FINITE PROJECTIVE PLANES

An incidence structure $\pi = (\mathcal{P}, \mathcal{L})$ is said to be a finite projective plane if the following axioms are satisfied.

**Axiom PP1** Every pair of distinct points are joined by exactly one line.

**Axiom PP2** Every pair of distinct lines meet on at least one point.

**Axiom PP3** There are four points, no three of which are collinear.

**Axiom PP4** $\mathcal{P}$ is a finite set.

Unless otherwise stated $\pi$ will denote a projective plane and it will be assumed that the set of points is denoted by $\mathcal{P}$ and the set of lines by $\mathcal{L}$. 
A set of four points of a projective plane no three of which are collinear, is called a **quadrangle** and usually denoted $Q$.

Let $Q = \{P_1, P_2, P_3, P_4\}$.

The points $D_1 = P_1P_2 \cap P_3P_4$, $D_2 = P_1P_3 \cap P_2P_4$ and $D_3 = P_1P_4 \cap P_2P_3$ are called the **diagonal elements** of $Q$.

Axiom PP3 asserts that a projective plane must possess a quadrangle.
EXAMPLE Let $\mathcal{P} = \{0, 1, \ldots, 6\}$ and $\mathcal{L} = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}$. 

FANO PLANE
**Lemma**  Let $\pi$ be finite projective plane. Every pair of distinct lines of $\pi$ meet at exactly one point of $\pi$.

**Proof:** Suppose that there exists two distinct lines $L$ and $M$ and a pair of distinct points $P$ and $Q$ both on $L$ and $M$. This is clearly impossible since $P$ and $Q$ are joined by two distinct lines and this contradicts Axiom PP1. Finally by Axiom PP2 every pair of distinct lines meets in at least one point and so the result follows.

**Lemma**  Let $\pi$ be a finite projective plane. There are four lines no three of which are concurrent.

**Proof:** By Axiom PP3, $\pi$ possesses a quadrangle $Q = \{P_1, P_2, P_3, P_4\}$. Let $L_{ij} = P_iP_j, \forall i, j, i \neq j$. Take the set of 4 lines $\{L_{12}, L_{23}, L_{34}, L_{41}\}$. WLOG, suppose that the lines $L_{12}, L_{23}, L_{34}$ are concurrent. Since $L_{12}, L_{23}$ meet at $P_2$ these three lines are concurrent at $P_2$. But this contradicts the fact that $Q$ is a quadrangle. It follows that this is a set of 4 lines no 3 of which are concurrent.
The dual of a projective plane $\pi = (\mathcal{P}, \mathcal{L})$ is the incidence structure $\sigma = (\mathcal{P}^d, \mathcal{L}^d)$ where

$$\mathcal{P}^d = \mathcal{L},$$
$$\mathcal{L}^d = \{\{L_1, \ldots, L_m\} \mid L_i \in \mathcal{P}^d, \text{ and } L_1, \ldots, L_m \text{ are all concurrent}\}$$

The dual of $\pi = (\mathcal{P}, \mathcal{L})$ is the incidence structure $\pi^d = (\mathcal{L}, \mathcal{P})$ where the points in $\pi^d$ correspond to lines in $\pi$ and lines in $\pi^d$ correspond to points in $\pi$, and $L$ is on $P$ in $\pi^d$ if and only if point $P$ is on line $L$ in $\pi$. 
**THEOREM**  The dual of a finite projective plane is a finite projective plane.

**Proof:** One can use the previous two lemmas to show that the dual of a finite projective plane satisfies Axioms PP1, PP2 and PP3.

Let \( \pi \) be a finite projective plane with point set \( \mathcal{P} \) and line set \( \mathcal{L} \). Let \( P(L) \) denote the set of points on line \( L \) and define a map \( \phi \)

\[
\phi : \quad L \rightarrow P(L).
\]

Then \( \{P(L) \mid L \in \mathcal{L}\} \) is a set of subsets of \( \mathcal{P} \).

Also since any two lines which contain two common points are in fact equal, \( \phi \) is a 1–1 mapping from \( \mathcal{L} \) to \( \{P(L) \mid L \in \mathcal{L}\} \). The set \( \{P(L) \mid L \in \mathcal{L}\} \) is finite, thus \( \mathcal{L} \) must be finite. Since the points set in the dual is \( \mathcal{L} \) the dual space satisfies Axiom PP4.
This is a powerful result as it says that if a statement follows from Axioms PP1, PP2, PP3, and PP4, then the dual of this statement will also follow.

As an example one may look at the next theorem and its corollary.

**THEOREM** There are at least three points on each line of a projective plane \( \pi \).

Proof Let \( Q = \{ P_1, P_2, P_3, P_4 \} \) be a quadrangle of \( \pi \) and \( L \) a line of \( \pi \).

There are three cases to consider. Case C1. \( L \) is through two points of \( Q \), Case C2. \( L \) is through one point of \( Q \), and Case C3. \( L \) is not through any points of \( Q \).
C1  *L is through two points of Q*. WLOG assume \( L = P_1P_2 \). Then the line \( P_3P_4 \) must meet \( L \) at a point distinct from \( P_1 \) and \( P_2 \). Hence there are at least three points on \( L \).

C2  *L is through one point of Q*. WLOG assume \( P_1 \) is the only point of \( Q \) on \( L \). Let \( X = L \cap P_2P_3 \) and \( Y = L \cap P_3P_4 \). Since \( Q \) is a quadrangle \( X \neq P_1 \) and \( Y \neq P_1 \). Now if \( X = Y \), then \( P_2P_3 \) and \( P_3P_4 \) are two lines through each of \( X \) and \( P_3 \), a contradiction. So we have \( P_1 \), \( Y \) and \( X \) all distinct points and all on \( L \).
C3 *L is through no point of Q.* In this case $P_1P_i, i = 2, 3, 4$ meet $L$ at three distinct points. Hence $L$ contains three distinct points.

**COROLLARY** There are at least three lines through each point of a projective plane $\pi$. 
Let $\mathcal{P}_0 \subseteq \mathcal{P}$. A line $L$ is said to be a $\mathcal{P}_0$–secant if $L$ is through more than one point of $\mathcal{P}_0$ and the set of such lines is denoted $\mathcal{P}_0(s)$.

A line $L$ is said to be a $\mathcal{P}_0$–tangent if $L$ is through precisely one point of $\mathcal{P}_0$ and the set of such lines is denoted $\mathcal{P}_0(t)$.

A line $L$ is said to be a $\mathcal{P}_0$–exterior line if $L$ is through no points of $\mathcal{P}_0$ and the set of such lines is denoted $\mathcal{P}_0(e)$. 
**Corollary** Let $\pi = (\mathcal{P}, \mathcal{L})$ be a finite projective plane. Then $\pi$ is a linear space.

**Proof:** We must show that Axioms L1 and L2 are satisfied. The fact that each line contains a least three points implies Axiom L1, and Axiom PP1 implies Axiom L2 (any two points are on precisely one line).
**LEMMA** Let $\pi$ be a finite projective plane and let $L, M \in \mathcal{L}$. Then there is a point $X$ of $\mathcal{P}$ which is on neither $L$ or $M$.

**Proof:** Let $P = L \cap M$. Also let $Q$ and $R$ be points on $L$ and $M$ respectively. Since there are least three point on each line of $\pi$ there is a third point $X$ on the line $QR$. Now since we chose $QR \neq L$, $X$ is not on $L$ and similarly $X$ is not on $M$. 

![Diagram](image-url)
**THEOREM** Let $\pi$ be a finite projective plane. Then $|L| = |M|$ for all $L, M \in \mathcal{L}$.

**Proof:** Consider $L, M \in \mathcal{L}$ and let $\{P_1, \ldots, P_L\}$ denote the set of points on $L$. By the previous lemma we know there is a point $P$ on neither $L$ or $M$. Let

$$PP_1 \cap M = X_1$$

and in general, for $i = 1, \ldots, L$,

$$PP_i \cap M = X_i.$$ 

Now if $X_i = X_j$, where $i \neq j$, then $P_i$ and $P_j$ are both on $PX_i = PX_j$ and this line must be $L$. But this is a contradiction. So $X_i \neq X_j$ for all $i \neq j$. It follows that $|M| \geq |L|$. But then when we interchange $L$ and $M$ in the above argument we get $|L| \geq |M|$. So

$$|L| = |M|.$$
Corollary Let $\pi$ be a finite projective plane. Then $|P| = |Q|$ for all $P, Q \in \mathcal{P}$.

THEOREM Let $\pi$ be a finite projective plane. Then there is an integer $n \geq 2$ such that

$|P| = n + 1$ for all $P \in \mathcal{P}$,

$|L| = n + 1$ for all $L \in \mathcal{L}$ and

$|\mathcal{P}| = |\mathcal{L}| = n^2 + n + 1$.

(Note the integer $n$ is said to be the order of the projective plane $\pi$.)

Proof: From the previous theorem and corollary we know there exists integers $n_l$ and $n_p$ such that $|L| = n_l + 1$ for all $L \in \mathcal{L}$ and $|P| = n_p + 1$ for all $P \in \mathcal{P}$. And since there are at least three points on a line and at least three lines through a point $n_l \geq 2$ and $n_p \geq 2$. Thus we need to show that $|P| = |L|$ for some $P \in \mathcal{P}$ and some $L \in \mathcal{L}$.
Take a point \( P \in \mathcal{P} \) not on \( L \), where \( L \in \mathcal{L} \). Let \( L = \{P_1, \ldots P_l\} \), where \( l = n_l + 1 \). If there exists \( i \neq j \) such that

\[
PP_i = PP_j,
\]
then \( PP_i = PP_j = L \), and \( P \) is on \( L \) which is a contradiction. So for all \( i \neq j \)

\[
PP_i \neq PP_j.
\]
It follows that \( |L| \leq |P| \). But then the dual argument shows that \( |P| \leq |L| \).

Finally, consider \( P \in \mathcal{P} \) and let \( Q \) be a point distinct from \( P \). Take the set

\[
\{(T, L) \mid (T \in \mathcal{P} \setminus P) \land (L = TP)\}.
\]
Note that it contains \( n(n+1) \) elements. Since for each \( Q \in \mathcal{P} \setminus P \) there exists a unique line \( PQ \) through \( P \) and \( Q \), we may define a mapping \( \phi \) as:

\[
\phi : \mathcal{P} \setminus P \rightarrow \{(T, L) \mid (T \in \mathcal{P} \setminus P) \land (L = TP)\}
Q \rightarrow (Q, PQ)
\]
However it is easy to show that \( \phi \) is in fact 1–1 and onto. So \( |\mathcal{P} \setminus P| = n^2 + n \) and \( |\mathcal{P}| = n^2 + n + 1 \). The fact that \( |\mathcal{L}| = n^2 + n + 1 \) follows by duality.
**Lemma** The diagonal elements of any quadrangle of a projective plane $\pi$ of order 2 are collinear.

**Proof:** Let $Q = \{P_1, P_2, P_3, P_4\}$ be a quadrangle in $\pi$. By definition the diagonal elements are

\[
D_1 = P_1P_2 \cap P_3P_4, \\
D_2 = P_1P_3 \cap P_2P_4, \text{ and} \\
D_3 = P_1P_4 \cap P_2P_3.
\]

The points $P_1, P_2, P_3, P_4, D_1, D_2, D_3$ are all distinct as are the lines $P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, P_3P_4$. This accounts for 7 points and 6 lines. If $\pi$ is a projective plane of order 2. But we know that $|P| = |L| = 2^2 + 2 + 1 = 7$. So we need one more line and consequently it must pass through the points $D_1, D_2, D_3$. 

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Let $\pi_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $\pi_2 = (\mathcal{P}_2, \mathcal{L}_2)$ be two finite projective planes of the same order. Let $\phi$ be a mapping such that

$$\phi : \mathcal{P}_1 \rightarrow \mathcal{P}_2.$$ 

For a line $L \in \mathcal{L}_1$, we use the notation $\phi(L)$ to represent the set of points $\{\phi(x) \mid x \in L\}$.

Let $\pi_1$ and $\pi_2$ be two projective planes of the same finite order, and assume the mapping $\phi$ is onto. Then $\phi$ is said to be an **isomorphism** from $\pi_1$ to $\pi_2$, if

$$\forall L \in \mathcal{L}_1, \phi(L) \in \mathcal{L}_2.$$ 

Note that since $\pi_1$ and $\pi_2$ have the same order and $\phi$ is onto it follows that $\phi$ is one to one.

$\pi_1$ and $\pi_2$ are said to be **isomorphic**.
**THEOREM** There is a projective plane of order two which is unique up to isomorphism.

**Proof:** Let \( \pi_1 = (\mathcal{P}_1, \mathcal{L}_1) \) and \( \pi_2 = (\mathcal{P}_2, \mathcal{L}_2) \) be two projective planes of order 2. Both \( \pi_1 \) and \( \pi_2 \) contain quadrangles, label these quadrangles, respectively, \( Q_1 = \{P_{11}, P_{12}, P_{13}, P_{14}\} \) and \( Q_2 = \{P_{21}, P_{22}, P_{23}, P_{24}\} \).

The remaining points of the projective planes are the diagonal elements. Let \( D_{11}, D_{12}, D_{13} \) represent the diagonal points of \( Q_1 \) be and \( D_{21}, D_{22}, D_{23} \) the diagonal points of \( Q_2 \). So

\[
\{D_{11}, D_{12}, D_{13}\} = L_{1\infty} \in \mathcal{L}_1, \quad \text{and} \\
\{D_{21}, D_{22}, D_{23}\} = L_{2\infty} \in \mathcal{L}_2
\]

Let \( \phi \) be a mapping from \( \pi_1 \) to \( \pi_2 \) where, for \( i = 1, \ldots, 4 \) and \( j = 1, 2, 3 \)

\[
\phi(P_{1i}) = P_{2i}, \quad \text{and} \\
\phi(D_{1j}) = D_{2j}.
\]

Note for all \( j, k \) where \( j \neq k \),

\[
\phi(P_{1j}P_{1k}) = P_{2j}P_{2k}, \quad \text{and} \\
\phi(L_{1\infty}) = L_{2\infty}.
\]

Hence \( \phi \) is an isomorphism from \( \pi_1 \) to \( \pi_2 \).
COROLLARY

Any projective plane of order 2 is isomorphic to its dual.

An isomorphism from a finite projective plane to its dual is called a correlation of $\pi$. If a finite projective plane has a correlation, then $\pi$ is said to be self–dual.

A projective plane of order 2 is self–dual.
Let $\chi$ be a correlation of a finite projective plane $\pi$.

If $P$ is a point on a line $L$,

then $\chi(L)$ is on $\chi(P)$. 
EXAMPLE  Take the following finite projective plane and its dual given below.

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>$\pi^d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{P} = {0, 1, \ldots, 6}$</td>
<td>$\mathcal{P}' = {L_0, \ldots, L_6}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mathcal{L}$</th>
<th>$\mathcal{L}'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_0 = {0, 1, 3}$</td>
<td>${L_0, L_1, L_5}$</td>
</tr>
<tr>
<td>$L_1 = {1, 2, 4}$</td>
<td>${L_3, L_5, L_6}$</td>
</tr>
<tr>
<td>$L_2 = {2, 3, 5}$</td>
<td>${L_0, L_2, L_3}$</td>
</tr>
<tr>
<td>$L_3 = {3, 4, 6}$</td>
<td>${L_0, L_4, L_6}$</td>
</tr>
<tr>
<td>$L_4 = {4, 5, 0}$</td>
<td>${L_1, L_2, L_6}$</td>
</tr>
<tr>
<td>$L_5 = {5, 6, 1}$</td>
<td>${L_2, L_4, L_5}$</td>
</tr>
<tr>
<td>$L_6 = {6, 0, 2}$</td>
<td>${L_1, L_3, L_4}$</td>
</tr>
</tbody>
</table>

Then $\chi$ from $\mathcal{P}$ to $\mathcal{P}'$ and $\mathcal{L}$ to $\mathcal{L}'$ defined as follows is a correlation of $\pi$ onto $\pi^d$. 

<table>
<thead>
<tr>
<th>$\mathcal{P}$</th>
<th>$\chi(\mathcal{P})$</th>
<th>$\mathcal{L}$</th>
<th>$\chi(\mathcal{L})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$L_1$</td>
<td>$L_0$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$L_5$</td>
<td>$L_1$</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>$L_3$</td>
<td>$L_2$</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>$L_0$</td>
<td>$L_3$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$L_6$</td>
<td>$L_4$</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>$L_2$</td>
<td>$L_5$</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>$L_4$</td>
<td>$L_6$</td>
<td>4</td>
</tr>
</tbody>
</table>
**Lemma** A projective plane of order $n$ is a balanced incomplete block design with parameters $(v, b, r, k, \lambda) = (n^2 + n + 1, n^2 + n + 1, n + 1, n + 1, 1)$.

**Proof:** See problem sheet.

We say a balanced incomplete block design is **symmetric** if $v = b$.

**Lemma** If $v = b$ in a balanced incomplete block design then $r = k$.

**Proof:** See problem sheet.

Let $(V, \mathcal{B})$ be a BIBD. Assume $V = \{u_1, u_2, \ldots, u_v\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$. We construct the **dual design** $(V^d, \mathcal{B}^d)$ with $V^d = \{w_1, w_2, \ldots, w_b\}$ and $\mathcal{B}^d = \{C_1, C_2, \ldots, C_v\}$, where

$$w_i \in C_j \iff u_j \in B_i.$$
Lemma Let \((V, \mathcal{B})\) be a BIBD, with incidence matrix \(A\). Then \(A^T\) is the incidence matrix of the dual design.

Let \((V, \mathcal{B})\) be a BIBD, with parameters \((v, b, r, k, \lambda)\). The BIBD is said to be **linked** if for all blocks \(B_1, B_2 \in \mathcal{B}\), \(|B_1 \cap B_2|\) is constant.

**Lemma** Let \((V, \mathcal{B})\) be a symmetric BIBD, with parameters \((v, v, k, k, \lambda)\). Then the BIBD is linked and for all blocks \(B_1, B_2 \in \mathcal{B}\),

\[
|B_1 \cap B_2| = \lambda.
\]

**Proof:**