MATHEMATICS DEPARTMENT: MATH3301 GEOMETRY

MATH3301: SOLUTIONS Problem Sheet 8

1.(i) To establish (i) it is sufficient to show that ϕ restricted to $\mathcal{P}_1$ is one–to–one as it is onto by definition. Since π_1 is a projective plane of order n $|\mathcal{P}_1| = n^2 + n + 1$. Assume that ϕ is not one-to-one, then there exists point P and Q of $\mathcal{P}$ such that $\phi(P) = \phi(Q)$. Now since ϕ is onto $\phi(\mathcal{P}) = \mathcal{P}_2$ and it follows that $|\mathcal{P}_2| \leq n^2 + n + 1 - 1$. But this is a contradiction as π_2 is a projective plane of order n and we must have $|\mathcal{P}_2| = n^2 + n + 1$. ϕ is one–to–one from $\mathcal{P}_1$ onto $\mathcal{P}_2$.

(ii) Similarly we can show ϕ is one–to–one from $\mathcal{L}_1$ onto $\mathcal{L}_2$.

2. Consider the mapping ϕ^* given by

$$\phi^* \ : \ (P_1, L_1) \rightarrow (\phi(P_1), \phi(L_1)),$$

which ϕ induces on $\mathcal{P}_1 \times \mathcal{L}_1$. Since ϕ was shown to be one–to–one and onto it follows that ϕ^* is one–to–one and onto from $\mathcal{P}_1 \times \mathcal{L}_1$ to $\mathcal{P}_2 \times \mathcal{L}_2$. Define a set $\mathcal{I}_1 \subseteq \mathcal{P}_1 \times \mathcal{L}_1$ where $\mathcal{I}_1 = \{(P_1, L_1) \mid P_1 \in \mathcal{P}_1, L_1 \in \mathcal{L}_1$ where point P_1 is on line $L_1\}$. Similarly we can define $\mathcal{I}_2$ and $\phi^*(\mathcal{I}_1)$. Since ϕ is incidence preserving, we have $\phi^*(\mathcal{I}_1) \subseteq \mathcal{I}_2$. Furthermore, since ϕ^* is one–to–one we have $|\mathcal{I}_1| = |\phi^*(\mathcal{I}_1)|$. But $|\mathcal{I}_1| = |\mathcal{I}_2| = (n+1)(n^2+n+1)$, (the number lines multiplied by the number of points on each line.) It follows that $\phi^*(\mathcal{I}_1) = \mathcal{I}_2$. But then we must have $(\mathcal{P}_2 \times \mathcal{L}_2) \setminus \phi^*(\mathcal{I}_1) = (\mathcal{P}_2 \times \mathcal{L}_2) \setminus \mathcal{I}_2$. It now follows that ϕ is non–incidence preserving.

3. If D_1, D_2, D_3 are collinear, then the structure which is the restriction of π, has points $\mathcal{Q} \cup \{D_1, D_2, D_3\}$ and lines $\mathcal{Q}(s) \cup L_\infty$ where $\mathcal{Q}(s)$ are the $\mathcal{Q}$ secants. These lines are

$$\begin{array}{lll} \{P_1, P_2, D_1\} & \{P_3, P_4, D_1\} & \{P_1, P_3, D_2\} \\ \{P_2, P_4, D_2\} & \{P_1, P_4, D_3\} & \{P_2, P_3, D_3\} \\ \{D_1, D_2, D_3\}. & & \end{array}$$

Which are the lines of a projective plane of order 2.

4. If axioms A1, A5, A6 and A7 are self–dual then any incidence structure satisfying axioms A1, A5, A6 and A7 also satisfies the dual of A1. Take the incidence structure with points $1, 2, 3, 4, 5, 6$ and lines $\{2, 4\}, \{2, 5\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{2, 6\}$. The dual of axiom A1 states that every pair of distinct lines meet in a point. Obviously the line $\{2, 4\}$ and $\{5, 6\}$ do not meet. Hence this set of axioms is not self–dual.

5. Try to prove this for yourself. It is a typical exam question. However, it is also proved in the lecture notes, so you can check your results.

6. There are n lines in a parallel class of L and $n^2 + n$ lines in all. So n^2 lines are not in the parallel class which contains L. But these lines each meet L in a unique point. So there are n^2 lines which meet L in a unique point.

7.(i) 9.

(ii) 12.

(iii) 4.

(iv) 3.

(v) 3.

(vi) 4.

(vii) Let the points of the affine plane be 1, 2, 3, 4, 5, 6, 7, 8, 9. The affine plane is isomorphic to

$$\begin{array}{ccc} L_{\alpha 1} = \{1,2,3\} & L_{\alpha 2} = \{4,5,6\} & L_{\alpha 3} = \{7,8,9\} \\ L_{\alpha 4} = \{1,4,7\} & L_{\alpha 5} = \{2,5,8\} & L_{\alpha 6} = \{3,6,9\} \\ L_{\alpha 7} = \{1,5,9\} & L_{\alpha 8} = \{2,6,7\} & L_{\alpha 9} = \{3,4,8\} \\ L_{\alpha 10} = \{1,6,8\} & L_{\alpha 11} = \{2,4,9\} & L_{\alpha 12} = \{3,5,7\} \end{array}$$

(viii) Let the points of the projective plane be 1, 2, 3, 4, 5, 6, 7, 8, 9, $c_1,\ c_2,\ c_3,\ c_4$. The projective plane is isomorphic to

$$\begin{array}{ccc} L_1 = \{1,2,3,c_1\} & L_2 = \{4,5,6,c_1\} & L_3 = \{7,8,9,c_1\} \\ L_4 = \{1,4,7,c_2\} & L_5 = \{2,5,8,c_2\} & L_6 = \{3,6,9,c_2\} \\ L_7 = \{1,5,9,c_3\} & L_8 = \{2,6,7,c_3\} & L_9 = \{3,4,8,c_3\} \\ L_{10} = \{1,6,8,c_4\} & L_{11} = \{2,4,9,c_4\} & L_{12} = \{3,5,7,c_4\} \\ & & L_{13} = \{c_1,c_3,c_3,c_4\} \end{array}$$

You can check that the axiom for a projective plane are satisfied in this example. You should also make sure that you can explain the method of construction. See lecture notes.