FINITE PROJECTIVE PLANES

An incidence structure $\pi = (\mathcal{P}, \mathcal{L})$ is said to be a **finite projective plane** if the following axioms are satisfied.

Axiom PP1 Every pair of distinct points are joined by exactly one line.

Axiom PP2 Every pair of distinct lines meet on at least one point.

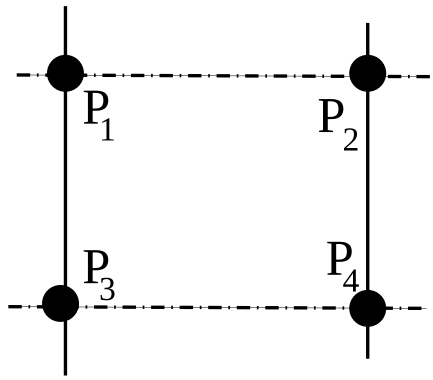
Axiom PP3 There are four points, no three of which are collinear.

Axiom PP4 $\mathcal{P}$ is a finite set.

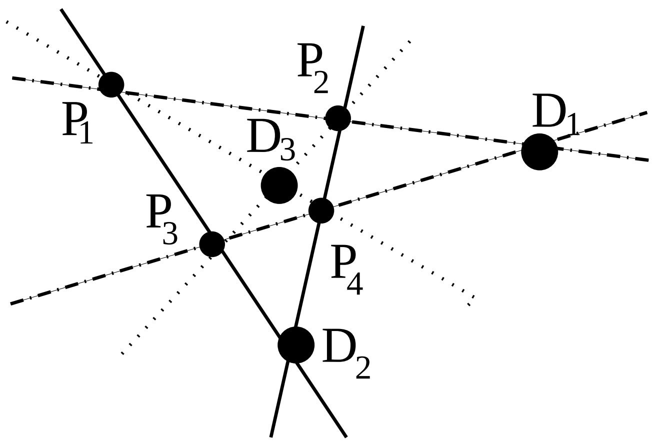
Unless otherwise stated π will denote a projective plane and it will be assumed that the set of points is denoted by $\mathcal{P}$ and the set of lines by $\mathcal{L}$.

A set of four points of a projective plane no three of which are collinear, is called a **quadrangle** and usually denoted $\mathcal{Q}$.

Let $\mathcal{Q} = \{P_1, P_2, P_3, P_4\}$.



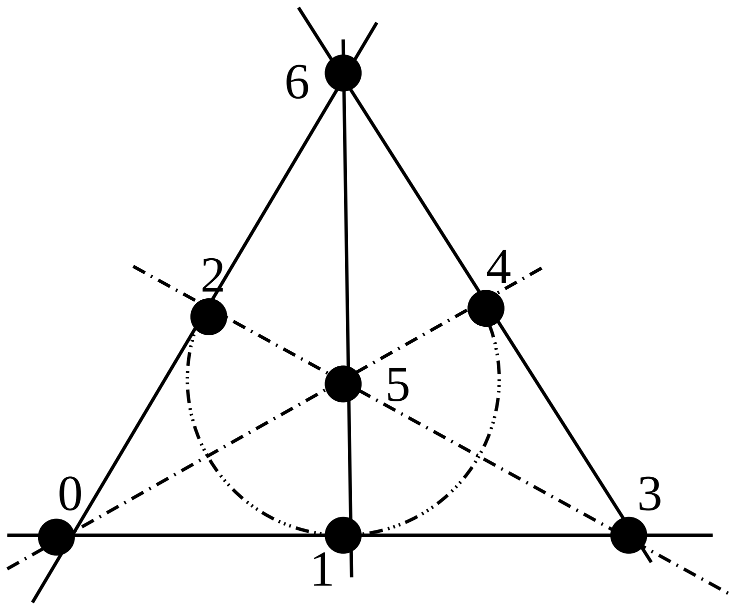
The points $D_1 = P_1P_2 \cap P_3P_4$, $D_2 = P_1P_3 \cap P_2P_4$ and $D_3 = P_1P_4 \cap P_2P_3$ are called the **diagonal elements** of $\mathcal{Q}$.



Axiom PP3 asserts that a projective plane must possess a quadrangle.

EXAMPLE Let $\mathcal{P} = \{0, 1, \ldots, 6\}$ and $\mathcal{L} = \{\{0, 1, 3\},$ $\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}$.

FANO PLANE



LEMMA Let π be finite projective plane. Every pair of distinct lines of π meet at exactly one point of π.

Proof: Suppose that there exists two distinct lines L and M and a pair of distinct points P and Q both on L and M. This is clearly impossible since P and Q are joined by two distinct lines and this contradicts Axiom PP1. Finally by Axiom PP2 every pair of distinct lines meets in at least one point and so the result follows.

LEMMA Let π be a finite projective plane. There are four lines no three of which are concurrent.

Proof: By Axiom PP3, π possesses a quadrangle $\mathcal{Q} = \{P_1, P_2, P_3, P_4\}$. Let $L_{ij} = P_iP_j$, $\forall i, j,\ i \neq j$. Take the set of 4 lines $\{L_{12}, L_{23}, L_{34}, L_{41}\}$. WLOG, suppose that the lines L_{12}, L_{23}, L_{34} are concurrent. Since L_{12}, L_{23} meet at P_2 these three lines are concurrent at P_2. But this contradicts the fact that $\mathcal{Q}$ is a quadrangle. It follows that this is a set of 4 lines no 3 of which are concurrent.

The dual of a projective plane $\pi = (\mathcal{P}, \mathcal{L})$ is the incidence structure $\sigma = (\mathcal{P}^d, \mathcal{L}^d)$ where

$$\begin{aligned} \mathcal{P}^d &= \mathcal{L}, \\ \mathcal{L}^d &= \{\{L_1, \ldots, L_m\} \mid L_i \in \mathcal{P}^d, \text{ and } L_1, \ldots, L_m \\ &\qquad \text{are all concurrent}\} \end{aligned}$$

The **dual** of $\pi = (\mathcal{P}, \mathcal{L})$ is the incidence structure $\pi^d = (\mathcal{L}, \mathcal{P})$ where the points in π^d correspond to lines in π and lines in π^d correspond to point in π, and L is on P in π^d if and only if point P is on line L in π.

THEOREM The dual of a finite projective plane is a finite projective plane.

Proof: One can use the previous two lemmas to show that the dual of a finite projective plane satisfies Axioms PP1, PP2 and PP3.

Let π be a finite projective plane with point set $\mathcal{P}$ and line set $\mathcal{L}$. Let $P(L)$ denote the set of points on line L and define a map ϕ

$$\phi : \quad L \to P(L).$$

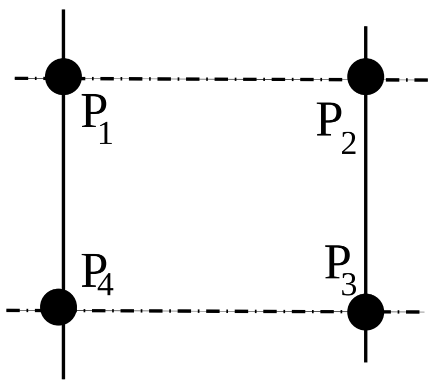
Then $\{P(L) \mid L \in \mathcal{L}\}$ is a set of subsets of $\mathcal{P}$. Also since any two lines which contain two common points are in fact equal, ϕ is a 1–1 mapping from $\mathcal{L}$ to $\{P(L) \mid L \in \mathcal{L}\}$. The set $\{P(L) \mid L \in \mathcal{L}\}$ is finite, thus $\mathcal{L}$ must be finite. Since the points set in the dual is $\mathcal{L}$ the dual space satisfies Axiom PP4.

This is a powerful result as it says that if a statement follows from Axioms PP1, PP2, PP3, and PP4, then the dual of this statement will also follow.

As an example one may look at the next theorem and its corollary.

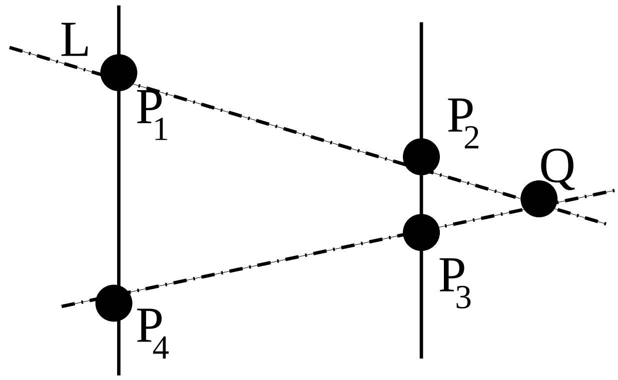
THEOREM There are at least three points on each line of a projective plane π.

Proof Let $\mathcal{Q} = \{P_1, P_2, P_3, P_4\}$ be a quadrangle of π and L a line of π.

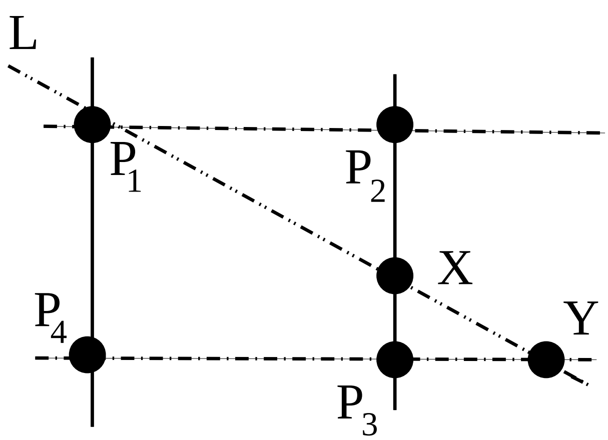


There are three cases to consider. Case C1. L is through two points of $\mathcal{Q}$, Case C2. L is through one point of $\mathcal{Q}$, and Case C3. L is not through any points of $\mathcal{Q}$.

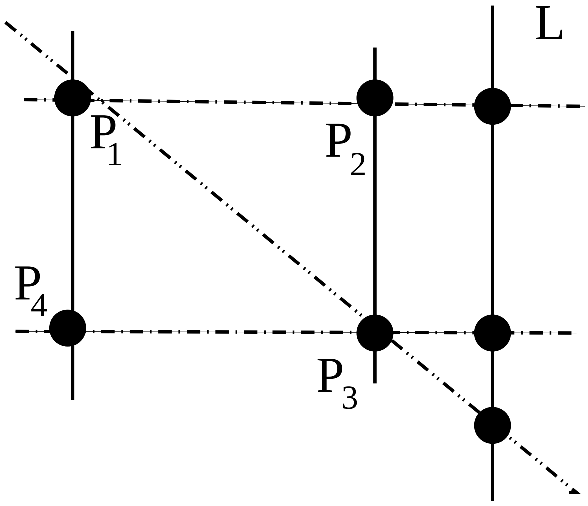
C1 *L is through two points of $\mathcal{Q}$.* WLOG assume $L = P_1P_2$. Then the line P_3P_4 must meet L at a point distinct from P_1 and P_2. Hence there are at least three points on L.



C2 *L is through one point of $\mathcal{Q}$.* WLOG assume P_1 is the only point of $\mathcal{Q}$ on L. Let $X = L \cap P_2P_3$ and $Y = L \cap P_3P_4$. Since $\mathcal{Q}$ is a quadrangle $X \neq P_1$ and $Y \neq P_1$. Now if $X = Y$, then P_2P_3 and P_3P_4 are two lines through each of X and P_3, a contradiction. So we have P_1, Y and X all distinct points and all on L.

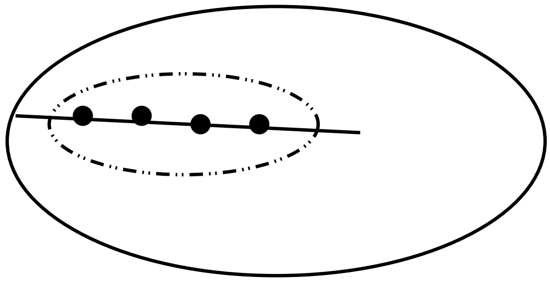


C3 *L is through no point of $\mathcal{Q}$.* In this case P_1P_i, $i = 2, 3, 4$ meet L at three distinct points. Hence L contains three distinct points.

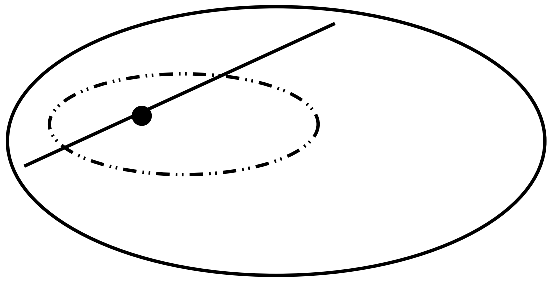


COROLLARY There are at least three lines through each point of a projective plane π.

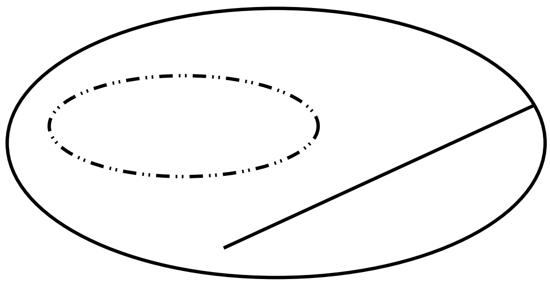
Let $\mathcal{P}_0 \subseteq \mathcal{P}$. A line L is said to be a $\mathcal{P}_0$–**secant** if L is through **more than one point** of $\mathcal{P}_0$ and the set of such lines is denoted $\mathcal{P}_0(s)$.



A line L is said to be a $\mathcal{P}_0$–**tangent** if L is through **precisely one point** of $\mathcal{P}_0$ and the set of such lines is denoted $\mathcal{P}_0(t)$.



A line L is said to be a $\mathcal{P}_0$–**exterior line** if L is through **no points** of $\mathcal{P}_0$ and the set of such lines is denoted $\mathcal{P}_0(e)$.

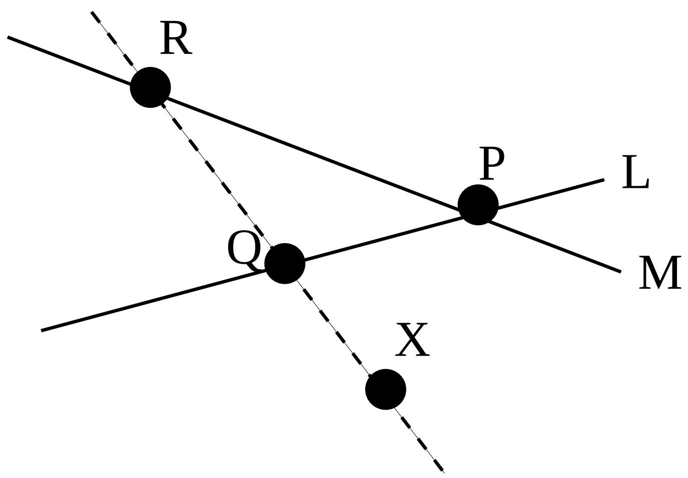


Corollary Let $\pi = (\mathcal{P}, \mathcal{L})$ be a finite projective plane. Then π is a linear space.

Proof: We must show that Axioms L1 and L2 are satisfied. The fact that each line contains a least three points implies Axiom L1, and Axiom PP1 implies Axiom L2 (any two points are on precisely one line).

LEMMA Let π be a finite projective plane and let $L, M \in \mathcal{L}$. Then there is a point X of $\mathcal{P}$ which is on neither L or M.

Proof: Let $P = L \cap M$. Also let Q and R be points on L and M respectively. Since there are least three point on each line of π there is a third point X on the line QR. Now since we chose $QR \neq L$, X is not on L and similarly X is not on M.



THEOREM Let π be a finite projective plane. Then $|L| = |M|$ for all $L, M \in \mathcal{L}$.

Proof: Consider $L, M \in \mathcal{L}$ and let $\{P_1, \ldots, P_L\}$ denote the set of points on L. By the previous lemma we know there is a point P on neither L or M. Let

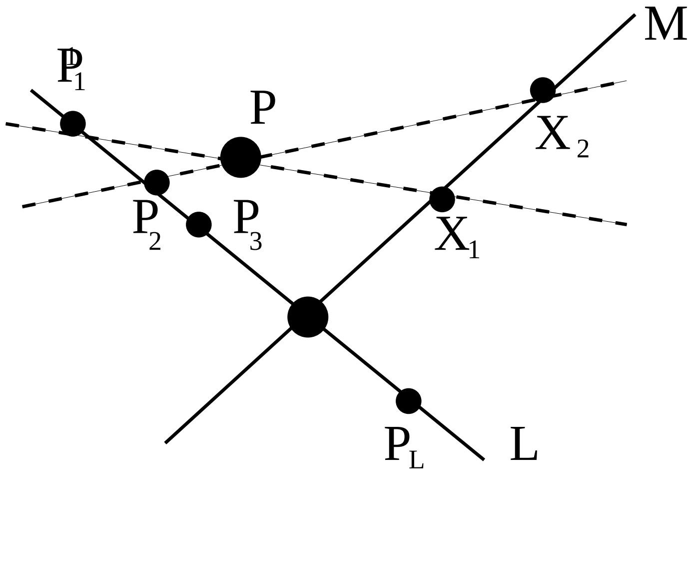
$$PP_1 \cap M = X_1$$

and in general, for $i = 1, \ldots, L$,

$$PP_i \cap M = X_i.$$

Now if $X_i = X_j$, where $i \neq j$, then P_i and P_j are both on $PX_i = PX_j$ and this line must be L. But this is a contradiction. So $X_i \neq X_j$ for all $i \neq j$. It follows that $|M| \geq |L|$. But then when we interchange L and M in the above argument we get $|L| \geq |M|$. So

$$|L| = |M|.$$



Corollary Let π be a finite projective plane. Then $|P| = |Q|$ for all $P, Q \in \mathcal{P}$.

THEOREM Let π be a finite projective plane. Then there is an integer $n \geq 2$ such that

$|P| = n + 1$ for all $P \in \mathcal{P}$,

$|L| = n + 1$ for all $L \in \mathcal{L}$ and

$|\mathcal{P}| = |\mathcal{L}| = n^2 + n + 1$.

(Note the integer n is said to be the **order** of the projective plane π.)

Proof: From the previous theorem and corollary we know there exists integers n_l and n_p such that $|L| = n_l + 1$ for all $L \in \mathcal{L}$ and $|P| = n_p + 1$ for all $P \in \mathcal{P}$. And since there are at least three points on a line and at least three lines through a point $n_l \geq 2$ and $n_p \geq 2$. Thus we need to show that $|P| = |L|$ for some $P \in \mathcal{P}$ and some $L \in \mathcal{L}$.

Take a point $P \in \mathcal{P}$ not on L, where $L \in \mathcal{L}$. Let $L = \{P_1, \dots P_l\}$, where $l = n_l + 1$. If there exists $i \neq j$ such that

$$PP_i = PP_j,$$

then $PP_i = PP_j = L$, and P is on L which is a contradiction. So for all $i \neq j$

$$PP_i \neq PP_j.$$

It follows that $|L| \leq |P|$. But then the dual argument shows that $|P| \leq |L|$.

Finally, consider $P \in \mathcal{P}$ and let Q be a point distinct from P. Take the set

$$\{(T, L) \mid (T \in \mathcal{P} \setminus P) \wedge (L = TP)\}.$$

Note that it contains $n(n+1)$ elements. Since for each $Q \in \mathcal{P} \backslash P$ there exists a unique line PQ through P and Q, we may define a mapping ϕ as:

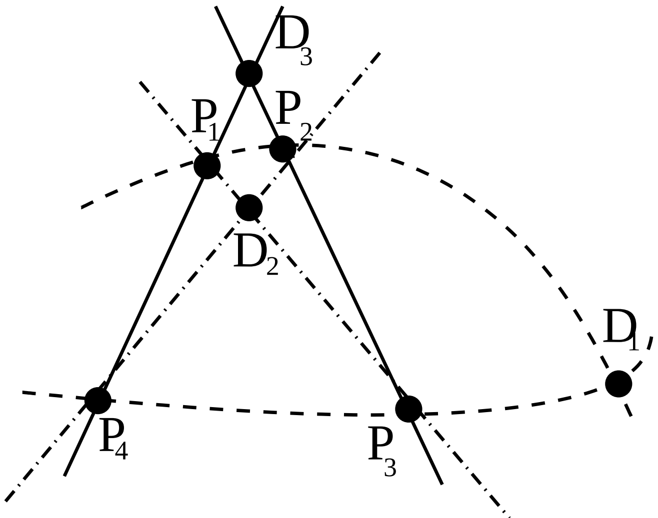
$$\begin{aligned} \phi \ : \ & \mathcal{P} \setminus P \rightarrow \{(T, L) \mid (T \in \mathcal{P} \setminus P) \wedge (L = TP)\} \\ & Q \rightarrow (Q, PQ) \end{aligned}$$

However it is easy to show that ϕ is in fact 1–1 and onto. So $|\mathcal{P} \setminus P| = n^2 + n$ and $|\mathcal{P}| = n^2 + n + 1$. The fact that $|\mathcal{L}| = n^2 + n + 1$ follows by duality.

LEMMA The diagonal elements of any quadrangle of a projective plane π of order 2 are collinear.

Proof: Let $\mathcal{Q} = \{P_1, P_2, P_3, P_4\}$ be a quadrangle in π. By definition the diagonal elements are

$$\begin{aligned} D_1 &= P_1P_2 \cap P_3P_4, \\ D_2 &= P_1P_3 \cap P_2P_4, \text{ and} \\ D_3 &= P_1P_4 \cap P_2P_3. \end{aligned}$$



The points $P_1, P_2, P_3, P_4, D_1, D_2, D_3$ are all distinct as are the lines $P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, P_3P_4$ This accounts for 7 points and 6 lines. If π is a projective plane of order 2. But we know that $|\mathcal{P}| = |\mathcal{L}| = 2^2 + 2 + 1 = 7$. So we need one more line and consequently it must pass through the points D_1, D_2, D_3.

Let $\pi_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $\pi_2 = (\mathcal{P}_2, \mathcal{L}_2)$ be two finite projective planes of the same order. Let ϕ be a mapping such that

$$\phi : \mathcal{P}_1 \to \mathcal{P}_2.$$

For a line $L \in \mathcal{L}_1$, we use the notation $\phi(L)$ to represent the set of points $\{\phi(x) \mid x \in L\}$.

Let π_1 and π_2 be two projective planes of the same finite order, and assume the mapping ϕ is onto. Then ϕ is said to be an **isomorphism** from π_1 to π_2, if

$$\forall L \in \mathcal{L}_1,\ \phi(L) \in \mathcal{L}_2.$$

Note that since π_1 and π_2 have the same order and ϕ is onto it follows that ϕ is one to one.

π_1 and π_2 are said to be **isomorphic**.

THEOREM There is a projective plane of order two which is unique up to isomorphism.

Proof: Let $\pi_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $\pi_2 = (\mathcal{P}_2, \mathcal{L}_2)$ be two projective planes of order 2. Both π_1 and π_2 contain quadrangles, label these quadrangles, respectively, $\mathcal{Q}_1 = \{P_{11}, P_{12}, P_{13}, P_{14}\}$ and $\mathcal{Q}_2 = \{P_{21}, P_{22}, P_{23}, P_{24}\}$.

The remaining points of the projective planes are the diagonal elements. Let D_{11}, D_{12}, D_{13} represent the diagonal points of $\mathcal{Q}_1$ be and D_{21}, D_{22}, D_{23} the diagonal points of $\mathcal{Q}_2$. So

$$\begin{aligned} \{D_{11}, D_{12}, D_{13}\} &= L_{1\infty} \in \mathcal{L}_1, \text{ and} \\ \{D_{21}, D_{22}, D_{23}\} &= L_{2\infty} \in \mathcal{L}_2 \end{aligned}$$

Let ϕ be a mapping from π_1 to π_2 where, for $i = 1, \ldots, 4$ and $j = 1, 2, 3$

$$\begin{aligned} \phi(P_{1i}) &= P_{2i}, \text{ and} \\ \phi(D_{1j}) &= D_{2j}. \end{aligned}$$

Note for all j, k where $j \neq k$,

$$\begin{aligned} \phi(P_{1j}P_{1k}) &= P_{2j}P_{2k}, \text{ and} \\ \phi(L_{1\infty}) &= L_{2\infty}. \end{aligned}$$

Hence ϕ is an isomorphism from π_1 to π_2.

COROLLARY

Any projective plane of order 2 is isomorphic to its dual.

An isomorphism from a finite projective plane to its **dual** is called a **correlation** of π. If a finite projective plane has a correlation, then π is said to be **self–dual**.

A projective plane of order 2 is self–dual.

Let χ be a correlation of a finite projective plane π.

If P is a point on a line L,

then $\chi(L)$ is on $\chi(P)$

EXMAPLE Take the following finite projective plane and its dual given below.

π	π^d
$\mathcal{P} = \{0, 1, \ldots, 6\}$	$\mathcal{P}' = \{L_0, \ldots, L_6\}$
$\mathcal{L}$	$\mathcal{L}'$
$L_0 = \{0, 1, 3\}$	$\{L_0, L_1, L_5\}$
$L_1 = \{1, 2, 4\}$	$\{L_3, L_5, L_6\}$
$L_2 = \{2, 3, 5\}$	$\{L_0, L_2, L_3\}$
$L_3 = \{3, 4, 6\}$	$\{L_0, L_4, L_6\}$
$L_4 = \{4, 5, 0\}$	$\{L_1, L_2, L_6\}$
$L_5 = \{5, 6, 1\}$	$\{L_2, L_4, L_5\}$
$L_6 = \{6, 0, 2\}$	$\{L_1, L_3, L_4\}$

Then χ from $\mathcal{P}$ to $\mathcal{P}'$ and $\mathcal{L}$ to $\mathcal{L}'$ defined as follows is a correlation of π onto π^d.

$\mathcal{P}$	$\chi(\mathcal{P})$	$\mathcal{L}$	$\chi(\mathcal{L})$
0	L_1	L_0	1
1	L_5	L_1	6
2	L_3	L_2	3
3	L_0	L_3	0
4	L_6	L_4	2
5	L_2	L_5	5
6	L_4	L_6	4

BLOCK DESIGNS

Lemma A projective plane of order n is a balanced incomplete block design with parameters $(v, b, r, k, \lambda) = (n^2+n+1, n^2+n+1, n+1, n+1, 1)$.

Proof: See problem sheet.

We say a balanced incomplete block design is **symmetric** if $v = b$.

Lemma If $v = b$ in a balanced incomplete block design then $r = k$.

Proof: See problem sheet.

Let $(V, \mathcal{B})$ be a BIBD. Assume $V = \{u_1, u_2, \ldots, u_v\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$. We construct the **dual design** $(V^d, \mathcal{B}^d)$ with $V^d = \{w_1, w_2, \ldots, w_b\}$ and $\mathcal{B}^d = \{C_1, C_2, \ldots, C_v\}$, where

$$w_i \in C_j \leftrightarrow u_j \in B_i.$$

Lemma Let $(V, \mathcal{B})$ be a BIBD, with incidence matrix A. Then A^T is the incidence matrix of the dual design.

Let $(V, \mathcal{B})$ be a BIBD, with parameters (v, b, r, k, λ). The BIBD is said to be **linked** if for all blocks $B_1, B_2 \in \mathcal{B}$, $|B_1 \cap B_2|$ is constant.

Lemma Let $(V, \mathcal{B})$ be a symmetric BIBD, with parameters (v, v, k, k, λ). Then the BIBD is linked and for all blocks $B_1, B_2 \in \mathcal{B}$,

$$|B_1 \cap B_2| = \lambda.$$

Proof: