MATH1061/7861 Semester 2, 2005 Solution Ass. 3

Exercise Set 3.1

Q9 True as we only have to exhibit one example, and $5^2 = 3^2 + 4^2$.

Q27 Assume $n = 2k+1$ and $m = 2l+1$, where $k, l \in \mathbb{Z}$. Then $n+m = 2k+1+2l+1 = 2(k+l+1)$. Since $k+l+1 \in \mathbb{Z}$ it follows that $n+m$ is even as required.

Q53 This statement is false. For example let $m = 2$ and $n = 18$. Then $m.n = 36 = 6^2$, but m and n are not perfect squares.

Exercise Set 3.2

Q27 If $\dfrac{ax+b}{cx+d} = 1$ then $ax+b = cx+d$ and $ax-cx = d-b$. Implying $x(a-c) = d-b$ or $x = \dfrac{d-b}{a-c}$. Note $a \neq c$ so $a-c \neq 0$. And since $a, b, c, d \in \mathbb{Z}$ we have $a-c, d-b \in \mathbb{Z}$. Thus $x = \dfrac{p}{q}$ where $q = a-c \in \mathbb{Z}$ and $p = d-b \in \mathbb{Z}$. Thus x is rational.

Q31 Let $f(x) = r_0 + r_1x + r_2x^2 \ldots + r_nx^n$, where $r_i \in Q, 0 \leq i \leq n$. Then $r_i = \dfrac{p_i}{q_i}$ for $0 \leq i \leq n$ and $p_i, q_i \in \mathbb{Z}$. So

$$f(x) = \frac{p_0}{q_0} + \frac{p_1}{q_1}x + \frac{p_2}{q_2}x^2 + \ldots + \frac{p_n}{q_n}x^n.$$

Since c is a root of the polynomial $f(x)$ we have $f(c) = 0$. Hence

$$f(c) = \frac{p_0}{q_0} + \frac{p_1}{q_1}c + \frac{p_2}{q_2}c^2 + \cdots + \frac{p_n}{q_n}c^n = 0.$$

Multiplying throughout by $q_0q_1q_2 \ldots q_n$ we obtain

$$q_1q_2 \ldots q_np_0 + q_0q_2 \ldots q_np_1c + q_0q_1q_3 \ldots q_np_2c^2 + \ldots + q_0q_1 \ldots q_{n-1}p_nc^n = 0.$$

Thus $g(c) = 0$ where $g(x) = s_0 + s_1x + \ldots + s_nx^n$, $s_i = q_0 \ldots q_{i-1}q_{i+1} \ldots q_np_i$ and $s_i \in \mathbb{Z}$ as required.

Exercise Set 3.3

Q2 Yes, since $54 = 3 \times 18$

Q5 Yes, since $6m(2m+10) = 4(3m(m+5))$

Q9 Yes, since $2a.34b = 2 \times a \times 2 \times 17b = 4(17ab)$

Q13 Yes. If $n - 1 = 4k + 3$ then

$$\begin{aligned} n^2 - 1 &= (4k+3)^2 - 1 \\ &= 16k^2 + 24k + 9 - 1 \\ &= 16k^2 + 24k + 8 \\ &= 8(2k^2 + 3k + 1) \end{aligned}$$

Q25 No. Let $a = 6, b = 9, c = 4$. Then $a|bc$ But $a \nmid b$ nor do we have $a \nmid c$.

Q39 c) $2 \times 5 = 10$. So $2^8 \times 5^8 = 10^8$. Hence $(20!)^2$ ends in 8 zeros.

Exercise Set 3.4

Q29 **Case 1:** $n = 3q$ $(d = 3,\ r = 0)$. Then $n^2 = (3q)^2 = 9q^2 = 3(3q^2)$. Since $3q^2 \in \mathbb{Z}$ $\quad n^2 = 3k$ for some $k \in \mathbb{Z}$.

Case 2: $n = 3q + 1$ $(d = 3,\ r = 1)$. Then $n^2 = (3q+1)^2 = 9q^2 + 6q + 1$. So $n^2 = 3(q^2 + 2q) + 1$ and since $q^2 + 2q \in \mathbb{Z}$. $n^2 = 3k + 1$, where $k \in \mathbb{Z}$.

Case 3: $n = 3q + 2$ $(d = 3,\ r = 2)$. Then $n^2 = (3q+2)^2 = 9q^2 + 12q + 4$. So $n^2 = 3(3q^2 + 4q + 1) + 1$. Since $q^2 + 4q + 1 \in \mathbb{Z}$. $n^2 = 3k + 1$ for some $k \in \mathbb{Z}$.