FINITE AFFINE PLANES

An incidence structure $\alpha = (\mathcal{P}_\alpha, \mathcal{L}_\alpha)$ is said to be a **finite affine plane** if the following axioms are satisfied.

A1 Every pair of distinct points are joined by exactly one line.

A2 For each $L \in \mathcal{L}_\alpha$ and $P \in \mathcal{P}_\alpha$ such that P is not on L there is a unique $M \in \mathcal{L}_\alpha$ such that P is on M and M does not meet L.

A3 There are three points of $\mathcal{P}_\alpha$ which are not collinear.

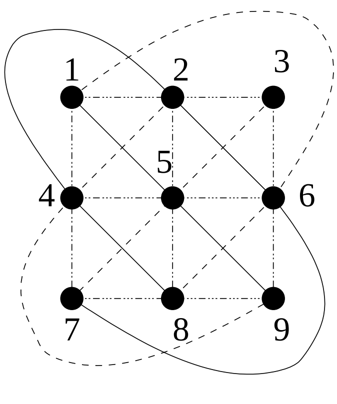
A4 $\mathcal{P}_\alpha$ is finite.

If α is an affine plane we denote its point set by $\mathcal{P}_\alpha$ and its line set by $\mathcal{L}_\alpha$.

Let $\mathcal{P}_\alpha = \{1,2,3,4,5,6,7,8,9\}$ and

$$\begin{aligned}\mathcal{L}_\alpha = \{ \; & \{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,4,7\},\\ & \{2,5,8\},\{3,6,9\},\{1,5,9\},\{4,8,3\},\\ & \{7,2,6\},\{1,8,6\},\{4,2,9\},\{7,5,3\}\}.\end{aligned}$$

Then $(\mathcal{P}_\alpha, \mathcal{L}_\alpha)$ is an affine plane.



LEMMA If an incidence structure satisfies Axiom A1, then $|(L) \cap (M)| \leq 1$ for all $L, M \in \mathcal{L}$ such that $L \neq M$.

Let $\pi = (\mathcal{P}_\pi, \mathcal{L}_\pi)$ be a projective plane of order n.

Let $L \in \mathcal{L}_\pi$.

Define a new incidence structure $\pi(L)$ with point set $\mathcal{P}_{\pi(L)} = \mathcal{P}_\pi \backslash L$ and line set $\mathcal{L}_{\pi(L)} = \{L' \cap (\mathcal{P}_\pi \backslash L) \mid L' \in \mathcal{L}_\pi\}$.

Then the substructure $\pi(L)$ is an affine plane called the **affine restriction of** π at L.

If π is a projective plane of order n, then we obtain $n^2 + n + 1$ affine restrictions from π .

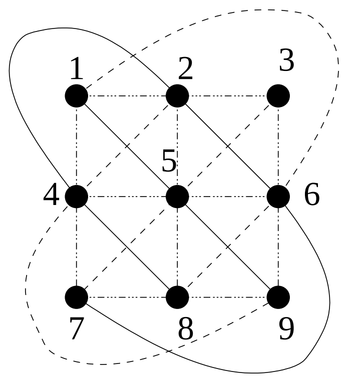
Let L be a line of an affine plane α. Let M be another line of α. If M does not meet L we say M is **parallel** to L.

LEMMA Let α be a finite affine plane. The relationship $\sim$ on $\mathcal{L}_\alpha$ defined $L \sim M$ if and only if $L = M$ or L is parallel to M is an equivalence relation.

Proof The relexivity and symmetry of $\sim$ are obvious. It is sufficient to consider distinct lines L, M and N such that L is parallel to M and M is parallel to N and then show that L is parallel to N. But if P is on L and N, then P is not on M and there are at least two lines on P which are parallel to M, a contradiction to Axiom A2.

Each equivalence class of $\mathcal{L}_\alpha$ under $\sim$ is called a **parallel class** of α.

In the affine plane

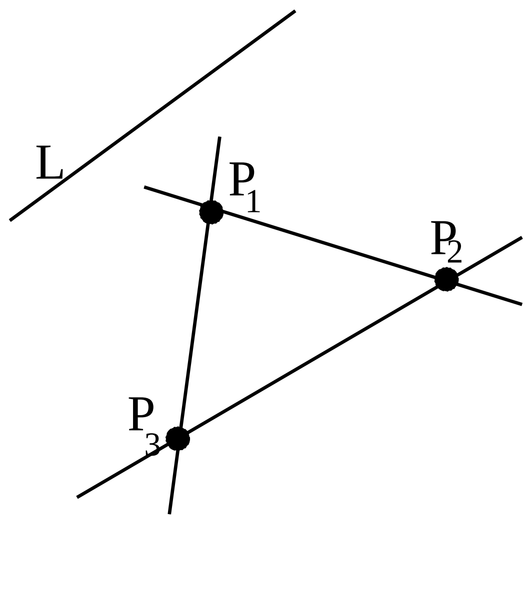


the parallel classes are

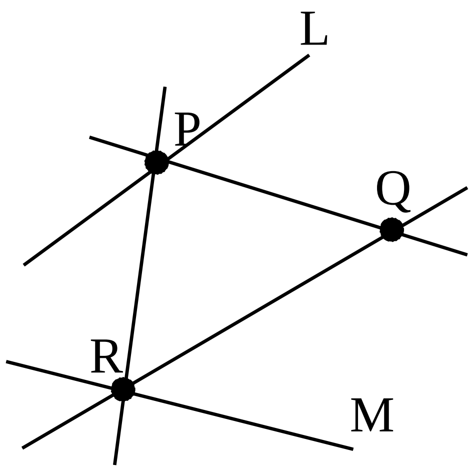
1 2 3	1 4 7	1 5 9	1 8 6
4 5 6	2 5 8	2 6 7	2 4 9
7 8 9	3 6 9	4 8 3	3 5 7

LEMMA There are at least two points on each line of a finite affine plane α.

Proof Suppose $L \in \mathcal{L}_\alpha$ is such that $(L) = \emptyset$. Let P_1, P_2, P_3 be a triangle of α. Now P_1 is not on L and P_1P_2 and P_1P_3 are distinct lines on P_1 which are parallel to L, contradicting Axiom A2. So $(L) \neq \emptyset$, for all $L \in \mathcal{L}_\alpha$.



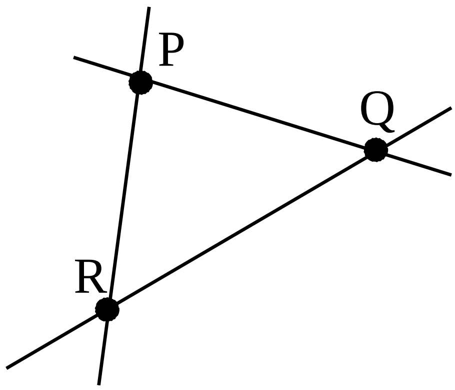
Suppose $L \in \mathcal{L}_\alpha$ is such that $(L) = \{P\}$ for some $P \in \mathcal{P}_\alpha$. By Axiom A3 α has a triangle. If there is a triangle not containing P, then there is also a triangle containing P as well. Let P, Q, R be a triangle of α. By Axiom A2 there is a line M on R distinct from RP and RQ and parallel to PQ. But then M and RQ are distinct lines on R parallel to L, contradicting Axiom A2.



Let α be a finite affine plane.

LEMMA 1 α has at least three parallel classes.

Proof α possesses a triangle. The lines joining the pairs of points of a triangle are in different parallel classes.



LEMMA 2 If L and M are in different parallel classes then they meet in exactly one point.

Proof If L and M are in different parallel classes then they meet in at least one point. But we know that two lines intersect in one or less points. Hence they meet in exactly one point.

A set of $\mathcal{L}'$ lines of an incidence structure is said to **partition** its point set $\mathcal{P}$ if

1. $(L) \neq \emptyset$ for all $L \in \mathcal{L}'$,

2. $(L) \cap (M) = \emptyset$ for all $L, M \in \mathcal{L}'$ such that $L \neq M$, and

3. $\cup_{L \in \mathcal{L}'} (L) = \mathcal{P}$

LEMMA 3 If C is a parallel class of α, then C partitions $\mathcal{P}_\alpha$ and there are at least two lines in C.

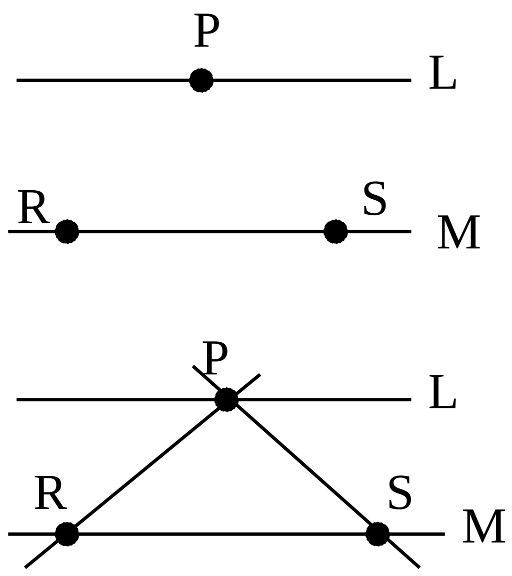
Proof Consider a parallel class C of α. We have $(L) \neq \emptyset$ for all $L \in C$. Clearly $(L) \cap (M) = \emptyset$ for all $L, M \in c$ such that $L \neq M$. Furthermore $\cup_{L \in C}(L) = \mathcal{P}_\alpha$ using Axiom A2. Using Axioms A2 and A3 we have that there are at least two lines in C.

LEMMA 4 α has at least 4 points.

Proof This part follows directly from the fact that there are at least two points on each line of an affine plane and Lemma 3 above.

LEMMA 5 There are at least three lines through each point of α.

Proof Consider a point P of α. Let L be a line on P and let M be a line parallel to L. Also let R and S be distinct points on M. Then L, PR, and PS are three different lines on P.

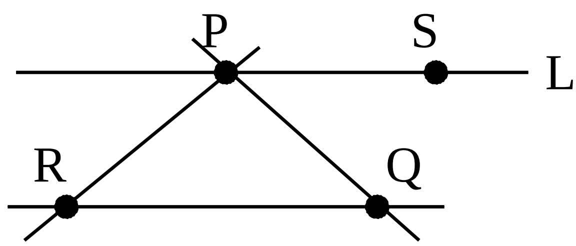


LEMMA 6 α has at least six lines.

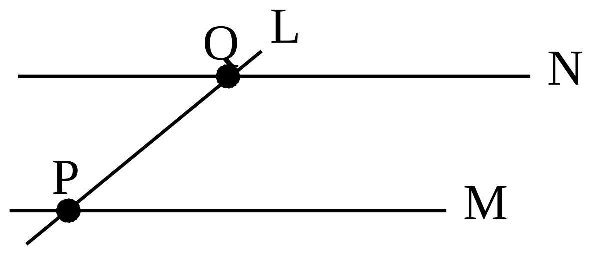
Proof This follows from Lemma (1) and (3) above.

THEOREM Any finite affine plane possesses a set of four points no three of which are collinear.

Proof Let P, Q, R be a triangle of α. Let L be a line through P and parallel to RQ and S be another point on the line L. Then P, Q, R, S are four points no three of which are collinear.



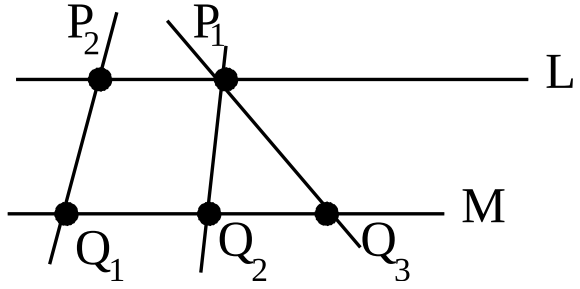
Consider an affine plane α which contains two lines L and M such that $(L) \cup (M) = \mathcal{P}_\alpha$.



Further suppose L and M meet at P. There is a point Q on L distinct from P. By Axiom A2 there is a line N through Q and parallel to M. But then $(N) = \{Q\}$, however each line of an affine plane must contain at least two points. So we have a contradiction and L and M are parallel.

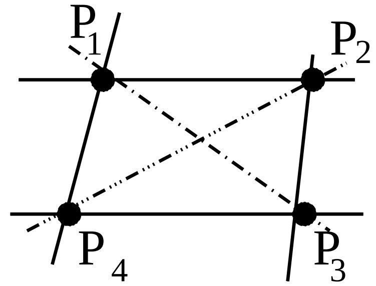
Therefore if there exists an affine plane α which contains two lines L and M such that $(L) \cup (M) = \mathcal{P}_\alpha$ then L and M are parallel.

So assumes there exists an affine plane α which contains two parallel lines L and M such that $(L) \cup (M) = \mathcal{P}_\alpha$.



Suppose at least one of L and M is through at least three points. WLOG assume that there are distinct points Q_1, Q_2, Q_3 on M. Let P_1 and P_2 be points on L. Since all points are on L or M we have that P_1Q_2 and P_1Q_3 are different lines on P_1 which are parallel to P_2Q_1. Since P_1 is not on P_2Q_1 we have a contradiction.

Hence L and M are through exactly 2 points and α is of the form:



In such an affine plane the following statements (1) to (6) can be verified when $n = 2$.

1. $|L| = n$ for all $L \in \mathcal{L}_\alpha$,

2. $|P| = n + 1$ for all $P \in \mathcal{P}_\alpha$,

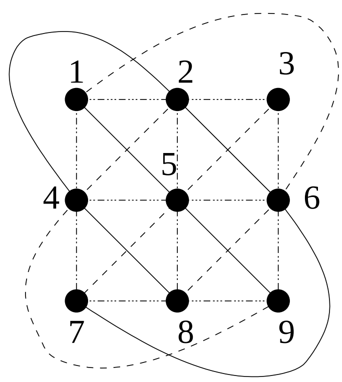
3. $|\mathcal{P}_\alpha| = n^2$,

4. $|C| = n$, for all parallel classes C of α,

5. there are $n + 1$ parallel classes of α, and

6. $|\mathcal{L}_\alpha| = n^2 + n$.

Consider the affine plan given by



Then $|L| = 3$, so let $n = 3$ and we see that $|P| = 4 = n + 1$, $|\mathcal{P}_\alpha| = 9 = n^2$, $|C| = 3 = n$, there are $4 = n + 1$ parallel classes and $|\mathcal{L}_\alpha| = 12 = n^2 + n$.

Next let α an affine plane which contains two parallel lines L and M and a point P not on L or M.

LEMMA A There exists an integer n such that $|L| = n$ for all $L \in \mathcal{L}_\alpha$.

Proof Let L and M be different lines of α, P a point on neither L nor M and $|P| = n + 1$. Let $P_1, \ldots, P_l$ be the points on L. The lines PP_i, $i = 1, \ldots, l$ are all distinct and there is a unique line on P which is parallel to L, hence $l = n$.

Similarly $|M| = n$. Since L and M are arbitrary lines of α we have $|N| = n$ for all lines N of α. And since there are at least two points on each line of an affine plane $n \geq 2$.

LEMMA B There exists an integer n such that $|P| = n + 1$ for all $P \in \mathcal{P}_\alpha$.

Proof Let P be any point of α. Let C be a parallel class and M be the line of C through P. Since $|C| \geq 2$, there is a line of C not through P. Let L be such a line and $(L) = \{P_1, \ldots, P_n\}$. The lines of α through P are PP_i, $i = 1, \ldots, n$, plus M. Thus $|P| = n + 1$.

LEMMA C There exists an integer n such that $|\mathcal{P}_\alpha| = n^2$,

Proof Consider a point P of α. Counting incidences of points with the lines on P we have $|\mathcal{P}_\alpha| = (n+1)(n-1)+1 = n^2$.

LEMMA D There exists an integer n such that $|C| = n$ for all parallel classes C of α,

Proof Consider a parallel class C of α. α has n^2 points, each line of C is through n points and the lines of C partition $\mathcal{P}_\alpha$.

LEMMA E There exists an integer n such that there are $n+1$ parallel classes of α.

Proof Consider a parallel class C of α. Each line of α on P determines a parallel class of α, and conversely. So the number of parallel classes of α equals $|P| = n+1$.

LEMMA F There exists an integer n such that $|\mathcal{L}_\alpha| = n^2 + n$.

Proof This follows from Lemmas (D) and (E) above.

THEOREM Let α be a finite affine plane. There is an integer $n \geq 2$ such that

(A) $|L| = n$ for all $L \in \mathcal{L}_\alpha$,

(B) $|P| = n + 1$ for all $P \in \mathcal{P}_\alpha$,

(C) $|\mathcal{P}_\alpha| = n^2$,

(D) $|C| = n$ for all parallel classes C of α,

(E) there are $n + 1$ parallel classes of α, and

(F) $|\mathcal{L}_\alpha| = n^2 + n$.

The proof is split into two separate cases.

Case 1 There are two lines L and M such that $(L) \cup (M) = \mathcal{P}_\alpha$.

Suppose L and M meet at P. There is a point Q on L distinct from P. By Axiom A2 there is a line N through Q and parallel to M. But then $(N) = \{Q\}$, however each line of an affine plane must contain at least two points. So we have a contradiction and L and M are parallel.

Suppose at least one of L and M is through at least three points. WLOG assume that there are distinct points Q_1, Q_2, Q_3 on M. Let P_1 and P_2 be points on L. Since all points are on L or M we have that P_1Q_2 and P_1Q_3 are different lines on P_1 which are parallel to P_2Q. Since P_1 is not on P_2Q we have a contradiction. The affine plane must have the following structure and (1) to (6) are verified when $n = 2$.

Case 2 For any two lines L and M there isa point P not on L or M.

(1) Let L and M be different lines of α, OP a point on neither L nor M and $|P| = n + 1$. Let $P_1, \ldots, P_l$ be the points on L. The lines PP_i, $i = 1, \ldots, l$ are all distinct and there is a unique line on P which is parallel to L, hence $l = n$.

Similarly $|M| = n$. Since L and M are arbitrary lines of α we have $|N| = n$ for all lines n of α. And since there are at least two points on each line of an affine plane $n \geq 2$.

(2) Let P be any point of α. Let c be a parallel class and M be the line of c through P. Since $|c| \geq 2$, there is a line of c not through P. Let L be such a line and $(L) = \{P_1, \ldots, P_n\}$. The lines of α through P are PP_i, $i = 1, \ldots, n$, plus M. Thus $|P| = n + 1$.

(3) Consider a point P of α. Counting incidences of points with the line on P we have $|\mathcal{P}_\alpha| = (n+1)(n-1)+1 = n^2$.

(4) Consider a parallel class c of α. α has n^2 points, each line of c is through n points and the lines of c partition $\mathcal{P}_\alpha$.

(5) Consider a parallel class c of α. Each line of α on P determines a parallel class of α, and conversely. So the number of parallel classes of α equals $|P| = n + 1$.

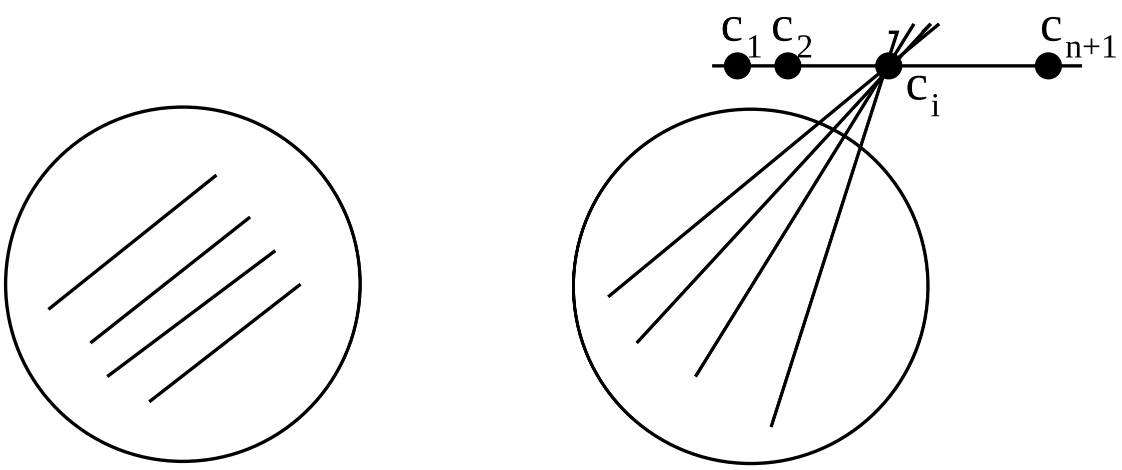
(6) This follows from parts (4) and (5) above.

The number n is said to be the **order** of the affine plane.

If there is a projective plane π such that an affine plane α is the affine restriction of π at L, then we say that π is a **projective completion** of α.

THEOREM Any finite affine plane α has a projective completion.

Proof Let C be the set of parallel classes of α, That is, $\mathcal{C} = \{C_1, \ldots, C_{n+1}\}$, where C_i $1 \leq i \leq n+1$ is a parallel class. The lines of the projective completion are going to be based on the lines of α and $\mathcal{C}$, which is taken to be a line on the points C_i, where C_i is a parallel class.



Define an incidence structure π with $\mathcal{P} = \mathcal{P}_\alpha \cup \{C_1, \ldots, C_{n+1}\}$ and $\mathcal{L} = \mathcal{L}_\alpha \cup \{\mathcal{C}\}$. Incidence for π is defined as follows.

1. In π a point P is on $\mathcal{C}$ if and only if P is a parallel class of $\mathcal{C}$.

2. In π a point P is on $L \in \mathcal{L}_\alpha$ if and only if $P \in \mathcal{P}_\alpha$ and P is on L in $\mathcal{L}_\alpha$ or the point P is a parallel class of $\mathcal{C}$ and then L is a member of that parallel class.

To complete the proof we show that π is a projective plane such that $\pi(L) = \alpha$.

Consider two points P and Q of $\mathcal{P} = \mathcal{P}_\alpha \cup \{C_1, \ldots, C_{n+1}\}$. The following argument verifies that there is a unique line through P and Q.

Case 1 $P, Q \in \mathcal{P}_\alpha$.

The unique line which joins P and Q in π is the unique line which joins P and Q in α.

Case 2 $P \in \mathcal{P}_\alpha$, $Q \in \mathcal{C}$.

Since Q is a parallel class of α there is a unique line of α in Q through P. This line is the unique line which joins P and Q in π.

Case 3 $P, Q \in \mathcal{C}$.

$\mathcal{C}$ is the unique line which joins P and Q in π.

Consider two lines L and M of π. The following argument verifies that the lines meet in a point.

(a) $L = \mathcal{C}$.
There is a unique parallel class C_i of $\mathcal{C}$ containing M. This parallel class C_i is the unique intersection point of L with M.

(b) L is parallel to M in α.
There is a unique parallel class C_i of $\mathcal{C}$ containing L and M. This parallel class is the unique intersection point of L and M in π.

(c) L is not parallel to M in α.
L and M meet at a unique point in α. This unique intersection point of L and M in α is the unique intersection point of L and M in π.

Finally α possesses a quadrangle, so π does also and so π is a finite projective plane.

Clearly $\alpha = \pi(\mathcal{C})$.

Two affine planes of the same order are said to isomorphic if there exists a one–to–one and onto mapping from the points and lines of the first to the points and lines of the second which preserves incidence.

An isomorphism from an affine plane onto itself is called a collineation.

LEMMA

Let π be an isomorphism from the affine plane α_1 of order n onto the affine plane α_2 of order n and L_1 and M_1 be parallel lines of α_1. Then $\pi(L_1)$ and $\pi(L_2)$ are parallel lines in α_2 and as a consequence if C_1 is a parallel class in α_1, then $\pi(C_1)$ is a parallel class of α_2. Furthermore, π takes the set of parallel classes of α_1 onto the set of parallel classes of α_2.

Proof Suppose L_1 and M_1 are parallel in α_1 and $\pi(L_1)$ and $\pi(M_1)$ meet at $\pi(P_1)$ in α_2. Then $\pi(P_1)$ is on $\pi(L_1)$ and $\pi(M_1)$ and so P_1 is on L_1 and M_1 a contradiction.

Consider a parallel class C_1 of α_1. The image under π of the lines in C_1 are mutually parallel in α_2, hence there is a parallel class C_2 of α_2 such that $\pi(C_1) \subseteq C_2$. But $|C_1| = |C_2| = n$ and π is one–to–one, so $\pi(C_1) = C_2$.

For $i = 1, 2$, let $\mathcal{C}_i$ be the set of parallel classes of α_i. Thus far we have $\pi(\mathcal{C}_1) \subseteq \mathcal{C}_2$. But $|\mathcal{C}_1| = |\mathcal{C}_2|$, hence $\mathcal{C}_1 = \mathcal{C}_2$.

THEOREM

Suppose that π_1 and π_2 are projective planes of order n and L_i is a line of π_i, for $i = 1, 2$. Then there is an isomorphism from $\pi_1(L_1)$ to $\pi_2(L_2)$ if and only if there is an isomorphism from π_1 onto π_2 which takes L_1 to L_2.

Proof Let $(L_1) = \{P_{1j} \mid j = 1, \ldots, n+1\}$ and C_{1j} be the parallel class of $\pi(L_1)$ containing the lines of $\pi(L_1)$ incident in π_1 with P_{1j}. Suppose $\overline{\pi}$ is an isomorphism from $\pi_1(L_1)$ onto $\pi_2(L_2)$. Now $\overline{\pi}$ takes the set of parallel classes of $\pi_1(L_1)$ onto the set of parallel classes of $\pi_2(L_2)$. For $j = 1, \ldots, n+1$, denote $\overline{\pi}(C_{1j})$ by C_{2j} and the points of π_2 on each of the lines of C_{2j} by P_{2j}.

To establish this part of our result we extend $\overline{\pi}$ to an isomorphism π from π_1 onto π_2 by taking L_1 and L_2 as follows:

Define π by

$$\pi(P) = \overline{\pi}(P), \text{ for all points } P \text{ of } \pi_1(L_1)$$
$$\pi(L) = \overline{\pi}(L), \text{ for all lines } L \text{ of } \pi_1(L_1)$$
$$\pi(L_1) = L_2, \text{ and}$$
$$\pi(P_{1j}) = \pi(P_{2j}), \text{ for } j = 1, \ldots, n+1.$$

So π is easily from π_1 onto π_2 but we must show π preserves incidence.

Consider the incident point/line pair Q_1, M_1 of π_1. Using the fact that $\overline{\pi}$ is an isomorphism from $\pi_1(L_1)$ onto $\pi_2(L_2)$ we see that it is immediate that $\pi(Q_1)$ is on $\pi(M_1)$ unless Q_1 is on L_1. So suppose $Q_1 = P_{1j}$ is on L_1 and $M_1 \neq L_1$. Then $P_{1j} = Q_1$ is on M_1 so

$$\begin{aligned} &\rightarrow \ M_1 \in C_{1j} \\ &\rightarrow \ \pi(M_1) \in \pi(C_{1j}) = C_{2j} \\ &\rightarrow \ P_{2j} = \pi(P_{1j}) = \pi(Q_1) \text{ is on } \pi(M_1). \end{aligned}$$

Suppose there is an isomorphism π from π_1 to π_2 which takes L_1 to L_2. Since π is one–to-one and preserves non–incidence we have $\pi((L_1)^c) = (L_2)^c$ and $\pi(\{L_1\}^c) = \{L_2\}^c$. Clearly π restricted to the set of points and lines of $\pi_1(L_1)$ (say $\overline{\pi}$) is from $\pi_1(L_1)$ onto $\pi_2(L_2)$. Since π is incidence–preserving, $\overline{\pi}$ is incidence–preserving. Thus $\overline{\pi}$ is an isomporphism from $\pi_1(L_1)$ onto $\pi_2(L_2)$.

COROLLARY Let α be a finite affine plane and π_1 and π_2 be projective completions of α. Any collineation of α extends to an isomorphism from π_1 onto π_2.

Proof Suppose $\alpha = \pi_1(L_1) = \pi_2(L_2)$, where L_2 and L_2 are lines of π_1 and π_2, respectively. From the proof the above Theorem any collineation of α extends to an isomorphism from π_1 onto π_2.