LINEAR SPACES

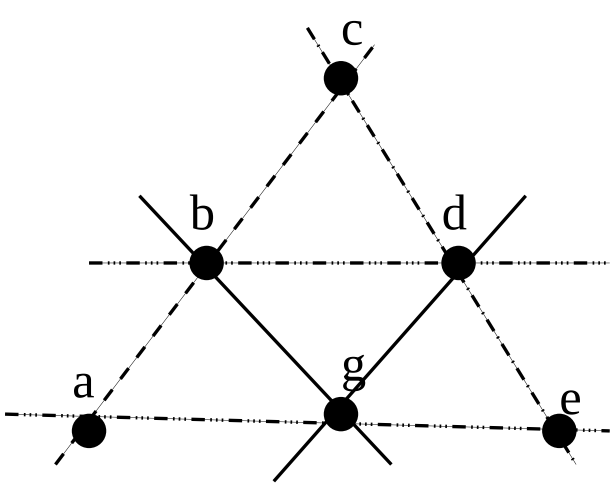
Define a **linear space** to be a near–linear space in which any two points are on a line.

A **linear space** is an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ such that

Axiom LS1: any line is incident with at least two points, and

Axiom LS2: any two points are on precisely one line.

Let S be a near–linear space where $\mathcal{P} = \{a, b, c, d, e, g\}$ and $\mathcal{L} = \{\{b, d\}, \{b, g\}, \{g, d\}, \{a, b, c\}, \{a, g, e\}, \{e, c, d\}\}$.



Then S is not a linear space since for example there is no line containing the points c and g.

BLOCK DESIGNS

A linear space is an example of a partially balanced design. That is a **partially balanced design** with index 1 is a finite set $V = \mathcal{P}$ and a collection, $\mathcal{B}$, of subsets of V called **blocks** (equivalent to lines) which satisfy the following condition:

- each pair of elements of V occurs in precisely one block.

More generally, a **partially balanced design** is a finite set $V = \mathcal{P}$ and a collection, $\mathcal{B}$, of subsets of V called blocks which satisfy the following condition:

- each pair of elements of V occurs in precisely λ blocks.

Example Let $V = \{1, 2, 3, 4\}$ and $\mathcal{B} = \{\{1, 2, 3\}, \{1, 4\}, \{1, 3, 4\}, \{1, 2\}, \{2, 3, 4\}, \{2, 4\}, \{3, 4\}\}$. Here $\lambda = 2$.

LEMMA Let S be a near–linear space with $|\mathcal{L}| \geq 1$. Take any point P_i and any line L_j such that P_i is not incident with L_j. If the connection number for P_i with L_j is $|L_j|$, then S is a linear space.

Proof Since $|\mathcal{L}| \geq 1$ there is at least one line say L_k. It must be shown that any two points P and Q are on a line.

If they are both on L_k we are finished.

However there are two other possibilities.

1) Assume P is on L_k and Q is not. The connection number for Q and L_k is $|L_k|$ so every point of L_k is collinear with Q.

2) Assume that neither P or Q are on L_k. Once again since the connection number of P with L_k is $|L_k|$ there exists a point U on L_k such that PU is a line of the space. Now if Q is on this line we are finished. If not we use the same argument and so each point of PU must be on a line with Q. Hence there is a line on P and Q and we are finished.

LEMMA In a near–linear space

$$\sum_{i=1}^{v} |P_i|(|P_i| - 1) \leq b(b-1).$$

PROOF: Recall that $|P_i|$ represents the number of lines on point P_i and b the total number of lines. There are $|P_i|(|P_i| - 1)$ ordered pairs of lines on the point P_i. So the left hand side of the inequality counts the number of ordered pairs of intersecting lines. Clearly there are $b(b-1)$ ordered pairs of lines, so the inequality holds.

LEMMA In a near–linear space any two lines meet if and only if

$$\sum_{i=1}^{v} |P_i|(|P_i| - 1) = b(b-1).$$

THEOREM S is a linear space if and only if

$$\sum_{j=1}^{b} |L_j|(|L_j| - 1) = v(v-1).$$

Proof: See problem sheets.

THEOREM Let $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ be any finite linear space with $b > 1$. Then $b \geq v$.

OUTLINE OF PROOF: We want to show that $b \geq v$, and we will argue by contradiction. It will be shown that we can label the lines $L_1, \ldots, L_b$, and identify a subset of distinct points $P_1, \ldots, P_b$ such that $|L_j| \leq |P_j|$ for $1 \leq j \leq b$.

Consequently

$$\sum_{j=1}^{b} |L_j| \leq \sum_{j=1}^{b} |P_j|.$$

But we know (see slide 19 of near–linear space notes) that

$$\sum_{j=1}^{v} |P_j| = \sum_{j=1}^{b} |L_j| \leq \sum_{j=1}^{b} |P_j|,$$

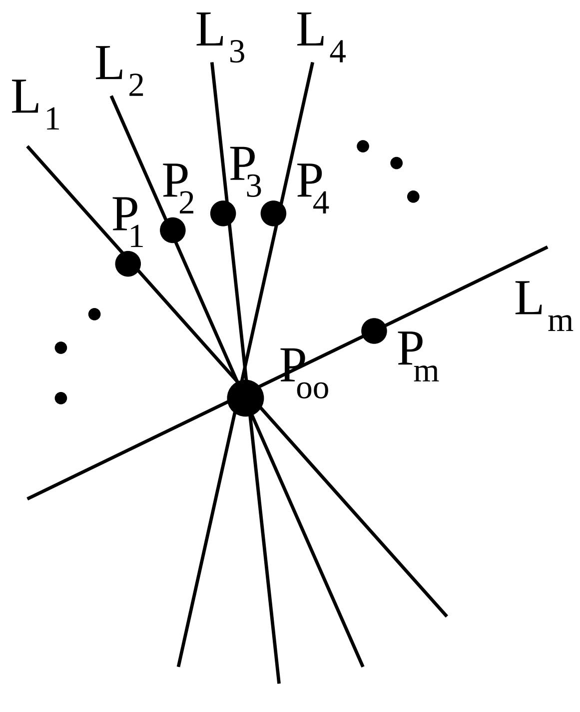
which is impossible if $b < v$, giving a contradiction. So it will follow that $b \geq v$.

PROOF: Assume $b < v$ and proceed as follows.

Let $m = min\{|P_i| \mid 1 \leq i \leq v\}$. Note by Axiom L2 $m > 1$, otherwise Axiom L1 would be violated or all points would be collinear.

Let P_∞ be a point with m lines through it and label these lines $L_1, \ldots, L_m$. Note $\cup_{x=1}^{m} L_x = \mathcal{P}$.

By Axiom L1, for $1 \leq i \leq m$, there exists $P_i \in L_i$ such that $P_i \neq P_\infty$. Also Axiom L2 implies $P_i \notin L_j$, for $i \neq j$.



So $|P_i| \geq |L_{i+1}|$, for $1 \leq i \leq m-1$ and $|P_m| \geq |L_1|$.

As $|P_\infty|$ is minimal, for all points P and for all lines L not on P_∞, we have

$$|L| \leq |P_\infty| \leq |P|.$$

For any L_j, $m+1 \leq j \leq b$, we know $P_\infty \notin L_j$.

So $|L_j| \leq |P_\infty| \leq |P|$ for any point P. Since we have assumed $b < v$ then $|L_j| \leq |P_j|$ for any $m+1 \leq j \leq b$. Therefore we have

$|L_1| \leq |P_m|$

$|L_2| \leq |P_1|$

$|L_3| \leq |P_2|$

$\vdots$

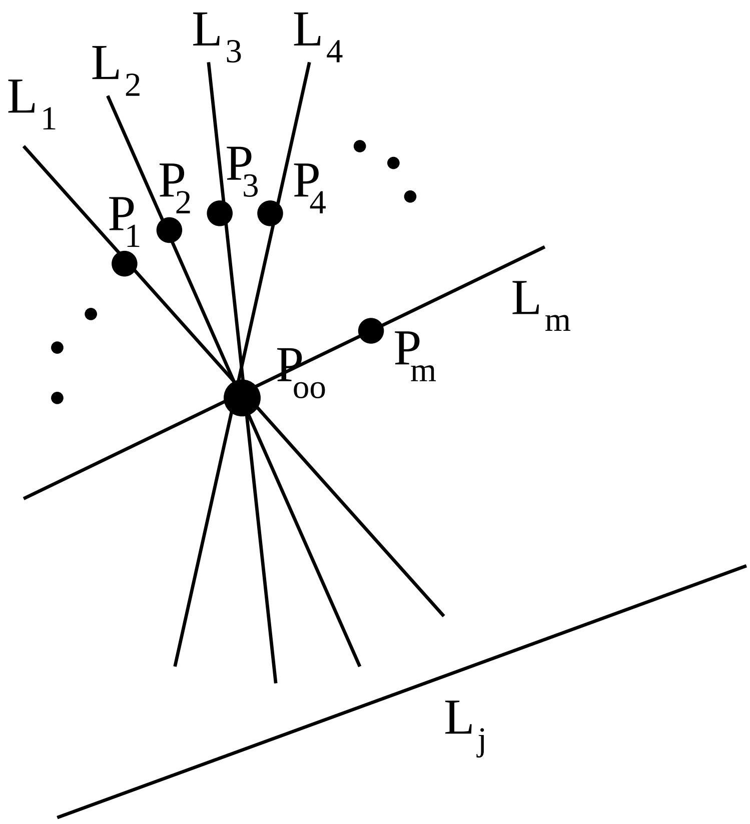
$|L_m| \leq |P_{m-1}|$

$|L_{m+1}| \leq |P_{m+1}|$

$|L_{m+2}| \leq |P_{m+2}|$

$\vdots$

$|L_b| \leq |P_b|$



Thus

$$
\begin{aligned}
\sum_{j=1}^{m} |L_j| + \sum_{j=m+1}^{b} |L_j| &\leq \sum_{i=1}^{m} |P_i| + \sum_{i=m+1}^{b} |P_i| \\
\sum_{j=1}^{b} |L_j| &\leq \sum_{i=1}^{b} |P_i| \leq \sum_{i=1}^{v} |P_i|.
\end{aligned}
$$

But $\displaystyle\sum_{i=i}^{v} |P_i| = \sum_{j=1}^{b} |L_j|.$

$$
\text{So, } \sum_{i=1}^{v} |P_i| \leq \sum_{i=1}^{b} |P_i| \leq \sum_{i=1}^{v} |P_i|
$$

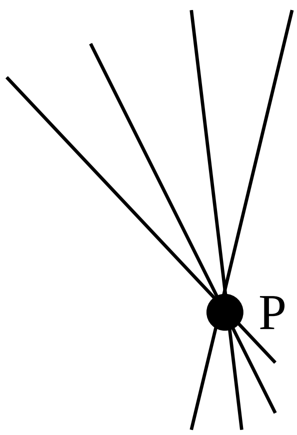
But this contradicts the fact that $v > b$.

Hence $b \geq v$.

If a linear space is point regular is it necessarily line regular?

LEMMA If a linear space is line regular then it is point regular.

Proof: Fix a point P. All the other points are joined to P by a line and these lines cover all points.



So if there are m lines through point P they all have say $k+1$ points on them. So we have

$$m((k+1)-1) = v-1 \text{ or } mk = v-1.$$

But v and k are independent of the point chosen so m must be independent of the point chosen and so the linear space is point regular.

LEMMA In a linear space $\mathcal{I} = (\mathcal{P}, \mathcal{L})$, where $|\mathcal{L}| \geq 2$, prove that $|P| \geq 2$, for all $P \in \mathcal{P}$.

Proof: Since $|\mathcal{L}| \geq 2$ there exists lines L and M and by the axioms of a linear space $|L| \geq 2$ and $|M| \geq 2$. So $|\mathcal{P}| \geq 2$. Let $P, Q \in L$. Take P.

1) If $P \in M$, then $|P| \geq 2$.

2)If $Q \in M$, then there exists $R \in M$ and $R \neq P$ and $R \notin L$. But R and P must be on a line so $|P| \geq 2$.

3) If $P, Q \notin M$, then there exists R and S in M, which gives $PR \neq PS$, so $|P| \geq 2$.

Let $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ be a linear space, with $|\mathcal{L}| \geq 2$. Then the **dual** of the linear space $\mathcal{I}$ is defined to be the incidence structure $\mathcal{J} = (\mathcal{P}^d, \mathcal{L}^d)$ where

$$
\begin{aligned}
\mathcal{P}^d &= \mathcal{L}, \\
\mathcal{L}^d &= \{\{L_1, \ldots, L_m\} \mid L_i \in \mathcal{P}^d, \text{ and } L_1, \ldots, L_m \\
&\qquad \text{are all concurrent at a point}\}.
\end{aligned}
$$

That is, the *dual* of a linear space is an incidence structure where

- the points of the dual correspond to lines in $\mathcal{I}$,

- lines of the dual correspond to point of $\mathcal{I}$, and

- L is on P in the dual if and only if point P is on line L in $\mathcal{I}$.

Lemma Let $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ be a linear space, with $|\mathcal{L}| \geq 2$. Then the dual of $\mathcal{I}$ is a near-linear space.

Proof: We need to show that in the dual space Axioms NL1 and NL2 are satisfied.

To prove Axiom NL1 for the dual space we need to show that any point in $\mathcal{I}$ is incident with at least two lines. That is, for all $P \in \mathcal{P}$, $|P| \geq 2$. However this follows from the previous lemma.

To prove Axiom NL2 we argue by contradiction. Assume it is not the case. That is, there exists two lines in the dual space which intersect in at least two point. The two lines in the dual space correspond to points, say P, Q, in $\mathcal{P}$ and if the lines in the dual intersect in two or more points then there exists lines L and M in $\mathcal{L}$ such L and M are both incident with P and Q. However, this contradicts the fact that $\mathcal{I}$ is a linear space.

Let $\mathcal{I}_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $\mathcal{I}_2 = (\mathcal{P}_2, \mathcal{L}_2)$ be two linear spaces. Let ϕ be a one to one and onto mapping such that

$$\phi : \mathcal{P}_1 \to \mathcal{P}_2.$$

Note that since ϕ is one to one and onto, ϕ^{-1} exists.

For a line $L_1 \in \mathcal{L}_1$, we use the notation $\phi(L_1)$ to represent the set of points $\{\phi(x) \mid x \in L_1\}$, and correspondingly when $L_2 \in \mathcal{L}_2$, $\phi^{-1}(L_2) = \{\phi^{-1}(x) \mid x \in L_2\}$.

The mapping ϕ is said to be an **isomorphism** from $\mathcal{I}_1$ to $\mathcal{I}_2$, if

$$\forall L \in \mathcal{L}_1, \phi(L) \in \mathcal{L}_2 \text{ and } \forall L \in \mathcal{L}_2, \phi^{-1}(L) \in \mathcal{L}_1$$

Example: Let $\mathcal{I}_1 = (\mathcal{P}_1, \mathcal{L}_1)$, $\mathcal{I}_2 = (\mathcal{P}_2, \mathcal{L}_2)$ and $\mathcal{I}_3 = (\mathcal{P}_2, \mathcal{L}_3)$, where

$$\begin{aligned}
\mathcal{P}_1 &= \{1, 2, 3\}, \\
\mathcal{P}_2 &= \{a, b, c\}, \\
\mathcal{L}_1 &= \{\{1, 2, 3\}\}, \\
\mathcal{L}_2 &= \{\{a, b, c\}\}, \text{ and} \\
\mathcal{L}_3 &= \{\{a, b\}, \{a, c\}, \{b, c\}\}.
\end{aligned}$$

It can be shown that $\mathcal{I}_1$ and $\mathcal{I}_2$ are isomorphic but $\mathcal{I}_1$ and $\mathcal{I}_3$ are not. (Take the mapping $\phi(1) = a$, $\phi(2) = b$ and $\phi(3) = c$.)

BLOCK DESIGNS

Recall that given a near-linear space which has point regularity r and line regularity k, then $vr = bk$.

Also in an earlier slide we defined a partially balanced design as a design in which each pair of elements of V occurs in λ blocks.

We will now consider partially balanced designs for which each block has constant size k. Such designs are called balanced incomplete block designs.

A **balanced incomplete block design** is a set V together with a collection of subset of V which satisfy the conditions:

– each block is of size k

– each pair of element of V occurs in precisely λ blocks.

We use the following conventions:

v	denotes	$\lvert V \rvert$, number of elements
$\mathcal{B}$	denotes	the collection of blocks
b	denotes	$\lvert \mathcal{B} \rvert$, number of blocks
k	denotes	the size of a block
$BIBD$	denotes	a balanced incomplete block design

Lemma Let $(V, \mathcal{B})$ be a BIBD. For all $x \in V$, x occurs in a constant number of blocks. This number is called the **replication number** and denoted r. Further,

$$\lambda(v-1) = r(k-1).$$

Proof: For a given $x \in V$ let r_x denote the number of block which contains element x. Each such block contains $k-1$ elements distinct from x. Hence these r_x blocks contain $r_x(k-1)$ pairs of the form xw, where $w \in V \setminus \{x\}$.

Also for each $w \in V \setminus \{x\}$, the pair xw occurs in λ blocks. Hence the number of such pairs is $\lambda(v-1)$. Equating these quantities gives:

$$r_x(k-1) \;=\; \lambda(v-1).$$

But k, λ, v are all independent of x, hence r_x is independent of x and so $r(k-1) = \lambda(v-1)$.

Lemma For a BIBD,

$$vr = bk.$$

Proof: We count the occurrence of elements in blocks of the BIBD in two different ways. First we note that each of the v elements occurs in r blocks, so in total we have vr occurrence of the elements in the blocks of the design. However, each of the b blocks contains k elements, hence in total we have bk occurrence of elements in blocks. And so

$$vr = bk.$$

Lemma (Fisher's inequality) For a BIBD

$$v \leq b.$$

The parameters of a BIBD are written (v, b, r, k, λ).

Blocks	Par. Bal.	BIBD	v	b	r	k	λ
1 2 3 2 5 6 3 4 6 1 4 5 1 6 3 5 2 4							
1 2 3 2 4 5 3 4 6 1 5 6 1 4 7 2 7 6 3 5 7							
1 2 3 4 1 2 5 6 3 4 5 6							
1 2 3 1 4 2 1 4 6 1 5 6 1 3 5 2 3 6 2 4 5 2 5 6 3 4 5 3 4 6							

Let $(V_1, \mathcal{B}_1)$ and $(V_2, \mathcal{B}_2)$ be two BIBDs. Let

$$f : V_1 \to V_2$$

be a one-to-one and onto function. For each block $B \in \mathcal{B}_1$ define $f(B) = \{f(x) \mid x \in B\}$. The BIBD $(V_1, \mathcal{B}_1)$ is said to be **isomorphic** to $(V_2, \mathcal{B}_2)$, if for all $B \in \mathcal{B}_1$

$$f(B) \in \mathcal{B}_2.$$

That is we can obtain one BIBD from the other by systematically relabelling the elements of the blocks.

If a function ρ maps a BIBD to itself then we say ρ is an **automorphism**.

Example

1 2 3	4 2 3	1 2 3	5 7 3
2 4 5	2 6 5	2 3 5	7 4 1
3 4 6	3 6 7	3 4 2	3 4 6
1 5 6	4 6 7	1 5 6	5 1 6
1 4 7	4 4 1	1 5 7	5 4 2
2 6 7	2 7 1	2 6 1	7 6 2
3 5 7	3 5 1	3 5 7	3 1 2

A balanced incomplete block design $(\mathcal{V}, \mathcal{B})$ may be represented by a $|\mathcal{V}| \times |\mathcal{B}|$ **incidence matrix** $A = [a_{i,j}]$, with row i corresponding to an element $x_i \in \mathcal{P}$ and column j corresponding to block $B_j \in \mathcal{B}$ and

$$a_{i,j} = \begin{cases} 0 & if \quad x_i \notin B_j \\ 1 & if \quad x_i \in B_j. \end{cases}$$

Note $a_{i,j} \in \{0, 1\}$. It is the entry in the i^{th} and j^{th} column of A and represents the occurrence, or not, of element x_i in block B_j.

Lemma Let A be an incidence matrix from a BIBD $(\mathcal{V}, \mathcal{B})$ with parameter set (v, b, r, k, λ). Let $\mathcal{V} = \{x_1, \ldots, x_v\}$ and $\mathcal{B} = \{B_1, \ldots, B_b\}$, where row i of A corresponds to element $x_i \in \mathcal{V}$ and column j of A corresponds to block $B_j \in \mathcal{B}$. For all $1 \leq i < h \leq v$ and $1 \leq j < g \leq b$

$$\sum_{s=1}^{b} a_{is}a_{hs} = \lambda$$

$$\sum_{s=1}^{v} a_{sg}a_{sj} = |B_g \cap B_j|$$

Proof: See problem sheet.