MATHEMATICS DEPARTMENT: MATH3301 GEOMETRY

Problem Sheet 7: SOLUTIONS

1. Let L and M belong to $\mathcal{L}$ and $P \in M$, but $P \notin L$, such a point always exists since $|M| \geq 2$, and even if $L \cap M \neq \emptyset$, we may choose $P \notin L \cap M$. If $P \in L$ then L and M meet. So assume $P \notin L$. There are $k+1$ lines on P and $k+1$ points on L. Each of the points on L must be joined to P by a line. So each of the lines on P must meet L. Hence M must meet L. Further all points must occur on one of the lines through P.

Fix P. $|\mathcal{P} \setminus P| = v - 1$ and $|P| = k + 1$ so

$$\frac{v-1}{k+1} = k + 1 - 1$$
$$\Rightarrow v = (k+1)k + 1 = k^2 + k + 1.$$

Now each point has $k+1$ lines through it and there are $k+1$ points on each line. Hence

$$v(k+1) = b(k+1)$$
$$\Rightarrow b = v.$$

2.(i) Yes for every pair of points there exist unique real numbers a, b, c such that $y = ax + b$ or $x = c$ and every line passes through at least two points.

(ii) Since we have excluded the boundary, every pair of points is joined by a line and every line passes through at least two points.

3. We know that $\lambda(v-1) = r(k-1)$ and since $k = 3$ and $\lambda = 1$, we have

$$\begin{aligned} \lambda(v-1) &= r(k-1) \\ v - 1 &= 2r \end{aligned}$$

Since v is an integer we obtain that v is odd. Also we know $vr = bk$ and since $k = 3$, and $2r = v - 1$ we obtain

$$\begin{aligned} vr &= bk \\ v(v-1) &= 3b. \end{aligned}$$

So $v(v-1)$ must be a multiple of 3 and we obtain $v \equiv 1, 3 (\text{mod } 6)$.

4. Let π be a finite projective plane. Every pair of distinct points are on exactly 1 line of π.

5. Let $L_1 = \{1,2,3\}, L_2 = \{1,4,5\}, L_3 = \{1,6,7\}, L_4 = \{2,4,6\}, L_5 = \{2,5,7\}, L_6 = \{3,4,7\}$, $L_7 = \{3,5,6\}$. Let $\pi^d = (\mathcal{P}', \mathcal{L}')$ where $\mathcal{P}' = \mathcal{L}$ and $\mathcal{L}' = \mathcal{P}$, so $\mathcal{P}' = \{L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$ and
$\mathcal{L}' = \{\{L_1, L_2, L_3\}, \{L_1, L_4, L_5\}, \{L_1, L_6, L_7\}, \{L_2, L_4, L_6\}, \{L_2, L_5, L_7\}, \{L_3, L_4, L_7\}, \{L_3, L_5, L_6\}\}$.
Yes the dual is a projective plane.

6. Let $\mathcal{Q} = \{P_1, P_2, P_3, P_4\}$ be a quadrangle of π, a projective plane of order 2. The quadrangle $\mathcal{Q}$ has 6 $\mathcal{Q}$–secants and one $\mathcal{Q}$–exterior line. There are seven lines in total. Hence there are no $\mathcal{Q}$–tangents.

7(i) A point P of π is said to be an $\overline{\mathcal{L}}$–secant if P is on more than one line of $\overline{\mathcal{L}}$.

(ii) A point P of π is said to be an $\overline{\mathcal{L}}$-tangent if P is on precisely one line of $\overline{\mathcal{L}}$.

(iii) A point P of π is said to be an $\overline{\mathcal{L}}$-exterior point if P is on no line of $\overline{\mathcal{L}}$.

8. Let the lines of this plane be labeled as above.

	Secants	Tangents	Exterior Lines
(a)		L_1, L_2, L_3	L_4, L_5, L_6, L_7
(b)	L_1	L_2, L_3, L_4, L_5	L_6, L_7
(c)	L_1	L_2, L_3, L_4, L_5, L_6, L_7	
(d)	L_1, L_3, L_4, L_7	L_2, L_5, L_6,	

9.(i) $9^2 + 9 + 1 = 91$. (ii) $9^2 + 9 + 1 = 91$. (iii) $5 + 1 = 6$.

10.(i) $n^2 + n + 1 = 273$ so $n(n + 1) = 272$. Therefore $n = 16$ or -17. So the projective plane π has order 16.

(ii) 17.

11.(a)(i) 3

(ii) 5-3=2

(iii) 4

(iv) 5-4=1

(v) $4 \times 2 + 3 \times 1 = 11$.

(b)(i) There are 21 lines in all of π, 9 of which are $\overline{\mathcal{Q}}$–secants and 11 of which are $\overline{\mathcal{Q}}$–tangents. So the number of lines which are $\overline{\mathcal{Q}}$–exterior lines is $21 - 9 - 11 = 1$.

(ii) The six $\mathcal{Q}$–secants meet L at 6 different points since $\mathcal{Q} \cap (L) = \emptyset$. So $|L| \geq 6$. But π is a projective plane of order 4 and so $|L| = 4 + 1 = 5$. Clearly we have a contradiction. So our supposition that the diagonal points D_1, D_2 and D_3 are not collinear is false. But $\mathcal{Q}$ is arbitrary and so the diagonal points of any quadrangle $\mathcal{Q}$ of π are collinear.