Let $\pi_0 = (\mathcal{P}_0, \mathcal{L}_0)$ be a substructure of a projective plane $\pi = (\mathcal{P}, \mathcal{L})$. $\pi_0$ is said to be a **projective subplane** of $\pi$ if $\pi_0$ is a projective plane.

$\pi$ is always a subplane of itself.

If $\mathcal{P}_0$ is a proper subset of $\mathcal{P}$, then $\pi_0$ is said to be **proper**.

A projective subplane of a finite projective plane is a finte projective plane.

**Theorem** Let $\pi_0$ be a projective subplane, of order $m$, of a projective plane $\pi$ of order $n$, where $m < n$. Then

$$n = m^2 \quad \text{or} \quad n \geq m^2 + m.$$

**Proof** See Sheet 5

A proper projective subplane $\pi_0$, of order $m$, of a projective plane $\pi$, of order $n$, is said to be a **Baer subplane** of $\pi$ if $n = m^2$. That is all the lines of $\pi$ meet $\pi_0$ in at least one point.

We may deduce that

If a finite projective plane of order $n$ contains a Baer subplane, $n$ must be a square.

The Baer subplane of a projective plane $\pi$ of order $m^2$ are precisely the projective subplanes of $\pi$ of order $m$.

The Baer subplanes of a finite projective plane $\pi$ are precisely the projective subplanes of $\pi$ for which the set $\mathcal{P} \setminus \mathcal{P}_0$ is the $\mathcal{L}_0$–tangents and the set $\mathcal{L} \setminus \mathcal{L}_0$ is the $\mathcal{P}_0$–tangents

Let $\alpha_0 = (\mathcal{P}_0, \mathcal{L}_0)$ be a substructure of a projective plane $\pi = (\mathcal{P}, \mathcal{L})$. $\alpha_0$ is said to be a **affine subplane** of $\pi$ if $\alpha_0$ is a affine plane.

An obvious situation in which an affine subplane $\alpha_0$ arises in a projective plane $\pi$ is when there is a substructure $\pi_0$ such that

(i) $\pi_0$ is a projective subplane of $\pi$ and

(ii) $\alpha_0$ is an affine restriction of $\pi_0$.

In this case we say that $\alpha_0$ is projective completable in $\pi$.

Recall from Problem Sheet 3, Question 4, we defined $\mathcal{L}_0$ secants, tangents and exterior points as follows:

Let $\pi = (\mathcal{P}, \mathcal{L})$ be a projective plane of order $n$. Let $\mathcal{L}_0 \subseteq \mathcal{L}$.

A point $P$ is said to be a $\mathcal{L}_0$-*secant* if there exists two or more lines of $\mathcal{L}_0$ which intersect at $P$. We use the notation $\mathcal{L}_0(s)$ to denote the set of all $\mathcal{L}_0$-secant.

A point $P$ is said to be a $\mathcal{L}_0$-*tangent* if there exists exactly one line $\mathcal{L}_0$ which is incident with $P$. We use the notation $\mathcal{L}_0(t)$ to denote the set of all $\mathcal{L}_0$-tangent.

A point $P$ is said to be a $\mathcal{L}_0$-*exterior* point if $P$ is not incident with any line of $\mathcal{L}_0$. We use the notation $\mathcal{L}_0(e)$ to denote the set of all $\mathcal{L}_0$-exterior.

**Theorem** Let $\alpha_0 = (\mathcal{P}_0, \mathcal{L}_0)$ be an affine subplane of order $m$ of a projective plane $\pi = (\mathcal{P}, \mathcal{L})$ of order $n$. Then

if $\mathcal{L}_0(e) \neq \emptyset$ (that is, there exists a point not on a line of $\mathcal{L}_0$) then $n \geq m^2 - 1$, and

if $\mathcal{L}_0(e) = \emptyset$ (that is, given any point $P$ of $\mathcal{P}$ there exists a line of $\mathcal{L}_0$ which is through $P$) then $n \leq m^2 - 1$.

**Proof** Consider a point $P$ in $\mathcal{P}\backslash\mathcal{P}_0$. For every point $q$ of $\mathcal{P}_0$ there is a unique line of $\pi$ through $P$ and $Q$. Let $r(P)$ denote the number of lines of $\alpha_0$ through $P$. Now the $r(P)$ lines through $P$ are in a parallel class of $\alpha_0$. So there are $r(P)m$ points of $\alpha_0$ on these lines of $\alpha_0$. The remaining $n+1-r(P)$ lines through $P$ are $\mathcal{P}_0$ tangents or $\mathcal{P}_0$ exterior lines. If $f$ of these lines are $\mathcal{P}_0$ tangents, then we can count the number of points of $\alpha_0$ on these lines. So $r(P)m+f=m^2$. But then clearly we have

$$r(P)m+n+1-r(P)\geq m^2.$$

But, if there exists a point not on a line of $\mathcal{L}_0$ then there is a point $P$ for which $r(P)=0$. Hence

$$n\geq m^2-1.$$

Suppose $\mathcal{L}_0(e) = \emptyset$. Consider $Q \in \mathcal{P}_0$.

Now $|\mathcal{P}_0(s) \cap (Q)| = |\mathcal{L}_0 \cap (Q)| = m + 1$ and $|Q| = n + 1$. So there is a line of $\pi$ through $Q$ which is not a $\mathcal{P}_0$–secant. Such a line must be a $\mathcal{P}_0$–tangent. Let $L$ be a $\mathcal{P}_0$–tangent through $Q$, $\overline{\mathcal{L}}$ be the set of lines of $\alpha_0$ which are not through $Q$ and $M \in \overline{\mathcal{L}}$. Now $\phi$, where $\phi(M) = L \cap M$, is a mapping from $\overline{\mathcal{L}}$ into $(L) \setminus \{Q\}$. In fact $\phi$ is onto $(L) \setminus \{Q\}$ since, by hypothesis $\mathcal{L}(e) = \emptyset$. But $|\overline{\mathcal{L}}| = m^2 + m - (m+1) = m^2 - 1$ and $|(L) \setminus \{Q\}| = n$. So we must have $n \leq m^2 - 1$.

**Theorem** Let $\alpha_0 = (\mathcal{P}_0, \mathcal{L}_0)$ be an affine subplane of order $m$ of a projective plane $\pi = (\mathcal{P}, \mathcal{L})$ of order $n > m$. If $\mathcal{L}_0(t) \neq \emptyset$, then $n \geq m^2 - m + 1$.

**Theorem** Let $\alpha_0 = (\mathcal{P}_0, \mathcal{L}_0)$ be an affine subplane of order $m$ of a projective plane $\pi = (\mathcal{P}, \mathcal{L})$ of order $n > m$. such that $\mathcal{L}_0(e) = \emptyset$.

If $\mathcal{L}_0(t) \neq \emptyset$, then $m^2 - m + 1 \leq n \leq m^2 + 1$.

If $\mathcal{L}_0(t) = \emptyset$, then $m = 3$ and $n = 4$.

## THEOREM

If the points of a finite projective plane are taken as the treatments and the lines as blocks, the result is a balanced incomplete block design with parameters

$$(n^2+n+1,\ n^2+n+1,\ n+1,\ n+1,\ 1),$$

for some $n$.

Proof A finite projective plane has $n$+1 points incident with each line and $n+1$ lines incident with each point. Hence $r$ and $k$ are verified. Since any two points are collinear on one line, any pair of treatments occur together in exactly on block and $\lambda = 1$. $v$ and $b$ follow.

## Bruck–Ryser THEOREM

If there is a finite projective plane $\pi$ of order $n$ such that $n \equiv 1$ or 2(mod 4), then $n$ is the sum of two integral squares.

It can be shown that a positive integer $n$ is the sum of two squares if and only if each of its prime factors congruent to 3 (modulo 4) appears to an even power.

Let $n$ be congruent to one or two modulo four. If there is a prime number congruent to three modulo four which is a divisor of the square free part of $n$, then there is no finite projective plane of order $n$.

There exists no projective plane of order

$$6,\ 14,\ 21,\ 22,\ 30,\ 33,\ 38,\ 42\ etc$$

## THEOREM

If the points of a finite affine plane are taken as the treatments and the lines as blocks, the result is a balanced incomplete block design with parameters

$$(n^2,\ n^2+n+,\ n+1,\ n,\ 1),$$

for some $n$.

Recall that we write $n(k)$ for the size of the largest possible set of pairwise orthogonal Latin squares of side $k$ for $k > 1$.

**THEOREM** If there exists an affine plane of order $k$ then $n(k) = k - 1$.

Proof The lines of the affine plane can be partitioned into $k+1$ parallel classes. Number the parallel classes $C_0, C_1, \ldots, C_k$ and the lines of parallel class $C$ $L_{C1}, L_{C2}, \ldots, L_{Ck}$, for $C = 0, \ldots, k$. Select two parallel classes for reference purposes, say class $C_0$ and class $C_k$. We shall construct a square array from each of the remaining classes. Select class $C$, for $1 \leq i \leq k-1$ and construct a $k \times k$ array $A_C$ as follows. Take a line say $L_{0x}$ of class 0 and a line say $L_{ky}$ of class $k$. These two lines are incident with a point say $P$ and this point lines on exactly one line of class $C$. Assume $P$ lies on line $L_{Cs}$ of $C$. Place an $s$ in cell $(x, y)$ of the array $A_C$. Repeat this process of each possible pair of lines $L_{0x}$ and $L_{ky}$ where $x, y \in \{1, \ldots, k\}$.

We need to show that the array is a latin square. Choose column $y$ of $A_C$. Line $L_{ky}$ intersects each of the lines $L_{01}, \ldots, L_{0k}$ of class 0 in precisely one point, so column $y$ of $A_C$ will contain each of the number $1, \ldots, k$ in some order. A similar argument applies to the rows of $A_C$ and so $A_C$ is a Latin square.

Write $A_i$ and $A_j$ for the two Latin squares obtained from parallel classes $C_i$ and $C_j$, respectively where $i \neq j$. The cells of $A_i$ which contain entry $s$ all arise from points on the line $L_{is}$. The lines $L_{j1}, \ldots, L_{jk}$ of class $C_j$ each intersect $L_{is}$ in a unique point. Hence the correponding cell of $A_j$ each contain the elements $1, \ldots, k$ once each. A similar argument applies to the rows. So $A_i$ and $A_j$ are orthogonal.