MATH3301 GEOMETRY

ASSIGNMENT 5

DUE DATE for ASSIGNMENT FOUR: 5pm TUESDAY 17TH October 2006

Questions 1, 3, 6, 9 and 11 from Sheet 7 given below.

SHEET 7
1. Let a linear space on \( v \) points and \( b \) lines have line and point regularity \( k + 1, k \geq 1 \). Prove that all lines meet and \( b = v = k^2 + k + 1 \).

2. Take the Euclidean plane \( \mathbb{R}^2 \). Let a point of \( \mathbb{R}^2 \) be an ordered pair \((x, y)\) and a line be the set of points \((x, y)\) which satisfy an equation \( y = ax + b \) or \( x = c \), where \( a, b \) and \( c \) are real numbers.
   
   i. Is \( \mathbb{R}^2 \) a linear space?
   
   ii. Let \( \mathcal{P}' = \{ (x, y) \mid x^2 + y^2 < 1 \} \). (Note \( \mathcal{P}' \) is the set of points inside the unit circle.) Let \( \mathcal{L}' \) be the set of restrictions of lines of \( \mathbb{R}^2 \) to \( \mathcal{P}' \). Is \( (\mathcal{P}', \mathcal{L}') \) a linear space?

3. Let \((v, b, r, k, \lambda)\) be the parameters of a balanced incomplete block design, where \( k = 3 \) and \( \lambda = 1 \). Prove that
   
   \[ v \equiv 1, 3 \pmod{6} \]

4. State the dual of the following result.
   
   Let \( \pi \) be a finite projective plane. Every pair of distinct lines of \( \pi \) meet at exactly one point of \( \pi \).

5. Write out the lines of the dual of the projective plane \( \pi = (\mathcal{P}, \mathcal{L}) \) where \( \mathcal{P} = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \mathcal{L} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\} \). Is the dual a projective plane?

6. Let \( \mathcal{Q} = \{P_1, P_2, P_3, P_4\} \) be a quadrangle of \( \pi \), a projective plane of order 2. Let \( L \) be a line in \( \pi \). Can \( |L \cap \mathcal{Q}| = 1 \)?

7. Define the terms
   
   (i) \( \mathcal{L} \)-secant,
   
   (ii) \( \mathcal{L} \)-tangent, and
   
   (iii) \( \mathcal{L} \)-exterior point,
   
   where \( \mathcal{L} \) is a set of lines of a finite projective plane.

8. Consider the projective plane given in Question 5 above. Partition the lines of \( \pi \) into
   
   (i) \( \mathcal{P} \)-secants,
   
   (ii) \( \mathcal{P} \)-tangents, and
(iii) $\mathcal{P}$-exterior lines,

where

(a) $\mathcal{P} = \{1\}$,

(b) $\mathcal{P} = \{1, 2\}$,

(c) $\mathcal{P} = \{1, 2, 3\}$,

(d) $\mathcal{P} = \{1, 2, 3, 6\}$.

9.(i) How many points does a projective plane of order 9 have?

(ii) How many lines does a projective plane of order 9 have?

(iii) How many lines are through a point of a projective plane of order 5?

10. A projective plane $\pi$ has 273 points.

(i) What is the order of $\pi$?

(ii) How many points are there on a line of $\pi$?

11. Let $Q = \{P_1, P_2, P_3, P_4\}$ be a quadrangle of a projective plane $\pi$ of order four, $\mathcal{D} = \{D_1, D_2, D_3\}$ be the set of diagonal points of $Q$ and $\mathcal{Q} = Q \cup \mathcal{D}$. In $\pi$ there are six $Q$-secants (namely $P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, P_3P_4$). Suppose that $D_1$, $D_2$ and $D_3$ are not collinear. Then there are three $D$-secants (namely $D_1D_2, D_1D_3, D_2D_3$). The $Q$-secants and $D$-secants are all of the $Q$-secants.

(a) Retaining the supposition that $D_1$, $D_2$ and $D_3$ are not collinear, answer the following questions:

(i) How many $Q$-secants are there on each point of $Q$?

(ii) How many $Q$-tangents are there on each point of $Q$?

(iii) How many $Q$-secants are there on each point of $\mathcal{D}$?

(iv) How many $Q$-tangents are there on each point of $\mathcal{D}$?

(v) How many $Q$-tangents are there in all?

(b) (i) Show that, if the diagonal points $D_1$, $D_2$ and $D_3$ are not collinear, then there is precisely one $Q$-exterior line.

(ii) Let $L$ be a $Q$-exterior line. (Such a line exists by (b)(i) if the diagonal points $D_1$, $D_2$ and $D_3$ are not collinear.) Show that $|L| \geq 6$. Deduce that the diagonal points of a quadrangle of a projective plane of order four are collinear.