MATH1061/7861 Semester 2, 2005 Solution Ass. 8

Exercise Set 10.2

Q17 $\forall\, m, n \in \mathbb{Z} \quad m0n \Leftrightarrow m - n$ is odd.

Not reflexive since $m - m = 0$ which is even $\forall\, m \in \mathbb{Z}$.

Yes, symmetric. To prove this, assume $m0n$ that is $m - n = 2p + 1$ for some p. It follows that $n - m = -2p - 1 = 2(-p - 1) + 1$, which is odd. So $n0m$.

Not transitive as $22 - 1$ is odd and $1 - 22$ is odd, but $22 - 22$ is even.

Exercise Set 10.3

Q24) $\forall\, m, n \in \mathbb{Z}, mDn \Leftrightarrow m^2 \equiv n^2 (mod\ 3)$.

We shall prove this is an equivalence relation.

First note $\forall\, m,\ m^2 - m^2 = 0 = 0 \times 3$. Hence $3|(m^2 - m^2)$. Thus $m^2 \equiv m^2 (mod\, 3)$ and the relation is reflexive.

Next we shall prove $\forall\, m, n \in \mathbb{Z}$ if $m^2 \equiv n^2 (mod\, 3)$ then $n^2 \equiv m^2 (mod\, 3)$. If $m^2 \equiv n^2 (mod\, 3)$ then $3|(m^2 - n^2)$ or $m^2 - n^2 = 3k$ where $k \in \mathbb{Z}$. But this implies that $n^2 - m^2 = 3l$ where $l = -k \in \mathbb{Z}$. Thus $3|(n^2 - m^2)$ and so $n^2 \equiv m^2 (mod\, 3)$. So the relation is symmetric.

Next we must prove the relation is transitive. That is, we must prove $\forall\, m, n, p \in \mathbb{Z}$ if $m^2 \equiv n^2 (mod\, 3)$ and $n^2 \equiv p^2 (mod\, 3)$ then $m^2 \equiv p^2 (mod\, 3)$. So assume that $m^2 \equiv n^2 (mod\, 3)$ and $n^2 \equiv p^2 (mod\, 3)$. Implying that $m^2 - n^2 = k3$ and $n^2 - p^2 \equiv l3$ where $k, l \in \mathbb{Z}$. By adding these equations we get $m^2 - n^2 + n^2 - p^2 = k3 + l3 = (k + l)3$, where $k + l \in \mathbb{Z}$. Hence $m^2 \equiv p^2 (mod\, 3)$ and the relation is transitive.

Since the relation is reflexive, symmetric and transitive, it is an equivalence relation.

Q19 Equivalence classes are:

$[0] = \{4p | p \in \mathbb{Z}\}$;

$[1] = \{4p + 1 | p \in \mathbb{Z}\}$;

$[2] = \{4p + 2 | p \in \mathbb{Z}\}$; and

$[3] = \{4p + 3 | p \in \mathbb{Z}\}$.

Q24 Equivalence classes are:

$[0] = \{q | q = 3p, \text{ for some } p \in \mathbb{Z}\}$

$[1] = \{q | q = 3p + 1 \text{ or } q = 3p + 2, \text{ for some } p \in \mathbb{Z}\}$.

Exercise Set 7.1

Q2 a) The domain is $\{1, 3, 5\}$ and the co-domain is $\{a, b, c, d\}$.

b) $g(1) = b$, $g(3) = b$, $g(5) = b$.

c) The range is $\{b\}$.

e) The inverse image of b is $\{1, 3, 5\}$.
The inverse image of c is $\emptyset$.

f) $\{(1, b), (3, b), (5, b)\}$

Q13 b) $F(\phi) = 0$, d) $F(\{2, 3, 4, 5\}) = 0$.

Exercise Set 7.2

Q7 b) G is not one-to-one as $G(a) = G(d) = y$ but $a \neq d$.
G is not onto as there exists no element m of X such that $G(m) = z$.

Q18 $f(x) = \dfrac{3x - 1}{x}$, $x \neq 0$. Assume

$$\begin{aligned} f(x_1) &= f(x_2) \\ \Rightarrow \quad \frac{3x_1 - 1}{x_1} &= \frac{3x_2 - 1}{x_2} \\ \Rightarrow \quad (3x_1 - 1)x_2 &= (3x_2 - 1)x_1 \\ \Rightarrow \quad 3x_1x_2 - x_2 &= 3x_2x_1 - x_1 \\ \Rightarrow \quad x_2 &= x_1 \end{aligned}$$

Hence f is one-to-one.

Q50 Note that if $y = \dfrac{3x - 1}{x}$ then $y = 3 - \dfrac{1}{x}$ or $x = \dfrac{1}{3 - y}$ provided $y \neq 3$. So for all $y \in \mathbb{R} - \{3\}$ there exists an $x \in \mathbb{R}$ such that $f(x) = y$, namely $x = \dfrac{1}{3 - y}$. Hence $f(x)$ is onto.

Since $f(x)$ is one–to–one and onto, $f(x)$ is a one-to-one correspondence. The inverse function is

$$\begin{aligned} f^{-1} \quad &: \quad \mathbb{R} - \{3\} \Rightarrow \mathbb{R} - \{0\} \\ f^{-1}(y) &= \frac{1}{3 - y}. \end{aligned}$$