FINITE GEOMETRIES

Let $\mathcal{P}$ be a set of elements called **points**. Define v to be the number of distinct points; that is,

$$|\mathcal{P}| = v.$$

Let $\mathcal{L}$ be a collection of subsets of $\mathcal{P}$, which are called **lines**. Define b to be the number of lines in the line set; that is,

$$|\mathcal{L}| = b.$$

We call $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ an **incidence structure**.

BLOCK DESIGNS

Alternatively called a **block design** $(\mathcal{P}, \mathcal{L}) = (V, \mathcal{B})$.

That is, we let $\mathcal{P} = V$ (points are sometimes called **treatments**) and $\mathcal{L} = \mathcal{B}$, with lines referred to as **blocks**.

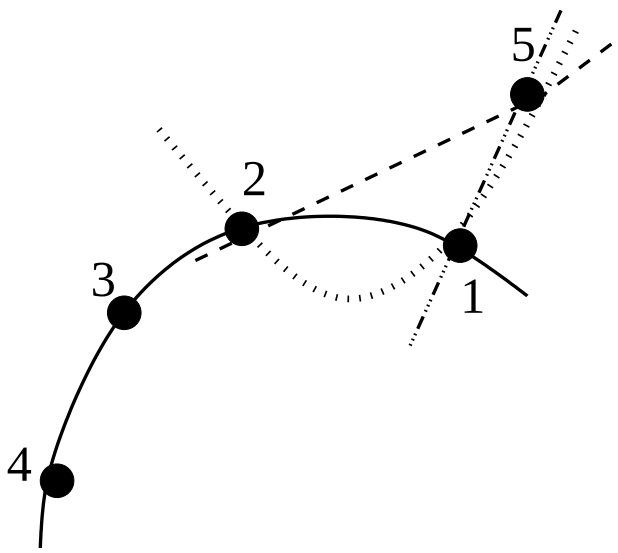
EXAMPLE Let

$\mathcal{P} = \{1, 2, 3, 4, 5\}$, be the point set. Then

$$|\mathcal{P}| = 5$$

$\mathcal{L} = \{\{1, 2, 3, 4\}, \{1, 2, 5\}, \{1, 5\}, \{2, 5\}\}$ be the line set, and

$$b = |\mathcal{L}| = 4.$$



$$\mathcal{I} = (\mathcal{P}, \mathcal{L}).$$

Sometimes when there can be no confusion, we use the convention that we delete the commas and brackets when listing lines. So in the above example $\mathcal{L} = \{1234, 125, 15, 25\}$.

Given $P, Q \in \mathcal{P}, \quad \{P, Q\} \subseteq L \in \mathcal{L}$

P is **incident with** L

P is **on** L

L is **incident with** P

L is **through** P

$(P) = \{L \in \mathcal{L} \mid L \text{ is through } P\}$

$|\ (P)\ |$ denotes the number of lines through P, and is abbreviated to $|\ P\ |$

$(L) = \{P \in \mathcal{P} \mid P \text{ is on } L\}$

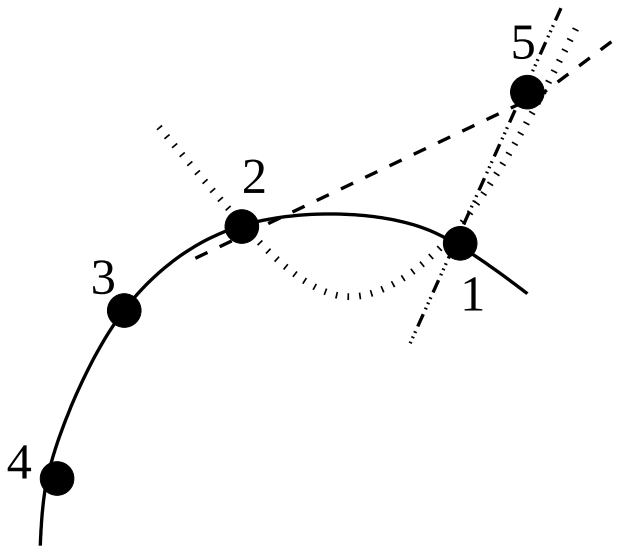
$|\ (L)\ |$ denotes the number of points on L and is abbreviated to $|\ L\ |$

P and Q distinct points, PQ denotes the line joining them.

If there exists a subset of two or more points all incident with line L then the points are said to be **collinear** on L or **joined** by L.

If there exists a subset of two or more lines all incident with the point P then the lines are **concurrent** at P or **meet** at P.

EXAMPLE Let $\mathcal{P} = \{1, 2, 3, 4, 5\}$ $\mathcal{L} = \{\{1, 2, 3, 4\}, \{1, 2, 5\}, \{1, 5\}, \{2, 5\}\}$.



points 1, 2, 3 are collinear on $\{1, 2, 3, 4\}$.

Lines $\{1, 5\}, \{2, 5\}$ are concurrent at point 5.

We are interested in studying the types of structures we obtain, once we place conditions on how we choose the lines.

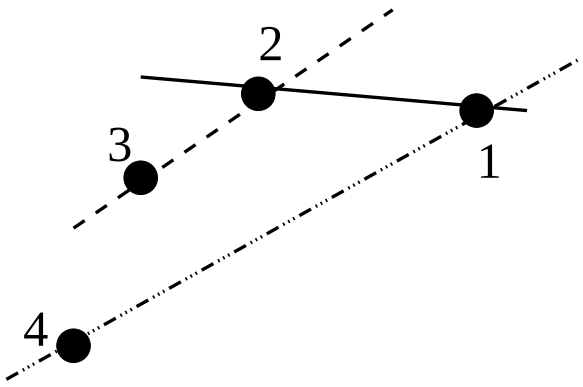
NEAR LINEAR SPACES

A **near–linear space** is an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ such that

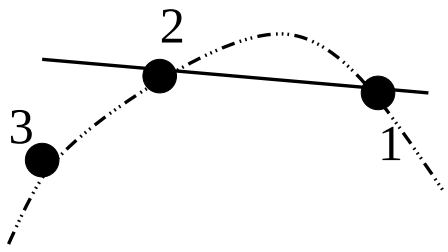
AXIOM NL1: any line is incident with at least two points, and

AXIOM NL2: two points are on at most one line.

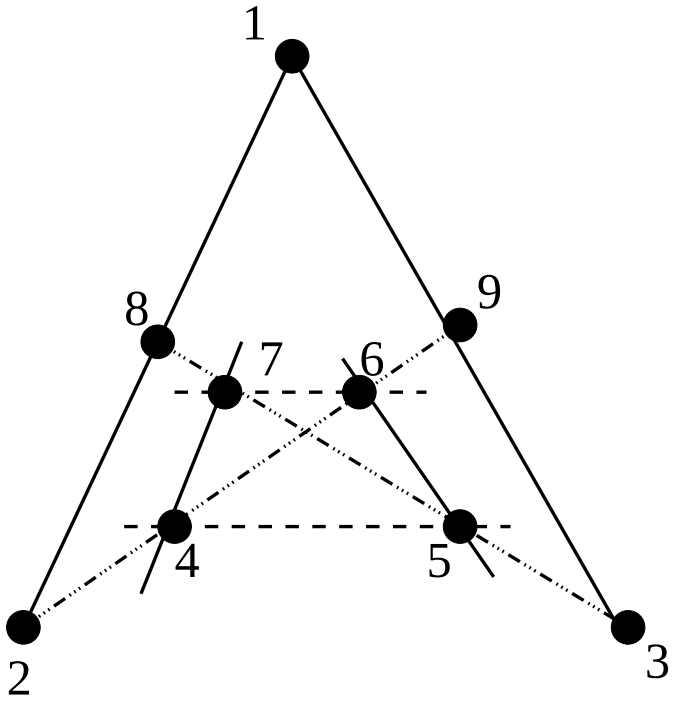
EXAMPLE Let $\mathcal{P} = \{1, 2, 3, 4\}$ and $\mathcal{L} = \{\{1, 4\}, \{1, 2\}, \{2, 3\}\}$. Then $(\mathcal{P}, \mathcal{L})$ is a near–linear space.



EXAMPLE Let $\mathcal{P} = \{1, 2, 3\}$ and $\mathcal{L} = \{\{1, 2, 3\}, \{1, 2\}\}$. Then $(\mathcal{P}, \mathcal{L})$ is not a near–linear space.



EXAMPLE Let $\mathcal{P} = \{1,2,3,4,5,6,7,8,9\}$ and $\mathcal{L} = \{\{1,2,8\},\{1,3,9\},\{2,4,6,9\},\{3,5,7,8\},\ \{4,5\},\ \{4,7\},\{5,6\},\{6,7\}\}$.



$\mathcal{I} = (\mathcal{P}, \mathcal{L})$ is a near–linear space.

The question is "What properties can we deduce from the axioms of a near linear space?"

LEMMA Two distinct lines of a near–linear space meet in at most one point.

Proof Suppose that there are two distinct lines L and M such that there is a pair of distinct points P and Q both on L and M. This is clearly impossible since P and Q are joined by two distinct lines and this contradicts Axiom NL2.

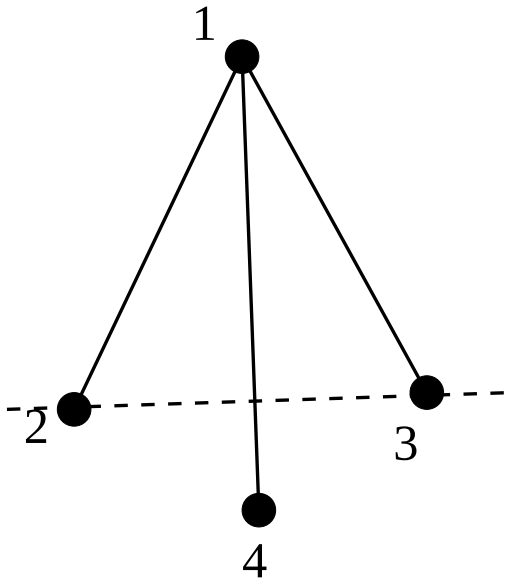
It has not been assumed that $\mathcal{P}$ is non–empty or in fact that $\mathcal{L}$ is non–empty. However from now on we shall assume that $\mathcal{P}$ and $\mathcal{L}$ are non-empty.

A subset $\mathcal{X} \subseteq \mathcal{P}$ is said to form a **subspace** $\mathcal{R} = (\mathcal{X}, \mathcal{L}_{\mathcal{X}})$ of the near–linear space $\mathcal{I} = (\mathcal{P}, \mathcal{L}_{\mathcal{P}})$ if the following condition is satisfied:

- for all points $P, Q \in \mathcal{X}$ and all lines $PQ = L \in \mathcal{L}_{\mathcal{P}}$, $L \subseteq \mathcal{X}$. That is, all points on the line PQ are contained in $\mathcal{X}$.

The incidence structure $\mathcal{R}$ consists of the lines of $\mathcal{I}$ restricted to the point set $\mathcal{X}$.

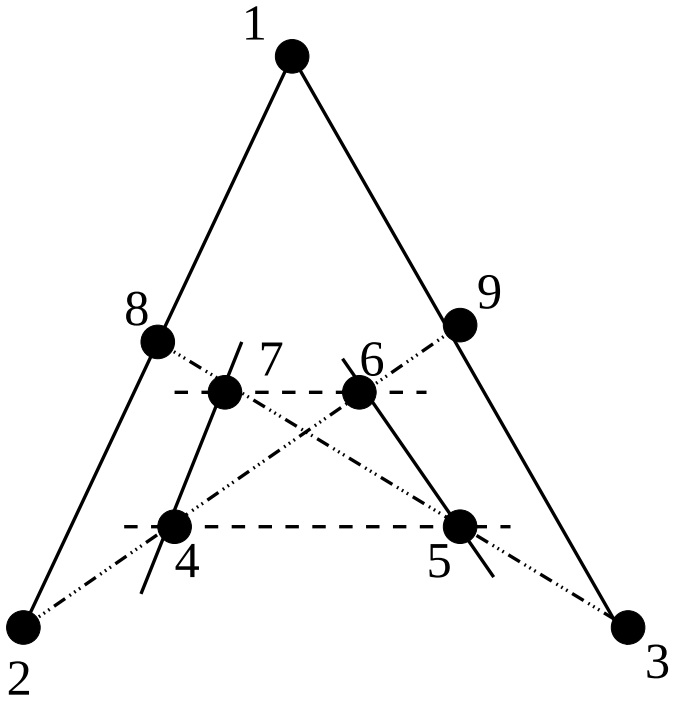
EXAMPLE Let $\mathcal{P} = \{1, 2, 3, 4\}$ and $\mathcal{L} = \{\{1, 4\}, \{1, 3\}, \{1, 2\}, \{2, 3\}\}$.



Let $\mathcal{X}_1 = \{1, 2, 3\}$. Then $(\mathcal{X}_1, \mathcal{L}_{\mathcal{X}_1})$ is a subspace of $(\mathcal{P}, \mathcal{L})$ with $\mathcal{L}_{\mathcal{X}_1} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

Let $\mathcal{X}_2 = \{1, 2, 4\}$. Then $(\mathcal{X}_2, \mathcal{L}_{\mathcal{X}_2})$ is a subspace of $(\mathcal{P}, \mathcal{L})$, with $\mathcal{L}_{\mathcal{X}_2} = \{\{1, 2\}, \{1, 4\}\}$.

EXAMPLE Let $\mathcal{P} = \{1,2,3,4,5,6,7,8,9\}$ and $\mathcal{L} = \{\{1,2,8\},\{1,3,9\},\{2,4,6,9\},\{3,5,7,8\},\{4,5\},\{4,7\},\{5,6\},\{6,7\}\}$.



Let $\mathcal{X}_1 = \{1,2,8\}$. Then $(\mathcal{X}_1, \mathcal{L}_{\mathcal{X}_1})$ is a subspace of $(\mathcal{P},\mathcal{L})$, with $\mathcal{L}_{\mathcal{X}_1} = \{\{1,2,8\}\}$.

Let $\mathcal{X}_2 = \{1,2,3,8\}$. Then $\mathcal{X}_2$ does not generate a subspace as $\{3,5,7,8\} \in \mathcal{L}$, but $\{3,5,7,8\} \not\subseteq \mathcal{X}_2$.

Let $\mathcal{X}_3 = \{1,2,3,8,9\}$. Then $\mathcal{X}_3$ does not generate a subspace as $\{3,5,7,8\} \in \mathcal{L}$, but $\{3,5,7,8\} \not\subseteq \mathcal{X}_3$.

LEMMA The intersection of any number of subspaces is a subspace.

Proof Take a collection of subspaces $\mathcal{R}_i$. Let $\mathcal{X}$ be the intersection of the point set of the subspaces.

It must be shown that if P and Q are points of $\mathcal{X}$ and P and Q are on a line PQ then $PQ \subset \mathcal{X}$.

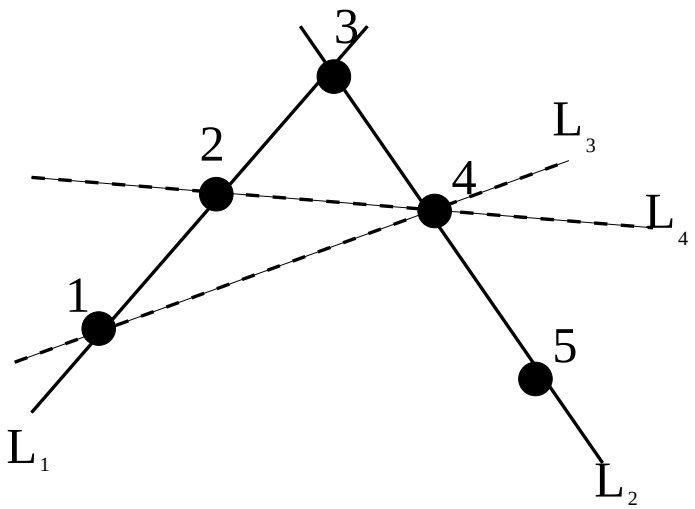
But any subspace containing $\mathcal{X}$ contains P and Q and by definition, the subspace contains the line PQ. Therefore the line PQ is in all subspace of which $\mathcal{X}$ is the intersection, and so PQ is a subset of $\mathcal{X}$.

Let $\mathcal{X}$ be any set of points of a near–linear space $\mathcal{I} = (\mathcal{P}, \mathcal{L})$. The **closure** of $\mathcal{X}$ is a subspace which contains $\mathcal{X}$ but does not properly contain any other subspace **on** $\mathcal{X}$. The closure of $\mathcal{X}$ is the smallest subspace containing $\mathcal{X}$. It is denoted $\langle \mathcal{X} \rangle$.

EXAMPLE Let $\mathcal{P} = \{1, 2, 3, 4, 5\}$ and
$\mathcal{L} = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4, 5\}, \{1, 4\}\}$.

$\langle \{1, 2\} \rangle$ generates the subspace on the set of points $\{1, 2, 3\}$ and

$\langle \{1, 2, 5\} \rangle$ generates the entire incidence structure $(\mathcal{P}, \mathcal{L})$ on the set of points $\{1, 2, 3, 4, 5\}$.



LEMMA The closure $\langle \mathcal{X} \rangle$ of a set $\mathcal{X}$ is the intersection of all subspaces on $\mathcal{X}$.

Proof We know that the intersection of the subspaces on $\mathcal{X}$ is a subspace. It is easy to see that the intersection of the subspaces on $\mathcal{X}$ gives the smallest subspace on $\mathcal{X}$.

Given a subset $\mathcal{X}$ of the point set of an incidence structure $(\mathcal{P}, \mathcal{L})$, it is a fact that the closure of $\mathcal{X}$ is unique.

$\mathcal{X}$ is said to **generate** its closure $\langle \mathcal{X} \rangle$.

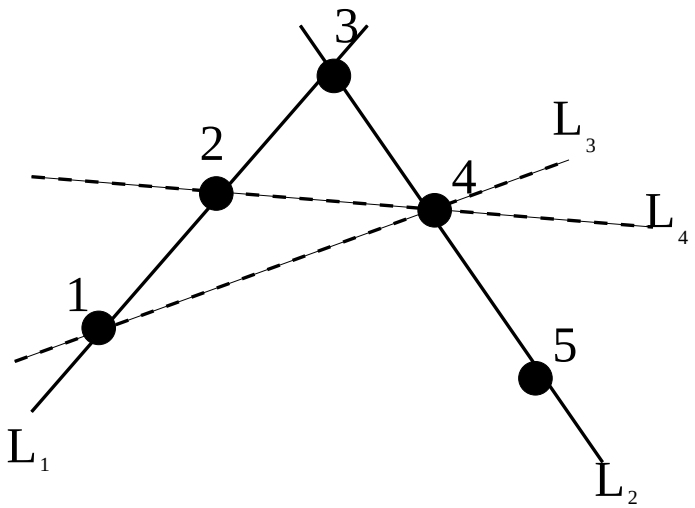
$\mathcal{X}$ is said to be the **generating set** for $\langle \mathcal{X} \rangle$.

An **independent set** $\mathcal{X}$ is a set of points such that for each $X \in \mathcal{X}$,

$$X \notin \langle \mathcal{X} \setminus \{X\} \rangle.$$

If a set is not independent then it is said to be **dependent**.

EXAMPLE Let $\mathcal{P} = \{1, 2, 3, 4, 5\}$ and $\mathcal{L} = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4, 5\}, \{1, 4\}\}$.

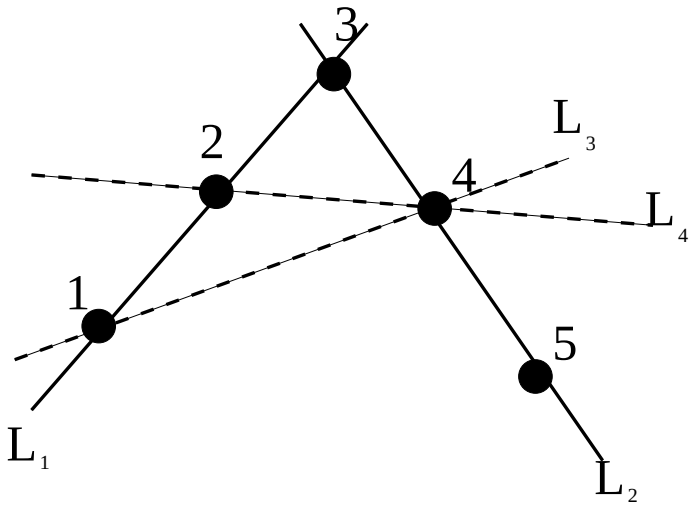


The sets $\{4, 5\}$ and $\{2, 4, 5\}$ are independent and the set $\{1, 2, 4, 5\}$ is dependent as $5 \in \langle \{1, 2, 4\} \rangle$.

A **basis** of a near–linear space S is an independent set of points of S which generates S.

A basis of a near–linear space is not necessarily unique.

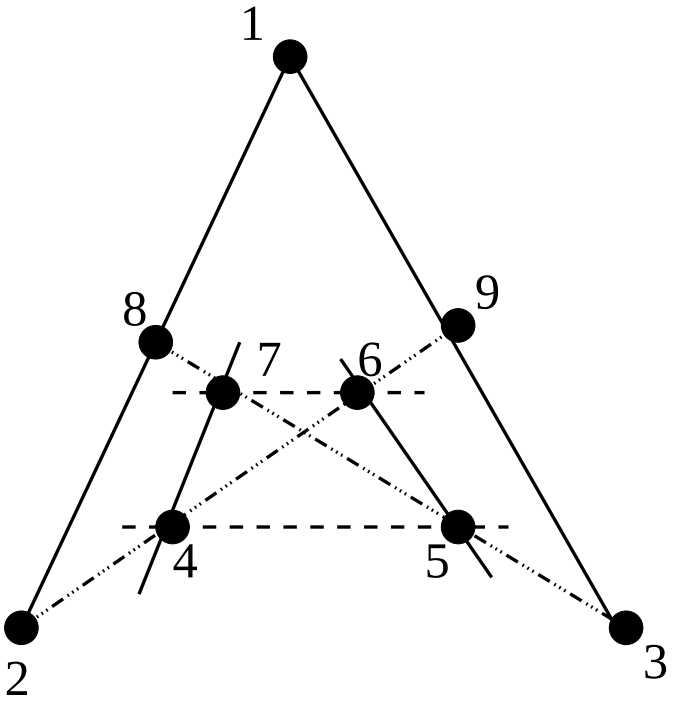
EXAMPLE Let $\mathcal{P} = \{1, 2, 3, 4, 5\}$ and $\mathcal{L} = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4, 5\}, \{1, 4\}\}$.



An example of a basis for $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ is the set $\{1, 2, 4\}$.

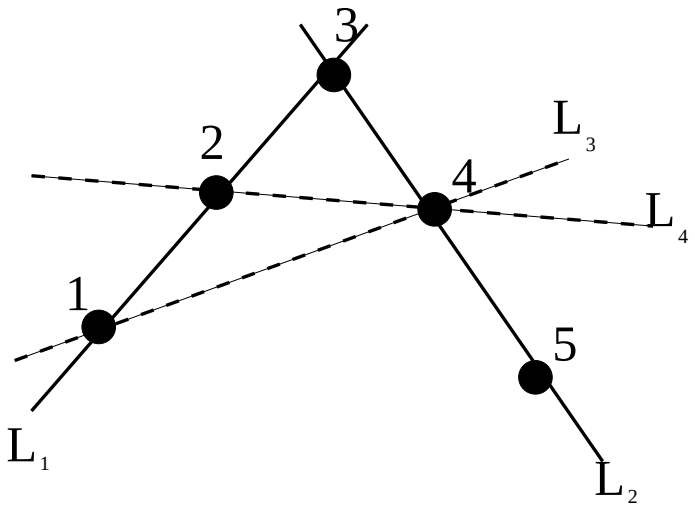
In a near–linear space not all the bases need to have the same number of elements.

EXAMPLE Let $\mathcal{P} = \{1,2,3,4,5,6,7,8,9\}$ and $\mathcal{L} = \{\{1,2,8\},\{1,3,9\},\{2,4,6,9\},\{3,5,7,8\},\{2,3\},\{4,5\},\{4,7\},\{5,6\},\{6,7\}\}$.



The set $\{1,2,3\}$ or $\{4,5,6,7\}$ form a basis for $\mathcal{I} = (\mathcal{P},\mathcal{L})$.

The **dimension** of a near–linear space S is **one less than min**$\{|\mathcal{B}| \mid \mathcal{B}$ basis for $S\}$.

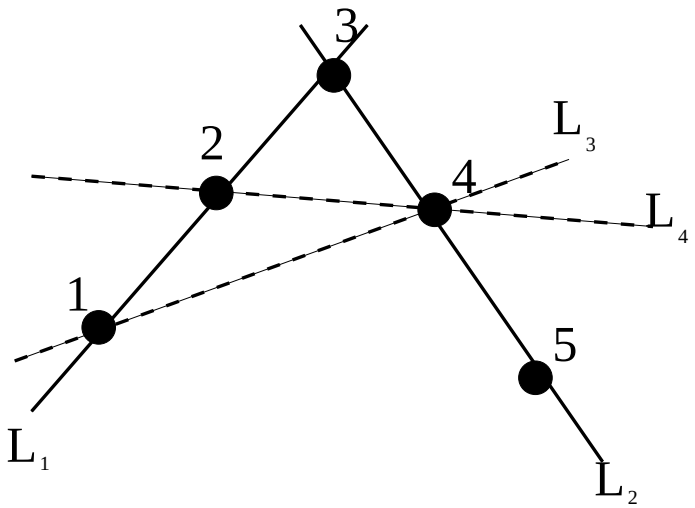


In the example above there is no set of points $\{P, Q\}$ which generates the entire space. However the set $\{1, 2, 4\}$ is a basis for S so the dimension of the near–linear space is 2.

Any finite near–linear space $(\mathcal{P}, \mathcal{L})$ can be represented by a matrix.

Each point corresponds to a row of the matrix and each line corresponds to a column. A 1 is placed in cell (i, j) of the matrix if point P_i is on the line L_j, otherwise a 0 is placed in this cell.

EXAMPLE Let $\mathcal{P} = \{1, 2, 3, 4, 5\}$ and $\mathcal{L} = \{L_1 = \{1, 2, 3\}, L_4 = \{2, 4\}, L_2 = \{3, 4, 5\}, L_3 = \{1, 4\}\}$.



	L_1	L_2	L_3	L_4
1	1	0	1	0
2	1	0	0	1
3	1	1	0	0
4	0	1	1	1
5	0	1	0	0

Let $S = (\mathcal{P}, \mathcal{L})$ be a finite near–linear space and let

$\mathcal{P} = \{P_i \mid i = 1, \ldots, v\}$ and

$\mathcal{L} = \{L_i \mid i = 1, \ldots, b\}$.

Let A be a $0, 1$-matrix and let $a_{i,j}$ denote the entry in the i^{th} and j^{th} column. Then

$$a_{i,j} = \begin{cases} 0 & if \quad P_i \notin L_j \\ 1 & if \quad P_i \in L_j. \end{cases}$$

The matrix A is said to be the **incidence matrix** of the near–linear space and a_{ij} is said to be the **incidence number** of P_i and L_j.

THEOREM Let $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ be a finite near–linear space and let $\mathcal{P} = \{P_i \mid i = 1, \ldots, v\}$ and $\mathcal{L} = \{L_j \mid j = 1, \ldots, b\}$ then

$$\sum_{i=1}^{v} |P_i| = \sum_{j=1}^{b} |L_j|.$$

Proof: Fix line L_j. The value, $|L_j|$, counts the number of 1's in column j of the incidence matrix of the near–linear space. If we sum over all possible columns, that is $\sum_{j=1}^{b} |L_j|$, we get the total number of 1's in the incidence matrix.

Similarly fix P_i. The value $|P_i|$ counts the number of 1's in row i. If we sum over all possible rows, that is $\sum_{i=1}^{v} |P_i|$, we get the total number of 1's in the incidence matrix.

The result now follows.

Given any matrix of 0's and 1's when does it represent a near–linear space?

We associate the rows of the matrix with the points and the columns with the lines and check that Axioms NL1 and NL2 are satisfied.

Axiom NL1 implies that each column must have at least two 1's in it.

Axiom NL2 implies there is at most one column j such that a_{ij} and a_{kj} are both 1 for distinct i and k. That is, the inner product of row i with row k of the incidence matrix must be less than or equal to 1.

A near–linear space is said to be **point regular** if $|\mathcal{P}| \geq 1$ and each point is on the same number of lines. This number is called the **point regularity** and denoted by r. That is, a near–linear space is said to be **point regular** if

$$|P_i| = r \text{ for all } i = 1, \ldots, v.$$

A near–linear space is said to be **line regular** if $|\mathcal{L}| \geq 1$ and each line is on the same number of points. This number is called the **line regularity** and denoted by k. That is, a near–linear space is said to be **line regular** if

$$|L_i| = k \text{ for all } i = 1, \ldots, b.$$

Lemma If a near-linear space $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ is point and line regular, with $\mid P_i \mid = r$, $\forall i \in \{1, \ldots, v\}$ and $\mid L_j \mid = k$, $\forall j \in \{1, \ldots, b\}$ then

$$vr = bk.$$

An incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ is said to be **symmetric** if

$$|\mathcal{P}| = |\mathcal{L}|.$$

Let P_i be a point not on the line L_j. Define the **connection number** of P_i with L_j to be the number of points on L_j joined by a line to P_i.

Because of Axiom NL2 this is the same as the number of lines on P_i which meet L_j.

If P_i is on L_j the connection number is defined to be 1.

LEMMA For any point P_i and line L_j the connection number is less than or equal to the number of points on the line L_j.

Proof: If P_i is not on L_j it follows easily from Axiom NL2. If P_i is on L_j it follows from Axiom NL1.

If P_i is a point not on a line L_j then the number of lines on P_i missing L_j is $|P_i|$ minus the connection number for P_i with L_j.