MATH1061/7861 Semester 2, 2005 Solutions Ass. 5

Exercise Set 4.2

Q7 Let $P(n)$ be the statement that $1 + 6 + 11 + \ldots + (5n - 4) = \frac{n(5n-3)}{2}$. That is, let $P(n)$ be the statement

$$\sum_{i=1}^{n} (5i - 4) = \frac{n(5n - 3)}{2}.$$

Then when $n = 1$ we have $\sum_{i=1}^{1} (5i - 4) = 1$ and $\frac{1(5\times 1-3)}{2} = 1$. Hence $P(1)$ is true.

Assume $P(k)$ is true. That is, assume $\sum_{i=1}^{k} (5i - 4) = \frac{k(5k-3)}{2}$.

Prove $P(k + 1)$ is true. That is, prove $\sum_{i=1}^{k+1} (5i - 4) = \frac{(k+1)(5(k+1)-3)}{2}$.

$$\begin{aligned} L.H.S = \sum_{i=1}^{k+1} (5i - 4) &= \sum_{i=1}^{k} (5i - 4) + 5(k + 1) - 4 \\ &= \frac{k(5k - 3)}{2} + 5(k + 1) - 4 \\ &= \frac{k(5k - 3) + 10(k + 1) - 8}{2} \\ &= \frac{5k^2 + 7k + 2}{2} \end{aligned}$$

$$\begin{aligned} R.H.S. = \frac{(k + 1)(5(k + 1) - 3)}{2} & \\ &= \frac{5k^2 + 7k + 2}{2} \end{aligned}$$

Hence L.H.S. = R.H.S. and $P(k+1)$ is true. Thus by the Principle of Mathematical Induction $P(n)$ is true for all $n \geq 1$.

Exercise Set 4.3

Q21 Let $P(n)$ be the statement that $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, $n \geq 2$. That is,

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}\,, \; n \geq 2.$$

When $n = 2$, $\sum_{i=1}^{2} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}} > \frac{\sqrt{1}+1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$. Hence $P(2)$ is true.

Assume $P(k)$ is true. That is, assume $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$.

Prove $P(k+1)$ is true. That is, prove $\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k+1}$.

$$
\begin{aligned}
\text{L.H.S.} = \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} &= \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \\
&> \sqrt{k} + \frac{1}{\sqrt{k+1}} \\
&= \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}} \\
&> \frac{\sqrt{k}\sqrt{k} + 1}{\sqrt{k+1}} \\
&= \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1} = R.H.S.
\end{aligned}
$$

Hence by the Principle of Mathematical Induction $P(n)$ is true, $\forall n \geq 2$.

Exercise Set 5.1

Q8

a)	Yes, $3 \in \{1, 2, 33\}$.	b)	No, $1 \not\subseteq \{1\}$.
c)	No, $\{2\} \notin \{1, 2\}$.	d)	Yes, $\{3\} \in \{1, \{2\}, \{3\}\}$.
e)	Yes, $1 \in \{1\}$.	f)	No, $\{2\} \not\subseteq \{1, \{2\}, \{3\}\}$.
g)	Yes, $\{1\} \subseteq \{1, 2\}$.	h)	No, $1 \notin \{\{1\}, 2\}$.
i)	Yes, $\{1\} \subseteq \{1, \{2\}\}$.	j)	Yes, $\{1\} \subseteq \{1\}$.

Q12

a) Yes, $\mathbb{Z}^+ \subseteq \mathbb{Q}$.

b) No, as $-\sqrt{2} \in \mathbb{R}^-$, but $-\sqrt{2} \notin Q$.

c) No, $\frac{1}{2} \in \mathbb{Q}$ but $\frac{1}{2} \notin \mathbb{Z}$.

d) No, $0 \in \mathbb{Z}$, but $0 \notin (\mathbb{Z}^- \cup \mathbb{Z}^+)$.

e) Yes, for any sets A, B where $A \subseteq B$, $A \cap B = A$ (see below).

f) Yes, for any sets A, B where $A \subseteq B$, $B \cup A = B$.

g) Yes, as in e) above.

h) No, see f) above for an explanation.