Then $\mathcal{C}_{\mathcal{H}}$ is a critical set in $\mathcal{H}$.

Proof. Fix f. Now using a similar argument to that used in the proof of Lemma 4.6 one can show that $\mathcal{C}_{\mathcal{H}}$ agrees with $\mathcal{H}$ in column $s(0)$ and columns $t(1)$ to $t(h-1)$. Next it can be shown that rows $s(f+1)$ to $s(h-1)$ are uniquely completable and therefore so is column $t(0)$. The unique completion of $\mathcal{C}_{\mathcal{H}}$ is now obvious.

The necessity of elements in quadrants 1, 3 and 4 follows from Lemmas 2.1, 2.2 and 2.3 and Theorem 2.4.

Similarly Lemma 2.3 and Corollary 2.4.1 can be used to show that each of the elements of the sets $\{(s(0), t(0); t(0))\}$ and $\{(s(i), t(j); t(j-i)) \mid i = 2, \ldots, h-1, j = 1, \ldots, i-1\}$ is necessary for the unique completion of $\mathcal{C}_{\mathcal{H}}$.

The remaining elements which need to be checked are $(s(e), t(0); t(h-e))$ where $e = f+1, \ldots, h-1$. Fix e and assume $(s(e), t(0); t(h-e))$ is removed from $\mathcal{C}_{\mathcal{H}}$. Choose a such that $e \equiv a (\text{mod } f)$ and consider the following cases

1. $a \neq 0$, and

2. $a = 0$.

Case 1. The remaining partial latin square completes to $\mathcal{H}$, but also to a latin square which agrees with $\mathcal{H}$ in all entries except

$$\begin{aligned}
&(s(a), s(h-a); s(0)), \\
&(s(a), t(0); t(h-a)), \\
&(s(a+fi), s(h-a-fi); s(0)), \quad \text{for} \quad i = 1, \ldots, \frac{e-a}{f} - 1, \\
&(s(a+fi), s(h-a-f(i-1); s(f)), \quad \text{for} \quad i = 1, \ldots, \frac{e-a}{f} - 1, \\
&(s(e), s(h-e); s(0)), \\
&(s(e), s(h-e+f); s(f)), \\
&(s(e), t(0); t(h-e)), \\
&(t(0), t(0); s(f)), \\
&(t(0), s(h-a-fi); t(h-a-fi)), \quad \text{for} \quad i = 1, \ldots, \frac{e-a}{f} - 1, \\
&(t(0), s(h-a); t(h-a)), \\
&(t(0), s(h-e); t(h-e)), \\
&(t(f), s(h-e); t(h-e+f)), \\
&(t(f), s(h-a-fi); t(h-a-f(i-1))), \quad \text{for} \quad i = 1, \ldots, \frac{e-a}{f} - 1, \\
&(t(f), t(0); s(0)).
\end{aligned}$$

The relevant entries in quadrant 1 are on or below the back diagonal and in quadrant 3 they are on or above the back diagonal. Therefore it is easily checked that these elements intersect $\mathcal{C}_{\mathcal{H}}$ in $(s(e), t(0); t(h-e))$ alone. Thus $(s(e), t(0); t(h-e))$ is necessary for the completion of $\mathcal{C}_{\mathcal{H}}$ to $\mathcal{H}$.

Case 2. When $a = 0$, a is repaced by f in the above argument and once again it can be shown that $(s(e), t(0); t(h-e))$ is necessary for the completion of $\mathcal{C}_{\mathcal{H}}$ to $\mathcal{H}$.

Consequently $\mathcal{C}_{\mathcal{H}}$ is a critical set in $\mathcal{H}$. □

6 More critical sets

In this section the techniques employed in Theorem 5.7 are adapted and used to construct critical sets for latin squares in which the latin subsquare L_4 is replaced by $\mathcal{L}$ of Section 3. Finally it will be shown in Theorem 6.10 that when each of the latin subsquares L_1, L_2, L_3 and L_4 is replaced by latin squares associated with $\mathcal{L}$, it is still possible to construct critical sets.

In Theorem 6.9 all arithmetic will be done modulo h unless otherwise stated.

THEOREM 6.9 *Let $\mathcal{L}_4$ be a latin square given by the set*

$$\mathcal{L}_4 = \{(u(i), u(j); u(i-j)(\text{mod } m))\} \ \cup \ \{(u(i), v(j); v(j-i)(\text{mod } m))\} \\ \cup \{(v(i), u(j); v(i-j)(\text{mod } m))\} \ \cup \ \{(v(i), v(j); u(j-i)(\text{mod } m))\},$$

where $i, j = 0, \ldots m-1$ and $m \geq 2$. Let $\mathcal{C}_{\mathcal{L}}$ be a partial latin square given by the set $\mathcal{C}_{\mathcal{L}} =$

$$\{(u(r), v(s); v(s-r)(\text{mod } m)) \mid r = 0, \ldots, m-1, s = 0, \ldots, m-1\} \quad \cup \\ \{(u(i), u(j); u(i-j)(\text{mod } m)), (v(i), u(j); v(i-j)(\text{mod } m)), \\ (v(i), v(j); u(j-i)(\text{mod } m)) \mid i = 0, \ldots, m-2, j = i+1, \ldots, m-1\}.$$

Let $s(i) = u(i)$, for $i = 0, \ldots, m-1$ and $s(i) = v(i-m)$, for $i = m, \ldots, 2m-1$. Then define $\mathcal{G}$ to be a latin square of order $2h$ where $h = 2m$ defined as follows:

$$\mathcal{G} = \{(s(i), s(j); s(i+j))\} \ \cup \ \{(s(i), t(j); t(j-i))\} \ \cup \\ \{(t(i), s(j); t(i+j))\} \ \cup \ \{(t(i), t(j); s(k)) \mid (s(i), s(j); s(k)) \in \mathcal{L}_4\},$$

where $i, j = 0, \ldots, h-1$. Let $\mathcal{C}_{\mathcal{G}}$ be the partial latin square represented by the set

$$\mathcal{C}_{\mathcal{G}} = \{(s(i), s(j); s(i+j)) \mid i = 0, \ldots, h-2, j = 0, \ldots, h-2-i\} \quad \cup \\ \{(s(0), t(0); t(0))\} \quad \cup$$

$$\{(s(i),t(j);t(j-i)) \mid i=1,\ldots,h-1, j=0,\ldots,i-1\} \quad \cup$$
$$\{(t(i),s(j);t(i+j)) \mid i=1,\ldots,h-1, j=h-i,\ldots,h-1\} \quad \cup$$
$$\{(t(i),t(j);s(k)) \mid (s(i),s(j);s(k)) \in \mathcal{C}_{\mathcal{L}}\}.$$

Then $\mathcal{C}_{\mathcal{G}}$ is a critical set in $\mathcal{G}$.

Proof.

The proof of Lemma 4.6 can be used to show that the completion of $\mathcal{C}_{\mathcal{G}}$ will agree with $\mathcal{G}$ in all entries in rows $s(0)$ to $s(h-1)$ of columns $t(0)$ to $t(h-1)$. At this point one can apply Lemmas 2.1, 2.2 and 2.3 and Theorems 2.4 and 3.5 to prove that $\mathcal{C}_{\mathcal{G}}$ has a unique completion to $\mathcal{G}$.

The necessity of elements in quadrants 1, 3 and 4 follows from Lemmas 2.1 and 2.3 and Theorems 2.4 and 3.5.

Lemmas 2.1 and 2.3 and Corollary 2.4.1 can be used to prove that each of the elements of the set $\{(s(i),t(j);t(j-i)) \mid i=2,\ldots,h-1, j=1,\ldots,i-1\}$ is necessary for the unique completion of $\mathcal{C}_{\mathcal{G}}$ to $\mathcal{G}$.

Next consider the elements of the set

$$\{(s(i),t(0);t(h-i)) \mid i=1,\ldots,h-1\}.$$

Fix i and assume that entry $(s(i),t(0);t(h-i))$ is removed from $\mathcal{C}_{\mathcal{G}}$. The result is a partial latin square that will complete to $\mathcal{G}$ and to a latin square which agrees with $\mathcal{G}$ in all entries except a 2×2 subsquare on $(s(i),s(h-i);s(0)),(s(i),t(0);t(h-i))$. Hence each of these entries is necessary for the unique completion of $\mathcal{C}_{\mathcal{G}}$ to $\mathcal{G}$.

Finally assume that $(s(0),t(0);t(0))$ is removed from $\mathcal{C}_{\mathcal{G}}$. The result is a partial latin square that will complete to $\mathcal{G}$ and to a latin square which agrees with $\mathcal{G}$ in all entries except

$$\begin{array}{cc}
(s(0),s(h-1);s(h-1)), & (s(0),t(0);t(0)), \\
(s(0),t(h-1);t(h-1)), & \\
(s(i),s(h-1-i);s(h-1)), & (s(i),s(h-i);s(0)), \\
(s(h-1),s(0);s(h-1)), & (s(h-1),s(1);s(0)), \\
(s(h-1),t(h-1);t(0)), & \\
(t(0),s(0);t(0)), & (t(0),s(h-1);t(h-1)), \\
(t(0),t(0);s(0)), & \\
(t(h-1),s(0);t(h-1)), & (t(h-1),t(0);s(h-1)), \\
(t(h-1),t(h-1);s(0)), &
\end{array}$$

where $i=1,\ldots,h-2$.

The relevant entries in quadrant 1 are all on or below the back diagonal. Thus it can be easily checked that these elements intersect $\mathcal{C}_{\mathcal{G}}$ in the element $(s(0),t(0);t(0))$ alone. Therefore $(s(0),t(0);t(0))$ is necessary for the unique completion of $\mathcal{C}_{\mathcal{G}}$ to $\mathcal{G}$.

Consequently $\mathcal{C}_{\mathcal{G}}$ is a critical set in $\mathcal{G}$. $\square$

In Theorem 6.10 all arithmetic will be done modulo m.

THEOREM 6.10 *Let $\mathcal{G}$ be a latin square with rows and columns indexed by the set $\{u(0),\ldots,u(m-1),v(0),\ldots,v(m-1),w(0),\ldots,w(m-1),z(0),\ldots,z(m-1)\}$ and entries as follows:*

$$\begin{array}{rcll}
\{(u(i),u(j);u(i+j))\} & \cup & \{(u(i),v(j);v(i+j))\} & \cup \\
\{(v(i),u(j);v(m-2-(i+j)))\} & \cup & \{(v(i),v(j);u(m-2-(i+j)))\} & \cup \\
\{(u(i),w(j);w(j-i))\} & \cup & \{(u(i),z(j);z(j-i))\} & \cup \\
\{(v(i),w(j);z(i-j))\} & \cup & \{(v(i),z(j);w(i-j))\} & \cup \\
\{(w(i),u(j);w(i+j))\} & \cup & \{(w(i),v(j);z(i+j))\} & \cup \\
\{(z(i),u(j);z(m-2-(i+j)))\} & \cup & \{(z(i),v(j);w(m-2-(i+j)))\} & \cup \\
\{(w(i),w(j);u(j-i))\} & \cup & \{(w(i),z(j);v(j-i))\} & \cup \\
\{(z(i),w(j);v(i-j))\} & \cup & \{(z(i),z(j);u(i-j))\}, &
\end{array}$$

for $i,j=0,\ldots,m-1$.

Let $\mathcal{C}_{\mathcal{G}}$ be a partial latin square with entries as follows:

$$\begin{array}{l}
\{(u(i),u(j);u(i+j)) \mid i,j=0,\ldots,m-1\} \quad \cup \\
\{(u(i),v(j);v(i+j)),(v(i),u(j);v(m-2-(i+j))), \\
(v(i),v(j);u(m-2-(i+j))) \mid i=0,\ldots,m-2, j=0,\ldots m-2-i\} \\
\cup \quad \{(u(0),w(0);w(0))\} \quad \cup \\
\{(u(i),w(j);w(j-i)) \mid i=2,\ldots,m-1, j=1,\ldots,i-1\} \quad \cup \\
\{(v(i),w(j);z(i-j)) \mid i,j=0,\ldots,m-1\} \quad \cup \\
\{(u(i),z(j);z(j-i)),(v(i),z(j);w(i-j)) \\
\mid i=1,\ldots,m-1, j=0,\ldots,i-1\} \quad \cup \\
\{(w(i),u(j);w(i+j)),(w(i),v(j);z(i+j)), \\
(z(i),u(j);z(m-2-(i+j))) \mid i=1,\ldots,m-1, j=m-i,\ldots,m-1\} \\
\cup \quad \{(z(i),v(j);w(m-2-(i+j))) \mid i,j=0,\ldots,m-1\} \quad \cup \\
\{(w(i),z(j);v(j-i)) \mid i,j=0,\ldots,m-1\} \quad \cup \\
\{(w(i),w(j);u(j-i)),(z(i),w(j);v(i-j)), \\
(z(i),z(j);u(i-j)) \mid i=0,\ldots,m-2, j=i+1,\ldots,m-1\}.
\end{array}$$

Then $\mathcal{C}_{\mathcal{G}}$ is a critical set in $\mathcal{G}$.

Proof.

The unique completion of $\mathcal{C}_{\mathcal{G}}$ can be verified using similar techniques to those used in earlier results.

The necessity of the elements in quadrants 1, 3 and 4 follows from Lemmas 2.1, 2.2 and 2.3 and Theorem 3.5. As can the necessity of all elements of quadrant 2 except $\{(v(i), w(j); z(i-j)) \mid i, j = 0, \ldots, m-1\}$.

If one removes any of the elements of the set $\{(v(i), w(0); z(i)) \mid i = 0, \ldots, m-2\}$, then the remaining partial latin square will complete to $\mathcal{G}$ and to a latin square which agrees with $\mathcal{G}$ in all entries except a 2×2 subsquare on $(u(1), w(0); w(m-1)), (u(1), z(i+1); z(i))$.

If one removes the element $(v(m-1), w(0); z(m-1))$ then the remaining partial latin square will complete to $\mathcal{G}$ and to a latin square which agrees with $\mathcal{G}$ in all entries except a 2×2 subsquare on $(v(m-1), v(m-1); u(0)), (v(m-1), w(0); z(m-1))$.

The fact that each of the elements of the set

$$\{(v(i), w(j); z(i-j)) \quad | \quad i = 0, \ldots, m-1, j = 1, \ldots, m-1\},$$

is necessary for the unique completion follows from the proof of Theorem 3.5.

Thus $\mathcal{C}_{\mathcal{G}}$ is a critical set in $\mathcal{G}$. □

7 Conclusions

Several new constructions have been presented for critical sets in latin squares. Theorem 5.8 can be used to construct critical sets for an infinte number of classes of latin squares.

One question which the work in this paper raises is "Can the critical set given in Theorem 6.10 be generalised in the same way that Theorem 5.8 generalised the critical set in Theorem 5.7?"

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