Closing a gap in the spectrum of critical sets

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Abstract

In 1998 Donovan and Howse proved that for all n there exist critical sets of order n and size s, where $\lfloor \frac{n^2}{4} \rfloor \leq s \leq \frac{n^2-n}{2}$ with the exception of the case $s = \frac{n^2}{4} + 1$ when n is even. In this paper we will present a construction for this case, where $n \geq 6$, based on the discovery of a critical set of size 17 for a Latin square of order 8. This verifies that there does exist a critical set of order n and size $\frac{n^2}{4} + 1$ when n is even.

1 Definitions

For the purposes of this paper, *Latin squares* of order n are $n \times n$ arrays of integers chosen from the set $X = \{0, ..., n-1\}$ such that each such integer occurs exactly once in each row and in each column. An example of a Latin square of order 6 is shown below.

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

A Latin square can also be written as a set of ordered triples $\{(i, j; k) \mid$ where symbol k occurs in cell (i, j) of the array$\}$. In this paper, for a Latin square of order n, there is an implicit "mod n" after the third element in each ordered triple; that is, the ordered triple $(i, j; k)$ refers to $(i, j; k \pmod{n})$.

A *back-circulant Latin square*, of order n, (referred to as BC_n) is a Latin square in which the cell (i, j) contains the symbol $i + j \pmod{n}$. In terms of ordered triples,

$$BC_n = \{(i, j; i + j) \mid 0 \leq i, j \leq n - 1\}.$$

A *partial Latin square* P, of order n, is an $n \times n$ array with entries chosen from a set X, of size n, such that each element of X occurs at most once in each row and each column. Let P be a partial Latin square of order n. Then $|P|$ is said to be the *size* of the partial Latin square and the set of positions $\mathcal{S}_P = \{(i, j) \mid (i, j; k) \in P, \exists k \in X\}$ is said to determine the *shape* of P.

A *critical set* of a Latin square L is a partial Latin square contained in L such that L is the only Latin square of order n with k in position (i, j) for every $(i, j, k) \in C$, and no proper subset of C satisfies this requirement. See [?] for further details.

For a given n, the spectrum of critical sets of order n is the set of values s for which there exists a critical set of size s and order n. Papers have appeared which provide some information about the spectrum of critical sets, but none of these have been able to settle the question of the existence of a critical set of size $\frac{(2m)^2}{4} + 1$ and order $2m$. The size $\frac{(2m)^2}{4} + 1$ has reasonable significance, as explained below. With this paper, we settle this outstanding question and close one of the gaps in the spectrum of critical sets. The construction pertaining to this case is given in Section 3. For the remainder of this section we provide some background information and relevant results.

Let $lcs(n)$ denote the size of the largest critical set in any Latin square of order n and $scs(n)$ denote the size of the smallest critical set in any Latin square of order n. It was conjectured by Nelder [?] that $lcs(n) = (n^2 - n)/2$ and $scs(n) = \lfloor n^2/4 \rfloor$. The bound for $lcs(n)$ was shown to be false in 1982, when Stinson and van Rees, [?], exhibited examples of critical sets of size greater than $(n^2 - n)/2$. Unfortunately, the research over the last twenty years has not added much information and in general an upper bound is given by $n^2 - n$. Fortunately, more is known about $scs(n)$. In 1978, Curran and van Rees, [?], showed that $scs(n) \leq \lfloor n^2/4 \rfloor$, and more recently Bate and van Rees [?] verified that for a special class of critical sets (strong critical sets, see [?] for definition) $scs(n) \geq \lfloor n^2/4 \rfloor$. This brings us closer to a lower bound, however in general, the size of the smallest critical set still needs to be established.

In 1998, Donovan and Howse [?], proved that for all n there exist critical sets of order n and size s, where $\lfloor \frac{n^2}{4} \rfloor \leq s \leq \frac{n^2-n}{2}$, with the exception of the case $s = \frac{n^2}{4} + 1$, when n is even. It is this case which is settled in the affirmative by this paper.

In order to validate the construction we require the definition of a Latin interchange and an association lemma.

Let P and P' be two partial Latin squares of the same order, with the same size and shape. Then P are P' are said to be *mutually balanced* if the entries in each row (and column) of P are the same as those in the corresponding row (and column) of P'. They are said to be *disjoint* if no cell in P' contains the same entry as the corresponding cell in P. A *Latin interchange* I is a partial Latin square for which there exists another partial Latin square I', of the same order, size and shape with the property that I and I' are disjoint and mutually balanced. The partial Latin square I' is said to be a *disjoint mate* of I. An example of a Latin interchange is given below.

9	.	.	.	.	3
3	4	5	6	7	8
4	5	6	7	8	9

I

3	.	.	.	.	9
4	5	6	7	8	3
9	4	5	6	7	8

I'

The following lemma clarifies the connection between critical sets and Latin interchanges.

Lemma 1.1 *A partial Latin square $C \subset L$, of size s and order n, is a critical set for a Latin square L if and only if the following hold:*

1. *C contains an element of every Latin interchange that occurs in L;*
2. *for each $(i, j; k) \in C$, there exists a Latin interchange I_r in L such that $I_r \cap C = \{(i, j; k)\}$.*

Proof.

1. If C does not contain an element from some Latin interchange I, where I has disjoint mate I', then C is also a partial Latin square of $L' = (L \setminus I) \cup I'$. Hence C is not uniquely completable.

2. If no such Latin interchange I_r can be found, then the position $(i,\ j;\ k)$ may be deleted from C and $C \setminus \{(i,\ j;\ k)\}$ will still be uniquely completable and thus a critical set for L.

2 Critical sets in Latin squares of orders 6 and 8

Let $\mathcal{A} = \{(i,j;i+j) \mid (0 \le i+j \le 1) \vee (8 \le i+j \le 10)\}$. Then $\mathcal{A}$ is a critical set of order 6 and size $\frac{6^2}{4} = 9$ in BC_6. Beginning with $\mathcal{A}$, if we remove $(5,4;3)$ and add $(3,2;5)$ and $(3,4;3)$ we get $\mathcal{A}'$, which is a critical set of size $\frac{6^2}{4}+1 = 10$ for $\mathcal{LA}$.

0	1	.	.	.	.
1	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	2
.	.	.	.	2	3
.	.	.	2	3	4

$\mathcal{A}$

0	1	.	.	.	.
1	.	.	.	.	.
.	.	.	.	.	.
.	.	5	.	3	2
.	.	.	.	2	3
.	.	.	2	.	4

$\mathcal{A}'$

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
4	0	5	1	3	2
5	4	1	0	2	3
3	5	0	2	1	4

$\mathcal{LA}$

Let $\mathcal{B} = \{(i,j;i+j) \mid (0 \le i+j \le 2) \vee (11 \le i+j \le 14)\}$. Then $\mathcal{B}$ is a critical set of order 8 and size $\frac{8^2}{4} = 16$ in BC_8. Beginning with $\mathcal{B}$, remove $(7,5;4)$ and $(7,6;5)$ and add $(4,3;7)$, $(4,5;4)$, and $(4,6;5)$. This gives $\mathcal{B}'$, a critical set of size $\frac{8^2}{4}+1 = 17$ for $\mathcal{LB}$:

0	1	2	.	.	.	.	.
1	2	.	.	.	.	.	.
2	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	3
.	.	.	.	.	.	3	4
.	.	.	.	.	3	4	5
.	.	.	.	3	4	5	6

$\mathcal{B}$

0	1	2	.	.	.	.	.
1	2	.	.	.	.	.	.
2	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	7	.	4	5	3
.	.	.	.	.	.	3	4
.	.	.	.	.	3	4	5
.	.	.	.	3	.	.	6

$\mathcal{B}'$

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
6	0	1	7	2	4	5	3
5	7	6	1	0	2	3	4
7	6	0	2	1	3	4	5
4	5	7	0	3	1	2	6

$\mathcal{LB}$

3 Critical sets in Latin squares of order n, n even

The above examples can be generalised to produce critical sets of size $\frac{n^2}{4}+1$, when n is even.

Theorem 3.1 *Take the critical set*

$$C = \{(i,j;i+j) \mid (0 \leq i+j \leq \frac{n}{2}-2) \vee (\frac{3n}{2}-1 \leq i+j \leq 2n-2)\}.$$

Construct the set

$$\begin{aligned} D &= (C \backslash \{(n-1,j;j-1) \mid \frac{n}{2}+1 \leq j \leq n-2\}) \\ &\quad \cup \{(\frac{n}{2},j;j-1) \mid \frac{n}{2}+1 \leq j \leq n-2\} \cup \{(\frac{n}{2},\frac{n}{2}-1;n-1)\}. \end{aligned}$$

Then D is a critical set of size $\frac{n^2}{4}+1$.

The proof is this result is presented below.

Henceforth, we shall refer to the completion of D as $\mathcal{LD}$. The following process outlines how D can be uniquely completed to $\mathcal{LD}$. In completing D, at each step in the completion process the given cell is forced to contain the specified symbol. If any other symbol were to be placed in the specified cell, the result would not be a partial Latin square.

We begin by filling row $\frac{n}{2}$ starting at column $j=0$ and moving right to column $j=\frac{n}{2}-2$. In row $\frac{n}{2}$, fill the cell in column j with

$n-2$, when $j=0$;

$j-1$, when $1 \leq j \leq \frac{n}{2}-2$;

$\frac{n}{2}-2$, when $j=\frac{n}{2}$.

We will fill rows $n-2$ to $\frac{n}{2}+1$ sequentially, from left to right in columns 0 to $\frac{n}{2}-2$, then column $\frac{n}{2}$, then column $\frac{n}{2}-1$. So, for $2 \leq x \leq \frac{n}{2}-1$, and $0 \leq j \leq \frac{n}{2}$ fill the cell in row $n-x$ and column j with

$(n-x)+j \pmod{n}$, when $j \neq x-1$ and $j \neq x-2$;

$n-1$, when $j=x-2$;

$n-2$, when $j = x-1$;

$\frac{n}{2} - 1 - x$, when $j = \frac{n}{2}$;

$\frac{n}{2} - x$, when $j = \frac{n}{2} - 1$.

When $n \geq 8$ the triangle bounded by the cells $(\frac{n}{2}+1, \frac{n}{2}+1)$, $(\frac{n}{2}+1, n-3)$, and $(n-3, \frac{n}{2}+1)$ is filled from bottom to top and left to right. If $n \geq 8$, for $3 \leq x \leq \frac{n}{2} - 2$ fill the cell in row $n-x$, column $j = \frac{n}{2}+1$ to $j = \frac{n}{2} + x - 2$ with $(n-x)+j \pmod{n}$.
For $0 \leq j \leq \frac{n}{2} - 3$, fill the cell in row $n-1$ and column j with $\frac{n}{2} + j \pmod{n}$. Fill

$n-1$, when $j = \frac{n}{2} - 2$ and

0, when $j = \frac{n}{2} - 1$ with 0.

For $\frac{n}{2} + 1 \leq j \leq n-2$, fill the cell in row $n-1$ and column j with $j - \frac{n}{2} \pmod{n}$.
For $0 \leq x \leq \frac{n}{2} - 1$, fill the cells in row x sequentially right to left from column $j = n-1$ to $j = \frac{n}{2} - 1 - x$ with $x + j$.

To prove the necessity of each of the symbols in the critical set D, three types of Latin interchanges will be used:

Type (1)
This Latin interchange uses only two rows and consequently the same symbols in each row. The disjoint mate is obtained by interchanging the rows. For example, the Latin interchange I and its disjoint mate I' can be represented as:

$$\begin{aligned} I &= \{(r_1, c_1; e_1), (r_1, c_2; e_2), ..., (r_1, c_{m-1}; e_{m-1}), (r_1, c_m; e_m)\} \\ &\quad \cup \{(r_2, c_1; e_2), (r_2, c_2; e_3), ..., (r_2, c_{m-1}; e_m), (r_2, c_m; e_1)\}, \text{ and} \\ I' &= \{(r_1, c_1; e_2), (r_1, c_2; e_3), ...(r_1, c_{m-1}; e_m), (r_1, c_m; e_1)\} \\ &\quad \cup \{(r_2, c_1; e_1), (r_2, c_2; e_2), ..., (r_2, c_{m-1}; e_{m-1}), (r_2, c_m; e_m)\}. \end{aligned}$$

Type (2)
This Latin interchange uses three rows, with the top row containing two elements. For example, the Latin interchange I and its disjoint mate I' can

be represented as:

$$\begin{aligned}
I &= \{(r_1, c_1; x), (r_1, c_{m+1}; y)\} \\
&\quad \cup \{(r_2, c_1; y), (r_2, c_2; e_1), (r_2, c_3; e_2), ..., (r_2, c_m; e_{m-1}), (r_2, c_{m+1}; e_m)\} \\
&\quad \cup \{(r_3, c_1; e_1), (r_3, c_2; e_2), (r_3, c_3; e_3), ..., (r_3, c_m; e_m), (r_3, c_{m+1}; x)\}, \text{ and} \\
I' &= \{(r_1, c_1; y), (r_1, c_{m+1}; x)\} \\
&\quad \cup \{(r_2, c_1; e_1), (r_2, c_2; e_2), (r_2, c_3; e_3), ..., (r_2, c_m; e_m), (r_2, c_{m+1}; y)\} \\
&\quad \cup \{(r_3, c_1; x), (r_3, c_2; e_1), (r_3, c_3, e_2), ..., (r_3, c_m; e_{m-1}), (r_3, c_{m+1}; e_m)\}.
\end{aligned}$$

Type (3)
These Latin interchanges take a variety of forms and cannot be written as simply as Type 1 and Type 2. Full details of these Latin interchanges are presented at the end of this section.

For $n = 6$, proving that the elements in the example given above are necessary is trivial. We assume $n \geq 8$ and prove the following. The following Latin interchanges (I_1 through I_{10}) exist in $\mathcal{LD}$:

I_1 is a Latin interchange of Type 1, and $I_1 \cap D = \{(\frac{n}{2}, \frac{n}{2} - 1; n - 1)\}$.

$$\begin{aligned}
I_1 &= \{(\frac{n}{2}, 0; n-2)\} \\
&\quad \cup \{(\frac{n}{2}, j; j-1) \mid 1 \leq j \leq \frac{n}{2} - 2\} \\
&\quad \cup \{(\frac{n}{2}, \frac{n}{2} - 1; n-1), (\frac{n}{2}, \frac{n}{2}; \frac{n}{2} - 2)\} \\
&\quad \cup \{(n-2, 0; n-1), (n-2, 1; n-2) \\
&\quad \cup \{(n-2, j; j-2) \mid 2 \leq j \leq \frac{n}{2} - 2\} \\
&\quad \cup \{(n-2, \frac{n}{2} - 1; \frac{n}{2} - 2), (n-2, \frac{n}{2}; \frac{n}{2} - 3)\}.
\end{aligned}$$

I_2 is a Latin interchange of Type 1, and $I_2 \cap D = \{(n-1, n-1; n-2)\}$.

$$\begin{aligned}
I_2 &= \{(\frac{n}{2} - 1, \frac{n}{2} - 1; n-2)\} \\
&\quad \cup \{(\frac{n}{2} - 1, j; \frac{n}{2} + j - 1) | \frac{n}{2} + 1 \leq j \leq n-1\} \\
&\quad \cup \{(n-1, \frac{n}{2} - 1; 0)\}
\end{aligned}$$

$$\cup \{(n-1, j; j-\frac{n}{2}) \mid \frac{n}{2}+1 \leq j \leq n-2\}$$
$$\cup \{(n-1, n-1; n-2)\}.$$

I_3 is a Latin interchange of Type 1, and $I_3 \cap D = \{(n-1, \frac{n}{2}; \frac{n}{2}-1)\}$.

$$\begin{aligned} I_3 \; = \; & \{(\frac{n}{2}-1, j; \frac{n}{2}+j-1) \mid 0 \leq j \leq \frac{n}{2}-2\} \\ & \cup \{(\frac{n}{2}-1, \frac{n}{2}; n-1)\} \\ & \cup \{(n-1, j; j+\frac{n}{2}) \mid 0 \leq j \leq \frac{n}{2}-3\} \\ & \cup \{(n-1, \frac{n}{2}-2; n-1), (n-1, \frac{n}{2}; \frac{n}{2}-1)\}. \end{aligned}$$

I_4 is a Latin interchange of Type 2, and for $\frac{n}{2}+2 \leq x \leq n-2$, $I_4 \cap D = \{(x, \frac{3n}{2}-1-x; \frac{n}{2}-1)\}$.

For $\frac{n}{2}+2 \leq x \leq n-2$, construct the Latin interchange

$$\begin{aligned} H \; = \; & \{(x-\frac{n}{2}-1, n-x; \frac{n}{2}-1), (x-\frac{n}{2}-1, \frac{3n}{2}-1-x; n-2)\} \\ & \cup \{(x-1, j; x-1+j) \mid \frac{n}{2}+1 \leq j \leq \frac{3n}{2}-1-x\} \\ & \cup \{(x-1, n-x; n-2)\} \\ & \cup \{(x-1, \frac{n}{2}-1; x-\frac{n}{2}-1), (x-1, \frac{n}{2}; x-\frac{n}{2})-2\} \\ & \cup \{(x, j; x+j) \mid \frac{n}{2}+1 \leq j \leq \frac{3n}{2}-1-x\} \\ & \cup \{(x, j; x+j) \mid n-x \leq j \leq \frac{n}{2}-2\} \\ & \cup \{(x, \frac{n}{2}-1; x+\frac{n}{2}), (x, \frac{n}{2}; x+\frac{n}{2}-1)\}. \end{aligned}$$

Then when $x = \frac{n}{2}+2$, let $I_4 = H$, and when $\frac{n}{2}+3 \leq x \leq n-2$, let $I_4 = H \cup \{(x-1, i; x-1+i) \mid n-x+1 \leq i \leq \frac{n}{2}-2\}$.

I_5 is a Latin interchange of Type 2, and for $\frac{n}{2}+1 \leq x \leq n-2$, $I_5 \cap D = \{(x, n-1; x-1)\}$.

For $\frac{n}{2}+1 \leq x \leq n-2$, construct the Latin interchange

$$I_5 \; = \; \{(x-\frac{n}{2}, j; x-\frac{n}{2}+j) \mid \frac{n}{2}-1 \leq j \leq n-1\}$$

$$\cup \{(x-\frac{n}{2}+1, j; x-\frac{n}{2}+1+j) \mid \frac{n}{2}-1 \le j \le n-1\}$$
$$\cup \{(x, \frac{n}{2}-1; x-\frac{n}{2}), (x, n-1; x-1)\}.$$

I_6 is a Latin interchange of Type 1, and $I_6 \cap D = \{(\frac{n}{2}+1, n-2; \frac{n}{2}-1)\}$.

If $4 \mid n$, construct the Latin interchange

$$\begin{aligned} I_6 \;=\; & \{(\frac{n}{2}-1, 2j; \frac{n}{2}-1+2j) \mid 0 \le j < \frac{n}{4}\} \\ & \cup \{(\frac{n}{2}-1; \frac{n}{2}-1; n-2)\} \cup \\ & \cup \{(\frac{n}{2}-1, 2j; \frac{n}{2}-1+2j) \mid \frac{n}{4} < j < \frac{n}{2}\} \\ & \cup \{(\frac{n}{2}+1, 2j; \frac{n}{2}+1+2j) \mid 0 \le j < \frac{n}{4}-1\} \\ & \cup \{(\frac{n}{2}+1, \frac{n}{2}-2; n-2), (\frac{n}{2}+1, \frac{n}{2}-1; 1)\} \\ & \cup \{(\frac{n}{2}+1, 2j; \frac{n}{2}+1+2j) \mid \frac{n}{4} < j < \frac{n}{2}\}. \end{aligned}$$

If $4 \nmid n$, construct the Latin interchange

$$\begin{aligned} I_6 \;=\; & \{(\frac{n}{2}-1, 2j; \frac{n}{2}-1+2j) \mid 0 \le j < \frac{n}{4}-1\} \\ & \cup \{(\frac{n}{2}-1, \frac{n}{2}; n-1)\} \\ & \cup \{(\frac{n}{2}-1, 2j; \frac{n}{2}-1+2j) \mid \frac{n}{4} < j < \frac{n}{2}\} \\ & \cup \{(\frac{n}{2}+1, 2j; \frac{n}{2}+1+2j) \mid 0 \le j < \frac{n}{4}-2\} \\ & \cup \{(\frac{n}{2}+1, \frac{n}{2}-3; n-1), (\frac{n}{2}+1, \frac{n}{2}; 0)\} \\ & \cup \{(\frac{n}{2}+1, 2j; \frac{n}{2}+1+2j) \mid \frac{n}{4} < j < \frac{n}{2}\}. \end{aligned}$$

I_7 is a Latin interchange of Type 1, and $I_7 \cap D = \{(\frac{n}{2}, n-1; \frac{n}{2}-1)\}$.

If $4 \mid n$, construct the Latin interchange

$$\begin{aligned} I_7 \;=\; & \{(\frac{n}{4}-1, j; \frac{n}{4}+j-1), (\frac{n}{4}, j; \frac{n}{4}+j) \mid \frac{n}{4} \le j \le n-1\} \\ & \cup \{(\frac{n}{2}, \frac{n}{4}; \frac{n}{4}-1), (\frac{n}{2}, n-1; \frac{n}{2}-1)\}. \end{aligned}$$

If $4 \nmid n$, construct the Latin interchange

$$\begin{aligned} I_7 &= \{(\frac{n-2}{4}, \frac{n-2}{4}; \frac{n}{2}-1), (\frac{n-2}{4}, n-1; \frac{n-6}{4})\} \\ &\quad \cup \{(\frac{n}{2}, \frac{n-2}{4}; \frac{n-6}{4}), (\frac{n}{2}, n-1; \frac{n}{2}-1)\}. \end{aligned}$$

For $\frac{n}{2}+1 \leq x \leq n-2$, I_8 is a Latin interchange of Type 1, and $I_8 \cap D = \{(\frac{n}{2}, x; x-1)\}$.

$$\begin{aligned} I_8 &= \{(\frac{n}{2}-1, x-\frac{n}{2}; x-1), (\frac{n}{2}-1, x; \frac{n}{2}+x-1)\} \\ &\quad \cup \{(\frac{n}{2}, x-\frac{n}{2}; \frac{n}{2}+x-1), (\frac{n}{2}, x; x-1)\}. \end{aligned}$$

For $(\frac{3n}{2} \leq x+y < 2n-2) \wedge (x \neq n-1) \wedge (y \neq n-1)$, I_9 is a Latin interchange of Type 1, and $I_9 \cap D = \{(y, x; y+x)\}$.

$$\begin{aligned} I_9 &= \{(y-\frac{n}{2}, x-\frac{n}{2}; y+x), (y-\frac{n}{2}, x; y+x-\frac{n}{2})\} \\ &\quad \cup \{(y, x-\frac{n}{2}; y+x-\frac{n}{2}), (y, x; y+x)\}. \end{aligned}$$

Where $0 \leq x+y \leq \frac{n}{2}-2$, there exists a Latin interchange I_{10} of Type 3, where $I_{10} \cap D = \{(y, x; y+x)\}$.

If $0 \leq x+y \leq \frac{n}{2}-2$, determine the Latin interchange I_{10} using results found in [?], as follows.

Let $\mathcal{A}$ denote the Latin subrectangle in $\mathcal{LD}^{\mathcal{T}}$ (the transpose of $\mathcal{LD}$) bounded by the cells $(x, y; y+x)$, $(n-1, y; y-1)$, $(x, \frac{n}{2}-1; \frac{n}{2}-1+x)$, and $(n-1, \frac{n}{2}-1; \frac{n}{2}-2)$. All future column and row references are relative to this subrectangle; that is, a reference to the cell $(i, j; k)$ means the cell $(i-x, j-y; k)$ in $\mathcal{LD}^{\mathcal{T}}$.

Let $c = \frac{n}{2}-y$, $r = n-x$, and $e = n+1-c$.

Define the sequence of numbers $\alpha_1, \alpha_2, ..., \alpha_P$ to be integers where

$$\begin{aligned} \alpha_1 &= c-1 \pmod{e} \text{ and, for } i \geq 2, \\ \alpha_i &= \alpha_{i-1} \pmod{e-\alpha_1-...-\alpha_{i-1}}. \end{aligned}$$

Let P be the value where $\alpha_P \neq 0$ and $\alpha_{P+i} = 0$ for all $i > 0$. For $i = 1, 2, ..., P$, let $\delta_i = \alpha_1 + \alpha_2 + ... + \alpha_i$. Define

$$\begin{aligned} A_0 &= \{(0, 0; x+y), (0, c-1; c-1+x+y)\} \text{ and if } \alpha_1 \neq c-1 \text{ define} \\ B_0 &= \{(c-1-ae, ae; c-1+x+y), (c-1-ae, (a+1)e; x+y) \\ &\qquad \mid 0 \leq a \leq \frac{c-1-\alpha_1}{e} - 1\}. \end{aligned}$$

If $\alpha_1 \neq 0$, define

$$A_1 = \{(e, c-1-\alpha_1; c-1+e-\alpha_1+x+y), (e, c-1; x+y)\},$$

and if $\alpha_1 \neq \alpha_2$ define

$$\begin{aligned} B_1 = \{&(\alpha_1 - a(e-\alpha_1), c-1-\alpha_1; c-1+x+y), \\ &(\alpha_1 - a(e-\alpha_1), c-1-\alpha_1+(a+1)(e-\alpha_1); c-1+e-\alpha_1+x+y) \\ &\mid 0 \leq a \leq \frac{\alpha_1-\alpha_2}{e-\alpha_1} - 1\}. \end{aligned}$$

If $P \geq 2$, for $2 \leq i \leq P$, define

$$\begin{aligned} A_i = \{&(e-\delta_{i-1}, c-1-\alpha_i; c-1+e-\delta_i+x+y), \\ &(e-\delta_{i-1}, c-1; c-1+e-\delta_{i-1}+x+y)\} \end{aligned}$$

and if $\alpha_i \neq \alpha_{i+1}$, define

$$\begin{aligned} B_i = \{&(\alpha_i - (e-\delta_i)a, c-1-\alpha_i+a(e-\delta_i); c-1+x+y), \\ &(\alpha_i - a(e-\delta_i), c-1-\alpha_i+(a+1)(e-\delta_i); c-1+e-\delta_i+x+y) \mid \\ &0 \leq a \leq \frac{\alpha_i-\alpha_{i+1}}{e-\delta_i} - 1\}. \end{aligned}$$

Then the set $I_{10} = A_0 \cup B_0 \cup A_1 \cup B_1 \cup ... \cup A_P \cup B_P$ is the required Latin interchange.

4 Conclusion

Further research could include determining those values $s > \frac{n^2-n}{2}$ for which there exists a critical set of order n and size s.

5 Acknowledgements

The authors wish to acknowledge the support of the Australian Resarch Council's small grant scheme - grant number 99/ARCS105. Richard Bean wishes to thank the Australian Government for the support of an Australian Postgraduate Award.