Solutions Assignment 3

Q1

By Erdos & Szekeres Th^m $r(m,n) \leq \binom{m+n-2}{m-1}$

Hence $r(3,n) \leq \binom{3+n-2}{3-1} = \binom{n+1}{2} = \frac{(n+1)n}{2}$.

Thus $r(3,n) \leq \frac{n^2+n}{2}$.

Q2.

By Erdos & Szekeres Th^m $r(m,n) \leq r(m-1,n) + r(m,n-1)$.

i) $r(5,5) \leq r(4,5) + r(5,4) + r(4,5) = 25$. So

i) $r(5,5) \leq 25 + 25 = 50$.

ii) $r(4,6) \leq r(4,5) + r(3,6) = 25 + 18 = 43$.

Q3. It is assumed $r(m,n) = p+1$. Thus if we 2-colour (red or blue) the edges of K_{p+1}, we will have a red K_m or a blue K_n.

Take a copy of K_p and adjoin a vertex, v, with edges to every vertex in K_p. So we have a copy of K_{p+1}. Two colour the edges of the K_{p+1} red or blue. Now we either have a red K_m or a blue K_n. If vertex v is in either the K_m or the K_n, we have a red K_{m-1} (excluding vertex v) or a blue K_{n-1} (excluding vertex v). So if we remove vertex v from K_{p+1} in the K_p we can be sure we have at least a red K_{m-1} or a blue K_{n-1}. So we can be sure we satisfy the stated conditions

Solutions to Assignment 3.

Q4 We will show that if n is even there is no 2-factorization, this is equivalent to showing if a 2-factorization exists then n is odd. So assume n is even. Then every vertex is of odd degree. A 2-factorization is a collection of 2-factors and in each 2-factor the degree of each vertex is 2. So if a 2-factorization exists each vertex has even degree, a contradiction. Hence no 2-factorization exists.

Next we will show that if n is odd a 2-factorization exists. To do this we label the vertices $1, 2, \ldots, n-1, \infty$ and then

$$(\infty, 1, 2, n-1, 3, n-2, \ldots \tfrac{n-1}{2}, \tfrac{n+3}{2}, \tfrac{n+1}{2})$$

is a Hamiltonian cycle. Hence a 2-factor. To obtain $\frac{n+1}{2}$ 2-factors we take the cycles

$$(\infty, 1+i, 2+i, n-1+i, 3+i, n-2+i, \ldots, \tfrac{n-1}{2}+i, \tfrac{n+3}{2}+i, \tfrac{n+1}{2}+i)$$

for $0 \leq i \leq \frac{n-1}{2}$. These cycles are edge disjoint and their union is K_n.

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Assignment 3 Solution (3)

Q5. Recall total degree of a graph is the sum of the degrees of each vertex which is twice the number of edges. So since the minimum degree of any vertex, $\delta(G) \geq m+1$ we have

$$\sum \text{degrees} = 2(\text{number of edges}) \geq 2m(m+1)$$

So.

$$\text{number of edges} \geq m(m+1).$$

We want to show G must contain a triangle so we can use Turan's theorem to calculate the minimum number of edges in G.

$$T(p,3) = T(2m,3)$$
$$= q(2m,2) \quad (\text{note } n=3, n-1=2, s=m, r=0)$$
$$= \binom{2m}{2} - (2-0)\binom{m}{2} - 0\binom{m+1}{2} + 1$$
$$= \frac{2m(2m-1)}{2} - 2\frac{(m(m-1))}{2} + 1$$
$$= m(2m-1) - m(m-1) + 1$$
$$= m(2m-1-m+1) + 1$$
$$= m^2 + 1.$$

Our graph has at least m^2+1 edges. Therefore by Turan's theorem it contains a triangle (K_3)