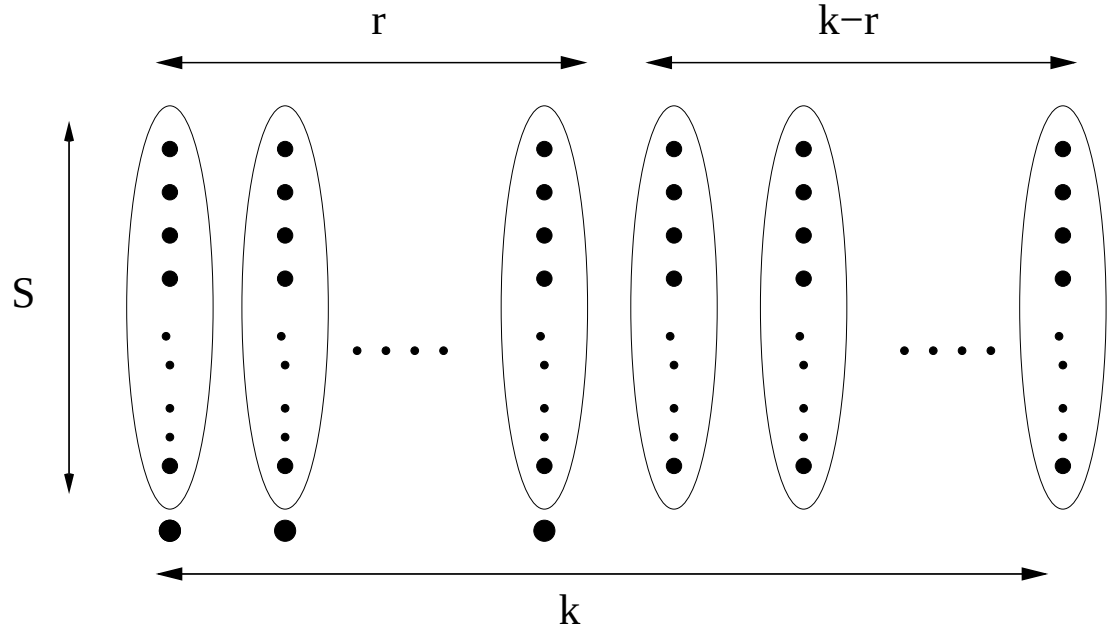
MATH3301 EXTREMAL GRAPH THEORY

Definition: A *near regular complete multipartite graph* is a complete multipartite graph with orders of its partite sets differing by at most 1.

$R(p,k) =$ the near regular complete k-partite graph of order p. Let $p = sk+r$, where s and r are positive integers such that $0 \leq r < k$.



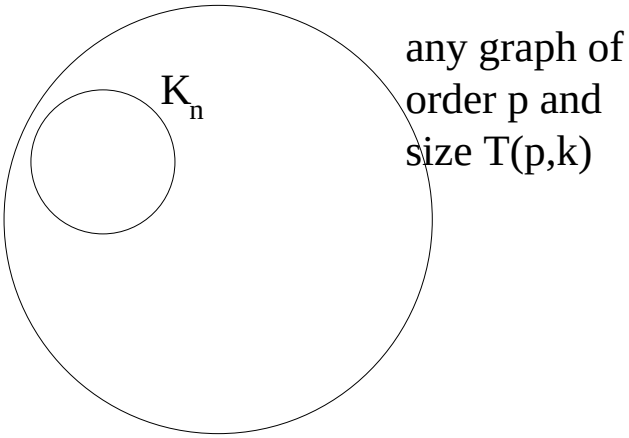
To calculate the number of edges in $R(p,k)$ we calculate the number of edges in K_p and delete edges between vertices in the same subset. So

$$\begin{aligned} \text{size of } R(p,k) &= q(p,k) \\ &= \binom{p}{2} - (k-r)\binom{s}{2} - r\binom{s+1}{2}. \end{aligned}$$

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Problem: $p \geq 2$, a fixed integer.
G, a graph of order at most p.
Find least integer m s.t. *every* graph of order p and size at least m contains G as a subgraph.

Notation: (T for Turán! 1941)
Let $T(p, n)$ denote the smallest integer s.t. every graph of order p and size $T(p, n)$ contains a subgraph isomorphic to K_n.

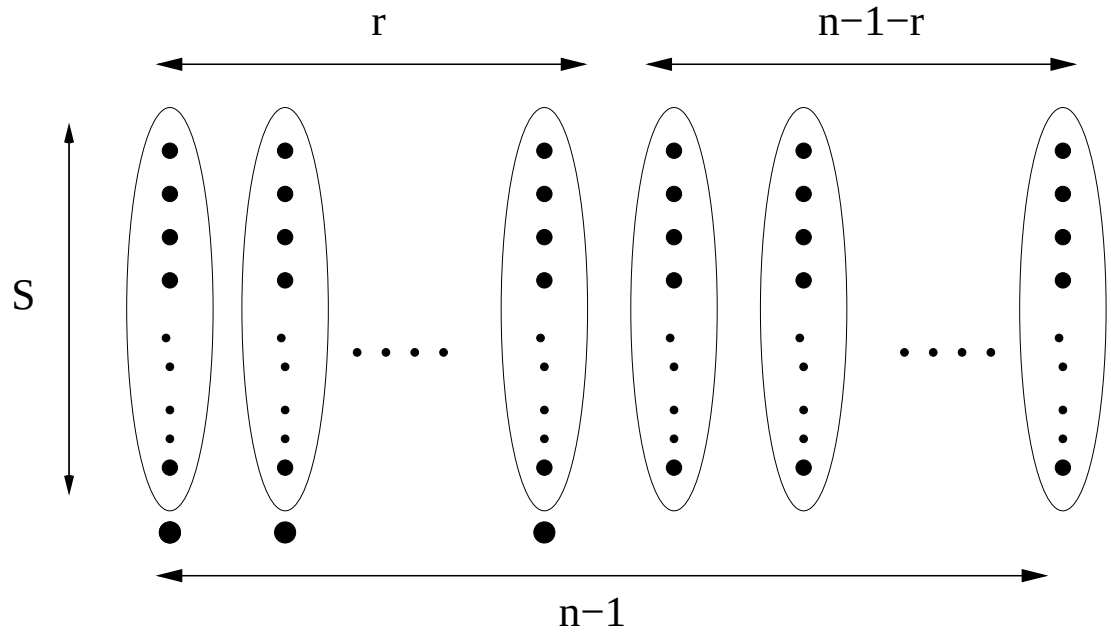


We'll find $T(p, n)$, and show that for all graphs of order p and size $T(p, n) - 1$ there is exactly one graph which does not contain a copy of K_n.

Turán's Theorem: Let $n, p \in \mathbb{N}$ with $2 \leq n \leq p$, and let $T(p,n) =$ least positive integer s.t. every graph of order p and size $T(p,n)$ contains a K_n. Then

(i) $T(p,n) = q(p,n-1)+1$.

(ii) If G is a graph of order p and size $T(p,n)-1$ that doesn't contain K_n, then $G \simeq R(p,n-1)$.



$$\begin{aligned} T(p,n) &= q(p,n-1)+1 \\ &= \binom{p}{2} - (n-1-r)\binom{s}{2} - r\binom{s+1}{2} + 1. \end{aligned}$$

Turán's theorem with $n = 3$, considers the existence of triangles or K_3's.

Corollary For $p \geq 3$, the smallest positive integer $T(p, 3)$ s.t. every graph of order p and size $T(p, 3)$ contains a triangle is

$$T(p, 3) = 1 + \left\lfloor \frac{p^2}{4} \right\rfloor.$$

Also the only triangle-free graph of order p and size $\left\lfloor \frac{p^2}{4} \right\rfloor$ is $R(p, 2) \simeq K_{\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil}$.

Proof of Corollary:
We need to calculate $T(p, 3)$. So let $n = 3$.

$T(p, 3) = q(p, 3-1)+1$ and we must show it equals $1 + \left\lfloor \frac{p^2}{4} \right\rfloor$.

If $n = 3$, then $p = 2s + r$, where $r = 0$ or 1.

For $r = 0$, $p = 2s$ and

$$T(p, 3) = T(2s, 3) = q(2s, 2) + 1.$$

For $r = 1$, $p = 2s + 1$ and

$$T(p, 3) = T(2s + 1, 3) = q(2s + 1, 2) + 1.$$

Case 1: $r = 0$,

$$q(p, n-1)+1 \;=\; \binom{p}{2} - (n-1-r)\binom{s}{2} - r\binom{s+1}{2} + 1$$

$$q(2s, 3-1)+1 = q(2s, 2)+1$$

$$\begin{aligned}
&= \binom{2s}{2} - (2-0)\binom{s}{2} - 0\binom{s+1}{2} + 1 \\
&= \binom{2s}{2} - 2\binom{s}{2} + 1 \\
&= \frac{2s(2s-1)}{2} - 2\frac{s(s-1)}{2} + 1 \\
&= s(2s-1) - s(s-1) + 1 \\
&= s(2s-1-s+1) + 1 \\
&= s^2 + 1
\end{aligned}$$

and

$$\begin{aligned}
\left\lfloor \frac{p^2}{4} \right\rfloor + 1 &= \left\lfloor \frac{(2s)^2}{4} \right\rfloor + 1 \\
&= \left\lfloor \frac{4s^2}{4} \right\rfloor + 1 \\
&= \left\lfloor s^2 \right\rfloor + 1 \\
&= s^2 + 1
\end{aligned}$$

Hence for $n = 3$ and $r = 0$, $T(p, 3) = \left\lfloor \frac{p^2}{4} \right\rfloor + 1$.

Case 2: $r = 1$,

$$q(p, n-1)+1 = \binom{p}{2} - (n-1-r)\binom{s}{2} - r\binom{s+1}{2} + 1$$

$$q(2s+1, 3-1)+1 = q(2s+1, 2)+1$$

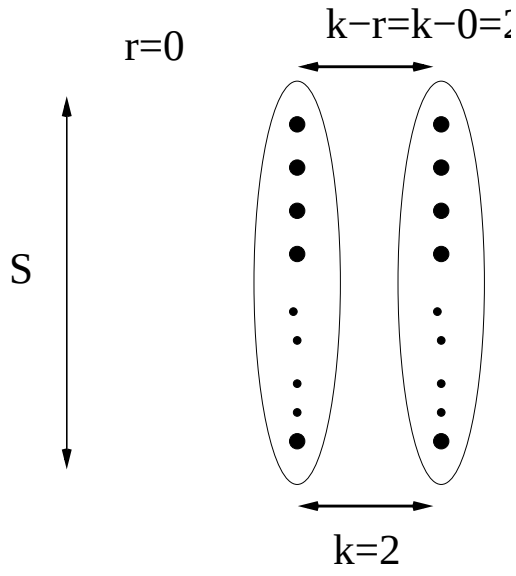
$$\begin{aligned}
&= \binom{2s+1}{2} - (2-1)\binom{s}{2} - 1\binom{s+1}{2} + 1 \\
&= \binom{2s+1}{2} - \binom{s}{2} - \binom{s+1}{2} + 1 \\
&= \frac{(2s+1)2s}{2} - \frac{s(s-1)}{2} - \frac{(s+1)s}{2} + 1 \\
&= s(2s+1) - \frac{s(s-1)+(s+1)s}{2} + 1 \\
&= s(2s+1) - \frac{s(2s)}{2} + 1 \\
&= s(2s+1) - s^2 + 1 \\
&= s^2 + s + 1
\end{aligned}$$

and

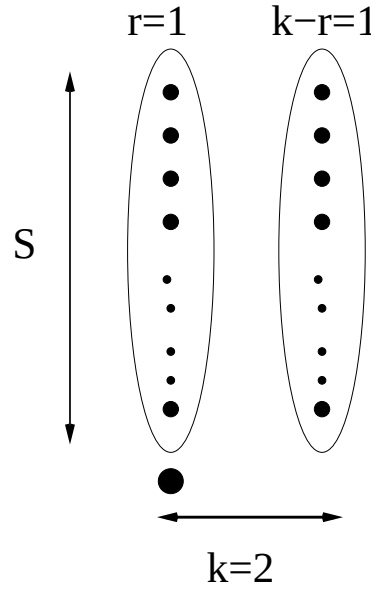
$$\begin{aligned}
\left\lfloor \frac{p^2}{4} \right\rfloor + 1 &= \left\lfloor \frac{(2s+1)^2}{4} \right\rfloor + 1 \\
&= \left\lfloor \frac{4s^2+4s+1}{4} \right\rfloor + 1 \\
&= \left\lfloor s^2 + s + 1/4 \right\rfloor + 1 \\
&= s^2 + s + 1
\end{aligned}$$

Hence for $n = 3$ and $r = 1$, $T(p, 3) = \left\lfloor \frac{p^2}{4} \right\rfloor + 1$.

For $r = 0$ we partition the vertices as two sets of size s, thus $k = 2$, $p = 2s$ and $R(p, 2) = K_{s,s} = K_{\lfloor p/2 \rfloor \lceil p/2 \rceil}$.



For $r = 1$ we partition the vertices as two sets of size s plus one extra point, thus $k = 2$, $p = 2s + 1$ and $R(p, 2) = K_{s+1,s} = K_{\lceil p/2 \rceil \lfloor p/2 \rfloor} = K_{\lfloor p/2 \rfloor \lceil p/2 \rceil}$.



Ramsey Numbers

Recall $\overline{G}$ is the complement of the graph G. That is, $V(\overline{G}) = V(G)$ and $\{x, y\} \in E(\overline{G})$ if and only if $\{x, y\} \notin E(G)$.

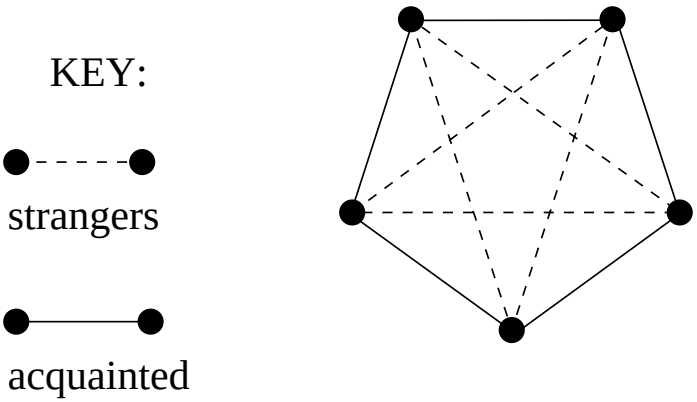
Def'n: $r(m, n)$ denotes a Ramsey number. It is the least positive integer p such that for *any* graph G on p vertices

- either G contains a copy of K_m,
- or else $\overline{G}$ contains a copy of K_n.

Note $G \cup \overline{G} = K_p$ so we may restate the definition as: $r(m, n)$ is the least positive integer p such that if we 2-colour the edges of K_p then either we have a mono-chromatic K_m or a mono-chromatic K_n.

Example

Can have 5 people, with no three of them all acquainted and no three of them all strangers.



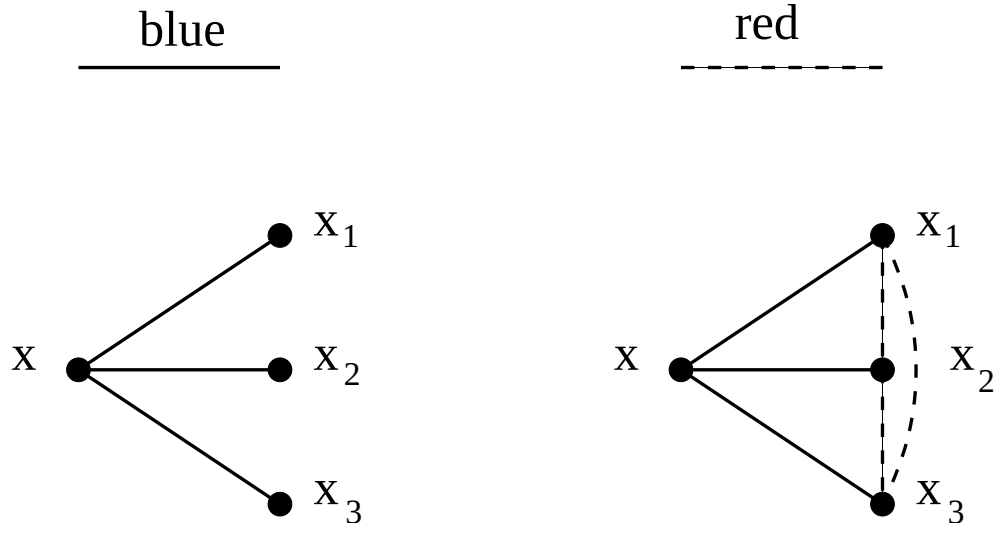
But with 6 people you *have* to have three of them, either all acquainted or all strangers!

Lemma If we 2-colour the edges of K_6, then there will always exist a monochromatic triangle; that is, $r(3,3) = 6$.

Proof: Assume this is not the case and that we can colour the edges of K_6 red or blue in such a way that there is no monochromatic triangle.

Let x be an arbitrary vertex. It has degree 5. Without loss of generality we may assume x is incident with 3 blue edges. Label these edges $\{x, x_1\}$, $\{x, x_2\}$, and $\{x, x_3\}$. If any edges $\{x_1, x_2\}$, $\{x_1, x_3\}$ or $\{x_2, x_3\}$ are coloured blue then we have a blue triangle, a contradiction. Hence the edges $\{x_1, x_2\}$, $\{x_1, x_3\}$ or $\{x_2, x_3\}$ are all coloured red but this implies there is a red triangle, another contradiction.

Thus the result follows.



Theorem $r(m,n) = r(n,m)$.

Proof: Let $r(m,n) = p$. Then by definition if G is a graph G of order p, then G contains K_m or $\overline{G}$ contains K_n.

We'll show that either G contains K_n or $\overline{G}$ contains K_m.

Note that $\overline{G}$ is a graph of order p, and since $r(m,n) = p$, then either $\overline{G}$ contains K_m or $\overline{\overline{G}} = G$ contains K_n. Hence $r(n,m) \leq p$.

Also we know that since $r(m,n) = p$ there exists a graph H, of order $p-1$, such that H contains no K_m and $\overline{H}$ contains no K_n.

Equivalently $\overline{H}$ is a graph of order $p-1$ which does not contains a K_n, and $\overline{\overline{H}} = H$ is a graph of order $p-1$ which does not contains a K_m, implying $r(n,m) \geq p$.

Given $r(n,m) \leq p$ and $r(n,m) \geq p$, it follows that

$$r(n,m) = p.$$

Fact $r(1, m) = r(m, 1) = 1$, for all $m \in \mathbb{N}$.

Proof: $r(1, m)$ represents the least positive integer, p, such that whenever we 2-colour the edges of K_p we obtain a monochromatic K_1 or a monochromatice K_m.

If $p = 1$ then the graph K_1 is a single vertex.

By default, K_1 always contains a monochromatic K_1.

Fact $r(2,m) = r(m,2) = m$, for all $m \in \mathbb{N}$.

Proof: Note $r(2,m)$ represents the least positive integer, p, such that all graphs G of order p contain K_2 or their complement contains K_m. Consider a graph G on m vertices, either G contains an edge (a K_2) or it contains no edges, in which case its complement contains a K_m.

To complete the proof we must show that there exists a graph G of order $m-1$ which contains no K_2 and its complement contains no K_m.

Let G be the graph on $m-1$ vertices but with no edges. This graph contains no edges, so no K_2. It complement is K_{m-1} which does not contain a K_m. Hence $r(2,m) \geq m$.

Since $r(2,m) \leq m$ and $r(2,m) \geq m$,

$$r(2,m) = m.$$

Known facts

$$r(3,3) = 6$$

$$r(3,4) = 9$$

$$r(3,5) = 14$$

$$r(3,6) = 18$$

$$r(3,7) = 23$$

$$r(3,8) = 28$$

found by McKay and Min in 1992

$$r(3,9) = 36$$

$$r(4,4) = 18$$

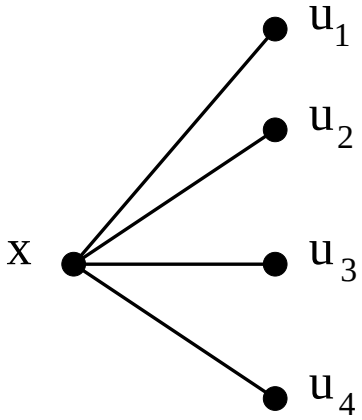
Lemma $r(3,4) = 9$.

Proof We will first show by contradiction that $r(3,4) \leq 9$, then show $r(3,4) > 8$.

Let G be a graph on 9 vertices such that G does not contain a K_3 and $\overline{G}$ does not contain a K_4.

Two cases to consider: Case 1: G contains a vertex, x of degree at least 4; Case 2: all vertices of G have degree 3 or less.

Case 1: Assume G contains a vertex, x, of degree at least 4.



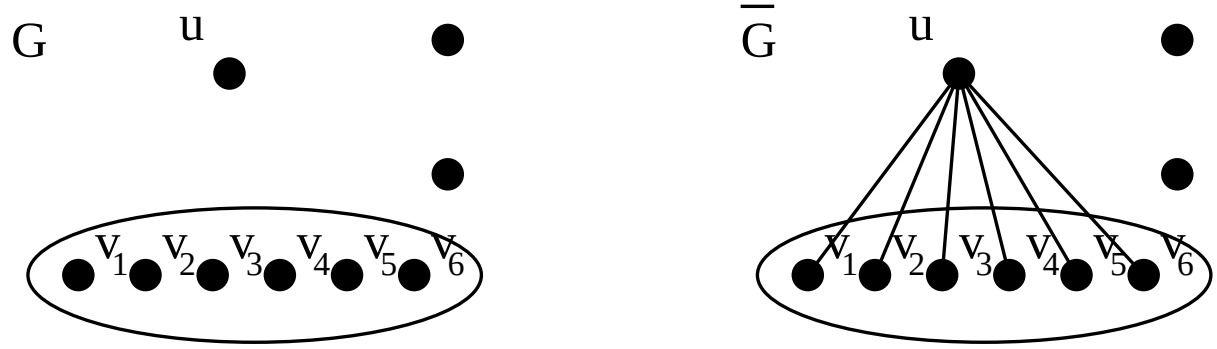
Since G does not contain a triangle the vertices u_1, u_2, u_3, u_4 are non adjacent in G. However this implies $\overline{G}$ contains a K_4 on the vertices u_1, u_2, u_3, u_4, contradicting our assumption. Thus Case 1 is not possible.

Case 2: The maximum degree of any vertex in G is 3.

Since G has 9 vertices and $\sum$ degrees $=$ even, there exists at least one vertex, say u, of even degree, so degree 0 or 2.

This implies there are at least six vertices, say $v_1, \ldots, v_6$, such that u and v_i are not adjacent in G but are adjacent in $\overline{G}$.

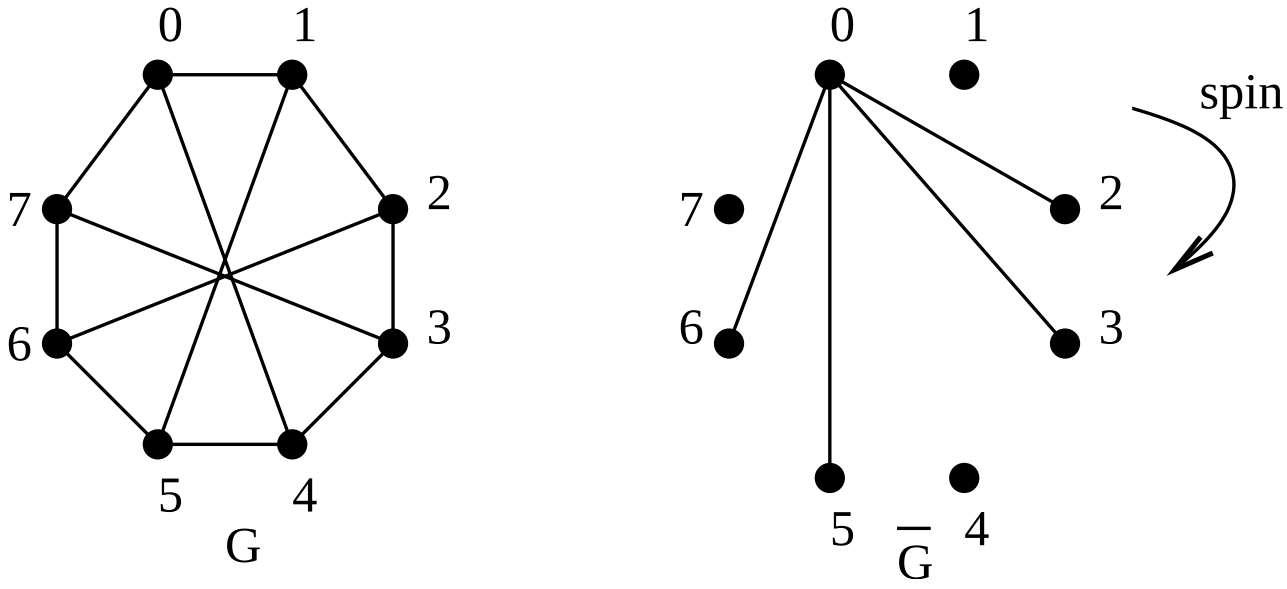
Focus on the vertices $v_1, \ldots, v_6$, and the induced subgraph in G.



Since $r(3,3) = 6$, either G or $\overline{G}$ contains a K_3 on these 6 vertices. If $\overline{G}$ contains a K_3 on these 6 vertices, then the additional edges from u to each of $v_1, \ldots, v_6$ in $\overline{G}$ constitute a K_4.

Hence $r(3,4) \leq 9$.

To prove $r(3,4) > 8$ we must exhibit a graph G of order 8, with no K_3 and such that $\overline{G}$ contains no K_4.



So $r(3,4) > 8$.

Therefore since $r(3,4) \geq 9$ and $r(3,4) \leq 9$, we have

$$r(3,4) = 9.$$

Theorem (Erdös and Szekeres)

For all $m, n \in \mathbb{N}$, $r(m,n)$ exists, and

(a) $r(m,n) \leq \binom{m+n-2}{m-1}$,

(b) $r(m,n) \leq r(m-1,n) + r(m,n-1)$

(if $m \geq 2$, $n \geq 2$.)

Proof: Use induction on $k = m + n$.

$m = 1$: $r(1,n) = 1$; $\binom{m+n-2}{m-1} = \binom{n-1}{0} = 1$;

$m = 2$: $r(2,n) = n$; $\binom{m+n-2}{m-1} = \binom{n}{1} = n$.

So equality when $m = 1$ or 2.

Similarly, when $n = 1$ or 2, $\forall\ m$, have equality.

So (a) holds for all $k = m + n = 2, 3, 4, 5$.

(5 is $2 + 3$ or $1 + 4$.)

Now assume $m \geq 3$ and $n \geq 3$.

Inductive hypothesis: $r(s,t)$ exists $\forall\ s, t$ with $s + t < k$, where $k \geq 6$, and $r(s,t) \leq \binom{s+t-2}{s-1}$.

Now take $m \geq 3$, $n \geq 3$ and $m + n \geq k$. From inductive hypothesis, $r(m-1,n)$ and $r(m,n-1)$ exist, and

$r(m-1,n) \leq \binom{m+n-3}{m-2}$, $r(m,n-1) \leq \binom{m+n-3}{m-1}$.

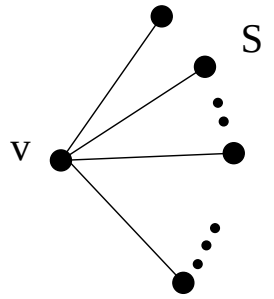
But $\binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1}$ (exercise!)

so let $p = r(m-1,n) + r(m,n-1) \leq \binom{m+n-2}{m-1}$. (*)

Consider K_p: colour its edges red and blue.

Claim: $\exists$ red K_m or blue K_n.

Let $v \in V(K_p)$.
deg $v = p-1 = r(m-1,n)+r(m,n-1)-1$. Either:
v incident with $\geq r(m-1,n)$ red edges:
$|S| \geq r(m-1,n)$.
Then $\langle S \rangle$ is a complete graph
of order at least $r(m-1,n)$,
so it contains a red K_{m-1}
or blue K_n.



So K_p has a red K_m ($\{v\} \cup \langle S \rangle$) or else a blue K_n.

Or: v incident with $\leq r(m-1,n)-1$ red edges.
Using deg v, get at least $r(m,n-1)$ blue edges at v — same argument.
$|T| \geq r(m,n-1)$
$\langle T \rangle$ has red K_m or blue K_{n-1}.
So K_p has red K_m or blue K_n.
So:
$r(m,n) \leq p = r(m-1,n) + r(m,n-1)$ (hence (b))

i.e. $r(m,n) \leq \binom{m+n-2}{m-1}$, from (*) (hence (a)).

Note: $m = n = 3$, also equality: $r(3,3) = 6$; $r(2,3) + r(3,2) = 3 + 3$.
Note: $r(4,5) = 25$ (McKay & Radziszowski 1993).
So can get bounds: $r(5,5) \leq 50$, $r(4,6) \leq 43$.

Generalized Ramsey Numbers

Let F, H be two graphs. The
Ramsey number $r(F, H)$ is the least $p \in \mathbb{N}$ s.t.
if G is *any* graph of order p, then either
F is a subgraph of G or H is a subgraph of $\overline{G}$.

So $r(K_m, K_n) \equiv r(m, n)$ earlier.

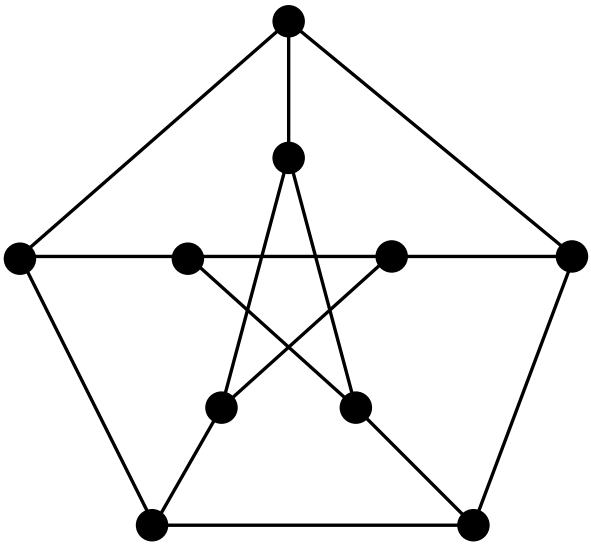
Note: $r(F, H)$ exists, for if F has order m and H has order n, then $r(F, H) \leq r(m, n)$, which exists.

Definition $\chi(G)$ is the chromatic number of a graph G; that is, the least number of colours which are needed to colour the vertices so that no two adjacent verices are coloured with the same colour.

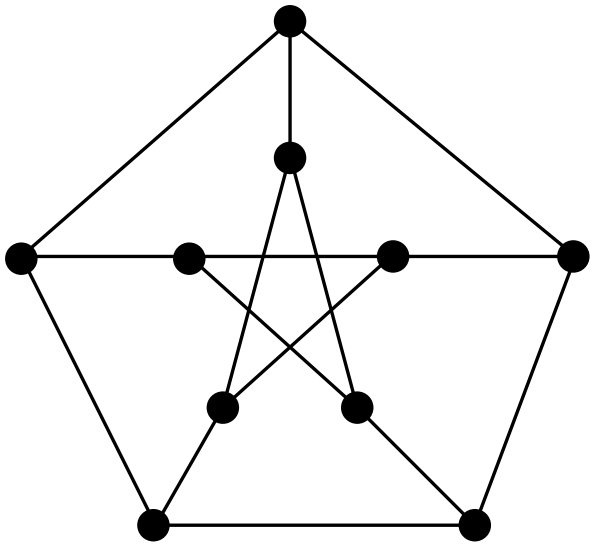
Note $\chi(G) \leq \Delta(G) + 1$ which is the maximum degree take over all vertices in G.

$\beta(G)$ is the independence number which is the greatest number of vertices in G with the property that none of these vertices are adjacent.

Example The Petersen graph P



$\chi(P)$



$\beta(G)$

$\chi(P) \leq \delta(G) = 3$. Considering the outside cycle $\chi(P) \geq 2$, hence $\chi(P) = 3$.

We can have at most 2 independent vertices in the outer cycle, and at most 2 independent vertices in the inner cycle, hence $\beta(P) \leq 4$. The above example shows $\beta(G) = 4$.

Note that in a vertex colouring of a graph each colour class gives an independent set.

Fact 1 Let G be a graph of order p, then $\chi(G)\beta(G) \geq p$.

Proof Take the p vertices, they can be partitioned into $\chi(G)$ colour classes and each colour class contains at most $\beta(G)$ vertices. Hence $p \leq \chi(G)\beta(G)$.

Fact 2 Let G be a graph of order $p = 1 + (m-1)(n-1)$ and $\beta(G) \leq n-1$. Then $\chi(G) \geq m$.

Proof We know $\chi(G)\beta(G) \geq p$, so

$$\begin{aligned}
\chi(G)\beta(G) &\geq 1 + (m-1)(n-1) \\
\chi(G)(n-1) &\geq 1 + (m-1)(n-1) \\
\chi(G) &\geq \frac{1}{n-1} + (m-1) \text{ and since } n \geq 2 \\
\chi(G) &\geq 1 + (m-1) = m
\end{aligned}$$

Given any graph G and any vertex v of G

$$\chi(G-v) \;=\; \begin{cases} \chi(G) \\ \chi(G)-1 \end{cases}$$

A graph G is critically n-chromatic if $\chi(G) = n$ and when we remove any vertex v

$$\chi(G-v) < \chi(G).$$

So a graph G is critically n-chromatic if $\chi(G) = n$ and $\forall v \in V(G)$, $\chi(G-v) = n-1$.

Given a graph G, recall that $\delta(G)$ denotes the minimum degree taken over all vertices of G.

Fact 3 Let G be a critically n-chromatic graph. Then $\delta(G) \geq n-1$.

Proof Assume G is critically n-chromatic. Hence for all vertices v, $\chi(G-v) = n-1$. Fix v.

Label the colour $C_1, \ldots, C_n$ and let S_i denote the set of vertices coloured C_i. Note that each S_i is an independent set.

Since $\chi(G-v) = n-1$, for each set S_i there exists an edge from v to at least one vertex in S_i. Hence $deg(v) \geq n-1$, and consequently, when we vary v across the vertex set $\delta(G) \geq n-1$.

Fact 4 If T is a tree of order m and G is a graph for which $\delta(G) \geq m-1$, then T is a subgraph of G.

Proof By induction.

K_1 and K_2 are subgraphs of all non-empty graphs.

Assume that for any tree T_1 on $m-1$ vertices and any graph H with $\delta(H) \geq m-2$, T_1 is a subgraph of H.

Let T have order m and G be a graph with $\delta(G) \geq m-1$. We wish to show that T is a subgraph of G. Let $v \in V(T)$ be a vertex of degree 1 adjacent to vertex, say, u.

Then $T-v$ is a tree of order $m-1$ and $\delta(G) \geq m-1 > m-2$.

By the inductive hypothesis $T-v$ is isomorphic to some subgraph T' of G. Let $u' \in V(T')$ be the vertex in T' which corresponds to u. Since $T-v$ is a tree of order $m-1$ and $deg_G(u') \geq m-1$ there exists $w \in V(G)$ such that w is adjacent to u', but $w \notin V(T')$. It follows that $T' + \{u', w\}$ is a tree isomorphic to T in G.

Theorem (Chvátal)
Let T be any tree of order $m \geq 1$, and let $n \in \mathbb{N}$. Then $r(T, K_n) = 1 + (m-1)(n-1)$.

Proof: True if either $m = 1$ or $n = 1$.

So assume $m \geq 2$ and $n \geq 2$.

Let $p = 1 + (m-1)(n-1)$, and let G be a graph of order p.
Claim: T is a subgraph of G, or K_n is a subgraph of $\overline{G}$. Say K_n is *not* a subgraph of $\overline{G}$. Then $\overline{G}$ doesn't contain n mutually non-adjacent vertices. So $\beta(G) \leq n-1$. By Fact 1,

$$\chi(G).\beta(G) \geq p \; = \; 1 + (m-1)(n-2).$$

Since $\beta(G) \leq n-1$, this means $\chi(G) \geq m$, see Fact 2.

Let F be a critically m-chromatic subgraph of G. By Fact 3, $\delta(F) \geq m-1$. Then by Fact 4, T (any tree of order M) is a subgraph of F, so T is a subgraph of G. Hence $r(T, K_n) \leq p$.

Now let $H \simeq (n-1)K_{m-1}$, so H has order $(n-1)(m-1) = p-1$. Since each component of H has order $m-1$, T isn't a subgraph of H. Since $\overline{H} \simeq$ complete $(n-1)$-partite graph $K_{m-1,m-1,\ldots,m-1}$, $\overline{H}$ doesn't contain K_n as a subgraph. Hence $r(T, K_n) \geq p$.

Further generalization:
Why stop at two graphs, or two colours for edges of K_n?

Let G_i, $1 \leq i \leq k$, $k \geq 2$ be graphs. Then $r(G_1, G_2, \ldots, G_k) =$ least $p \in \mathbb{N}$ s.t. for any factorization $K_p = F_1 \oplus F_2 \oplus \cdots \oplus F_k$, G_i is a subgraph of F_i for at least one i.
[Note $r(G_1, G_2, \ldots, G_k)$ exists.]

Write $r(n_1, n_2, \ldots, n_k)$ for $r(K_{n_1}, K_{n_2}, \ldots, K_{n_k})$. The only such Ramsey number known, where $k \geq 3$ and $n_i \geq 3$ for each i is $r(3,3,3) = 17$.
So K_{16} has a 3-colouring of its edges, with no monochromatic triangle, but K_{17} must have a triangle in at least one of the three colours.